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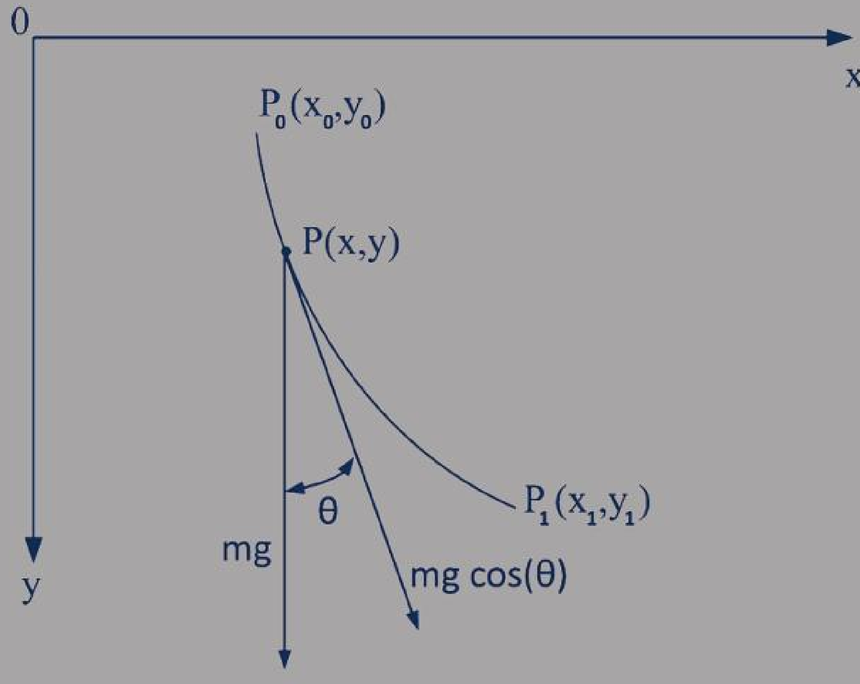
Nonlinear Optimal Control Theory



Leonard D. Berkovitz
Negash G. Medhin

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Nonlinear Optimal Control Theory

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Leonard D. Berkovitz
Purdue University

Negash G. Medhin
North Carolina State University

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Foreword

This book provides a thorough introduction to optimal control theory for nonlinear systems. It is a sequel to Berkovitz's 1974 book entitled *Optimal Control Theory.* In optimal control theory, the Pontryagin principle, Bellman's dynamic programming method, and theorems about existence of optimal controls are central topics. Each of these topics is treated carefully. The book is enhanced by the inclusion of many examples, which are analyzed in detail using Pontryagin's principle. These examples are diverse. Some arise in such applications as flight mechanics, and chemical and electrical engineering. Other examples come from production planning models and the classical calculus of variations.

An important feature of the book is its systematic use of a relaxed control formulation of optimal control problems. The concept of relaxed control is an extension of L. C. Young's notion of generalized curves, and the related concept of Young measures. Young's pioneering work in the 1930s provided a kind of "generalized solution" to calculus of variations problems with nonconvex integrands. Such problems may have no solution among ordinary curves. A relaxed control, as defined in Chapter 3, assigns at each time a probability measure on the space of possible control actions. The approach to existence theorems taken in Chapters 4 and 5 is to prove first that optimal relaxed controls exist. Under certain Cesari-type convexity assumptions, optimal controls in the ordinary sense are then shown to exist. The Pontryagin maximum principle (Chapters 6 and 7) provides necessary conditions that a relaxed or ordinary control must satisfy. In the relaxed formulation, it turns out to be sufficient to consider discrete relaxed controls (see Section 6.3). This is a noteworthy feature of the authors' approach.

In the control system models considered in Chapters 2 through 8, the state evolves according to ordinary differential equations. These models neglect possible time delays in state and control actions. Chapters 10, 11, and 12 consider models that allow time delays. For "hereditary systems" as defined in Chapter 10, Pontryagin's principle takes the form in Theorem 10.3.1. Hereditary control problems are effectively infinite dimensional. As explained in Section 10.6, the true state is a function on a time interval $[-r, 0]$ where r represents the maximum time delay in the control system. Chapter 11 considers hereditary system models, with the additional feature that states are constrained by given bounds. For readers interested only in control systems

without time delays, necessary conditions for optimality in bounded state problems are described in Section 11.6.

The dynamic programming method leads to first order nonlinear partial differential equations, which are called Hamilton-Jacobi-Bellman equations (or sometimes Bellman equations). Typically, the value function of an optimal control problem is not smooth. Hence, it satisfies the Hamilton-Jacobi-Bellman equation only in a suitable "generalized sense." The Crandall-Lions Theory of viscosity solutions provides one such notion of generalized solutions for Hamilton-Jacobi-Bellman equations. Work of A. I. Subbotin and co-authors provides another interesting concept of generalized solutions. Chapter 12 provides an introduction to Hamilton-Jacobi Theory. The results described there tie together in an elegant way the viscosity solution and Subbotin approaches. A crucial part of the analysis involves a lower Dini derivate necessary condition derived in Section 12.4.

The manuscript for this book was not quite in final form when Leonard Berkovitz passed away unexpectedly. He is remembered for his many original contributions to optimal control theory and differential games, as well as for his dedicated service to the mathematics profession and to the control community in particular. During his long career at Purdue University, he was a highly esteemed teacher and mentor for his PhD students. Colleagues warmly remember his wisdom and good humor. During his six years as Purdue Mathematics Department head, he was resolute in advocating the department's interests. An obituary article about Len Berkovitz, written by W. J. Browning and myself, appeared in the January/February 2010 issue of *SIAM News*.

Wendell Fleming

Preface

This book is an introduction to the mathematical theory of optimal control of processes governed by ordinary differential and certain types of differential equations with memory and integral equations. The book is intended for students, mathematicians, and those who apply the techniques of optimal control in their research. Our intention is to give a broad, yet relatively deep, concise and coherent introduction to the subject. We have dedicated an entire chapter to examples. We have dealt with the examples pointing out the mathematical issues that one needs to address.

The first six chapters can provide enough material for an introductory course in optimal control theory governed by differential equations. Chapters 3, 4, and 5 could be covered with more or less details in the mathematical issues depending on the mathematical background of the students. For students with background in functional analysis and measure theory, Chapter 7 can be added. Chapter 7 is a more mathematically rigorous version of the material in Chapter 6.

We have included material dealing with problems governed by integrodifferential and delay equations. We have given a unified treatment of bounded state problems governed by ordinary, integrodifferential, and delay systems. We have also added material dealing with the Hamilton-Jacobi Theory. This material sheds light on the mathematical details that accompany the material in Chapter 6.

The material in the text gives a sufficient and rigorous treatment of finite dimensional control problems. The reader should be equipped to deal with other types of control problems such as problems governed by stochastic differential equations and partial differential equations, and differential games.

I am very grateful to Mrs. Betty Gick of Purdue University and Mrs. Annette Rohrs of Georgia Institute of Technology for typing the early and final versions of the book. I am very grateful to Professor Wendell Fleming for reading the manuscript and making valuable suggestions and additions that improved and enhanced the quality of the book as well as avoided and removed errors. I also wish to thank Professor Boris Mordukovich for reading the manuscript and making valuable suggestions. All or parts of the material up to the first seven chapters have been used for optimal control theory courses in Purdue University and North Carolina State University.

This book is a sequel to the book *Optimal Control Theory* by Leonard

D. Berkovitz. I learned control theory from this book taught by him. We decided to write the current book in 1994 and we went through various versions.

L. D. Berkovitz was my teacher and a second father. He passed away on October 13, 2009 unexpectedly. He was caring, humble, and loved mathematics. He is missed greatly by all who were fortunate enough to have known him. This book was completed before his death.

Negash G. Medhin
North Carolina State University

Chapter 1

Examples of Control Problems

1.1 Introduction

Control theory is a mathematical study of how to influence the behavior of a dynamical system to achieve a desired goal. In optimal control, the goal is to maximize or minimize the numerical value of a specified quantity that is a function of the behavior of the system. Optimal control theory developed in the latter half of the 20th century in response to diverse applied problems. In this chapter we present examples of optimal control problems to illustrate the diversity of applications, to raise some of the mathematical issues involved, and to motivate the mathematical formulation in subsequent chapters. It should not be construed that this set of examples is complete, or that we chose the most significant problem in each area. Rather, we chose fairly simple problems in an effort to illustrate without excessive complication.

Mathematically, optimal control problems are variants of problems in the calculus of variations, which has a 300+ year history. Although optimal control theory developed without explicit reference to the calculus of variations, each impacted the other in various ways.

1.2 A Problem of Production Planning

The first problem, taken from economics, is a resource allocation problem; the Ramsey model of economic growth. Let $Q(t)$ denote the rate of production of a commodity, say steel, at time t. Let $I(t)$ denote the rate of investment of the commodity at time t to produce capital; that is, productive capacity. In the case of steel, investment can be thought of as using steel to build new steel mills, transport equipment, infrastructure, etc. Let $C(t)$ denote the rate of consumption of the commodity at time t. In the case of steel, consumption can be thought of as the production of consumer goods such as automobiles. We assume that all of the commodity produced at time t must be allocated

to either investment or consumption. Then

$$Q(t) = I(t) + C(t) \qquad I(t) \geq 0 \quad C(t) \geq 0.$$

We assume that the rate of production is a known function F of the capital at time t. Thus, if $K(t)$ denotes the capital at time t, then

$$Q(t) = F(K(t)),$$

where F is a given function. The rate of change of capital is given by the capital accumulation equation

$$\frac{dK}{dt} = \alpha I(t) - \delta K(t) \quad K(0) = K_0, \ K(t) \geq 0,$$

where the positive constant α is the growth rate of capital and the positive constant δ is the depreciation rate of capital. Let $0 \leq u(t) \leq 1$ denote the fraction of production allocated to investment at time t. The number $u(t)$ is called the savings rate at time t. We can therefore write

$$\begin{aligned} I(t) &= u(t)Q(t) = u(t)F(K(t)) \\ C(t) &= (1 - u(t))Q(t) = (1 - u(t))F(K(t)), \end{aligned}$$

and

$$\begin{aligned} \frac{dK}{dt} &= \alpha u(t)F(K(t)) - \delta K(t) \\ K(t) &\geq 0 \quad K(0) = K_0. \end{aligned} \tag{1.2.1}$$

Let $T > 0$ be given and let a "social utility function" U, which depends on C, be given. At each time t, $U(C(t))$ is a measure of the satisfaction society receives from consuming the given commodity. Let

$$J = \int_0^T U(C(t))e^{-\gamma t} \, dt,$$

where γ is a positive constant. Our objective is to maximize J, which is a measure of the total societal satisfaction over time. The discount factor $e^{-\gamma t}$ is a reflection of the phenomenon that the promise of future reward is usually less satisfactory than current reward.

We may rewrite the last integral as

$$J = \int_0^T U((1 - u(t))F(K(t)))e^{-\gamma t} dt. \tag{1.2.2}$$

Note that by virtue of (1.2.1), the choice of a function $u\colon [0, T] \to u(t)$, where u is subject to the constraint $0 \leq u(t) \leq 1$ determines the value of J. We have here an example of a *functional*; that is, an assignment of a real number to

every function in a class of functions. If we relabel K as x, then the problem of maximizing J can be stated as follows:

Choose a savings program u over the time period $[0, T]$, that is, a function u defined on $[0, T]$, such that $0 \leq u(t) \leq 1$ and such that

$$J(u) = -\int_0^T U((1 - u(t))F(\varphi(t)))e^{-\gamma t} dt \tag{1.2.3}$$

is minimized, where φ is a solution of the differential equation

$$\frac{dx}{dt} = \alpha u(t)F(x) - \delta x \quad \varphi(0) = x_0,$$

and φ satisfies $\varphi(t) \geq 0$ for all t in $[0, T]$. The problem is sometimes stated as

Minimize:

$$J(u) = -\int_0^T U((1 - u(t))F(x))e^{-\gamma t} dt$$

Subject to:

$$\frac{dx}{dt} = \alpha u(t)F(x) - \delta x, \quad x(0) = x_0, \quad x \geq 0, \quad 0 \leq u(t) \leq 1$$

1.3 Chemical Engineering

Let $x^1(t), \ldots, x^n(t)$ denote the concentrations at time t of n substances in a reactor in which n simultaneous chemical reactions are taking place. Let the rates of the reactions be governed by a system of differential equations

$$\frac{dx^i}{dt} = G^i\left(x^1, \ldots, x^n, \theta(t), p(t)\right) \qquad x^i(0) = x_0^i \quad i = 1, \ldots, n. \tag{1.3.1}$$

where $\theta(t)$ is the temperature in the reactor at time t and $p(t)$ is the pressure in the reactor at time t. We control the temperature and pressure at each instance of time, subject to the constraints

$$\begin{aligned} \theta_b &\leq \theta(t) \leq \theta_a \\ p_b &\leq p(t) \leq p_a \end{aligned} \tag{1.3.2}$$

where θ_a, θ_b, p_a, and p_b are constants. These represent the minimum and maximum attainable temperature and pressure.

We let the reaction proceed for a predetermined time T. The concentrations at this time are $x^1(T), \ldots, x^n(T)$. Associated with each product is an economic value, or price c^i, $i = 1, \ldots, n$. The price may be negative, as in the

case of hazardous wastes that must be disposed of at some expense. The value of the end product is

$$V(p,\theta) = \sum_{i=1}^{n} c^i x^i(T). \tag{1.3.3}$$

Given a set of initial concentrations x_0^i, the value of the end product is completely determined by the choice of functions p and θ if the functions G^i have certain nice properties. Hence the notation $V(p,\theta)$. This is another example of a functional; in this case, we have an assignment of a real number to each pair of functions in a certain collection.

The problem here is to choose piecewise continuous functions p and θ on the interval $[0,T]$ so that (1.3.2) is satisfied and so that $V(p,\theta)$ is maximized.

A variant of the preceding problem is the following. Instead of allowing the reaction to proceed for a fixed time T, we stop the reaction when one of the reactants, say x^1, reaches a preassigned concentration x_f^1. Now the final time t_f is not fixed beforehand, but is the smallest positive root of the equation $x^1(t) = x_f^1$. The problem now is to maximize

$$V(p,\theta) = \sum_{i=2}^{n} c^i x^i(t_f) - k^2 t_f.$$

The term $k^2 t_f$ represents the cost of running the reactor.

Still another variant of the problem is to stop the reaction when several of the reactants reach preassigned concentrations, say $x^1 = x_f^1$, $x^2 = x_f^2, \ldots, x^j = x_f^j$. The value of the end product is now

$$\sum_{i=j+1}^{n} c^i x^i(t_f) - k^2 t_f.$$

We remark that in the last two variants of the problem there is another question that must be considered before one takes up the problem of maximization. Namely, can one achieve the desired final concentrations using pressure and temperature functions p and θ in the class of functions permitted?

1.4 Flight Mechanics

In this problem a rocket is taken to be a point of variable mass whose moments of inertia are neglected. The motion of the rocket is assumed to take place in a plane relative to a fixed frame. Let $y = (y^1, y^2)$ denote the position vector of the rocket and let $v = (v^1, v^2)$ denote the velocity vector of the rocket. Then

$$\frac{dy^i}{dt} = v^i \qquad y^i(0) = y_0^i \qquad i = 1, 2, \tag{1.4.1}$$

where $y_0 = (y_0^1, y_0^2)$ denotes the initial position of the rocket.

Let $\beta(t)$ denote the rate at which the rocket burns fuel at time t and let $m(t)$ denote the mass of the rocket at time t. Thus,

$$\frac{dm}{dt} = -\beta. \tag{1.4.2}$$

If $a > 0$ denotes the mass of the vehicle, then $m(t) \geq a$.

Let $\omega(t)$ denote the angle that the thrust vector makes with the positive y^1-axis at time t. The burning rate and the thrust angle will be at our disposal subject to the constraints

$$0 \leq \beta_0 \leq \beta(t) \leq \beta_1 \qquad \omega_0 \leq \omega(t) \leq \omega_1, \tag{1.4.3}$$

where β_0, β_1, ω_0, and ω_1 are fixed.

To complete the equations of motion of the rocket we analyze the momentum transfer in rectilinear rocket motion. At time t a rocket of mass m and velocity v has momentum mv. During an interval of time δt let the rocket burn an amount of fuel $\delta\mu > 0$. At time $t + \delta t$ let the ejected combustion products have velocity v'; their mass is clearly $\delta\mu$. At time $t + \delta t$ let the velocity of the rocket be $v + \delta v$; its mass is clearly $m - \delta\mu$. Let us consider the system which at time t consisted of the rocket of mass m and velocity v. At time $t + \delta t$ this system consists of the rocket and the ejected combustion products. The change in momentum of the system in the time interval δt is therefore

$$(\delta\mu)v' + (m - \delta\mu)(v + \delta v) - mv.$$

If we divide the last expression by $\delta t > 0$ and then let $\delta t \to 0$, we obtain the rate of change of momentum of the system, which must equal the sum of the external forces acting upon the system. Hence, if F is the resultant external force per unit mass acting upon the system we have

$$Fm - (v' - v)\,\frac{d\mu}{dt} = m\,\frac{dv}{dt}.$$

If we assume that $(v' - v)$, the velocity of the combustion products relative to the rocket is a constant c, and if we use $d\mu/dt = \beta$, we get

$$F - c\beta/m = dv/dt.$$

If we apply the preceding analysis to each component of the planar motion we get the following equations, which together with (1.4.1), (1.4.2), and (1.4.3) govern the planar rocket motion

$$\begin{aligned} \frac{dv^1}{dt} &= F^1 - \frac{c\beta}{m}\cos\omega \\ \frac{dv^2}{dt} &= F^2 - \frac{c\beta}{m}\sin\omega \qquad v^i(0) = v_0^i, \quad i = 1, 2. \end{aligned} \tag{1.4.4}$$

Here, the components of the force F can be functions of y and v. This would be the case if the motion takes place in a non-constant gravitational field and if drag forces act on the rocket.

The control problems associated with the motion of the rocket are of the following type. The burning rate control β and the thrust direction control ω are to be chosen from the class of piecewise continuous functions (or some other appropriate class) in such a way that certain of the variables t, y, v, m attain specific terminal values. From among the controls that achieve these values, the control that maximizes (or minimizes) a given function of the remaining terminal values is to be determined. In other problems, an integral evaluated along the trajectory in the state space is to be extremized.

To be more specific, consider the "minimum fuel problem." It is required that the rocket go from a specified initial point y_0 to a specified terminal point y_f in such a way that the fuel consumed is minimized. This problem is important for the following reason. Since the total weight of rocket plus fuel plus payload that can be constructed and lifted is constrained by the state of the technology, it follows that the less fuel consumed, the larger the payload that can be carried by the rocket. From (1.4.2) we have

$$m_f = m_0 - \int_{t_0}^{t_f} \beta(t)dt,$$

where t_0 is the initial time, t_f is the terminal time (time at which y_f is reached), m_f is the final mass, and m_0 is the initial mass. The fuel consumed is therefore $m_0 - m_f$. Thus, the problem of minimizing the fuel consumed is the problem of minimizing

$$P(\beta, \omega) = \int_{t_0}^{t_f} \beta(t)dt \tag{1.4.5}$$

subject to (1.4.1) to (1.4.4). This problem is equivalent to the problem of maximizing m_f. In the minimum fuel problem the terminal velocity vector v_f will be unspecified if a "hard landing" is permitted; it will be specified if a "soft landing" is required. The terminal time t_f may or may not be specified.

Another example is the problem of rendezvous with a moving object whose position vector at time t is $z(t) = (z^1(t), z^2(t))$ and whose velocity vector at time t is $w(t) = (w^1(t), w^2(t))$, where z^1, z^2, w^1, and w^2 are continuous functions. Let us suppose that there exist thrust programs β and ω satisfying (1.4.3) and such that rendezvous can be effected. Mathematically this is expressed by the assumption that the solutions y, v of the equations of motion corresponding to the given choice of β and ω have the property that the equations

$$\begin{aligned} y(t) &= z(t) \\ v(t) &= w(t) \end{aligned} \tag{1.4.6}$$

have positive solutions. Such controls (β, ω) will be called admissible. Since for

each admissible β and ω the corresponding solutions y and v are continuous, and since the functions z and w are continuous by hypothesis, it follows that for each admissible pair (β, ω) there is a smallest positive solution $t_f(\beta, \omega)$ for which (1.4.6) holds. The number $t_f(\beta, \omega)$ is the rendezvous time. Two problems are possible here. The first is to determine from among the admissible controls one that delivers the maximum payload; that is, to maximize $m_f = m_f(t_f(\beta, \omega))$. The second is to minimize the rendezvous time $t_f(\beta, \omega)$.

1.5 Electrical Engineering

Example 1.5.1. A control surface on an airplane is to be kept at some arbitrary position by means of a servo-mechanism. Outside disturbances such as wind gusts occur infrequently and are short with respect to the time constant of the servo-mechanism. A direct-current electric motor is used to apply a torque to bring the control surface to its desired position. Only the armature voltage v into the motor can be controlled. For simplicity we take the desired position to be the zero angle and we measure deviations in the angle θ from this desired position. Without the application of a torque the control surface would vibrate as a damped harmonic oscillator. Therefore, with a suitable normalization the differential equation for θ can be written as

$$\frac{d^2\theta}{dt^2} + a\,\frac{d\theta}{dt} + \omega^2\theta = u \qquad \theta(0) = \theta_0 \qquad \theta'(0) = \theta_0'. \tag{1.5.1}$$

Here u represents the restoring torque applied to the control surface, the term $a d\theta/dt$ represents the damping effect, and ω^2 is the spring constant. If no damping occurs, then $a = 0$. Since the source of voltage cannot deliver a voltage larger in absolute value than some value v_0, the restoring torque must be bounded in absolute value. Hence it follows that we must have

$$|u(t)| \le A, \tag{1.5.2}$$

where A is some positive constant.

If we set

$$x^1 = \theta \qquad x^2 = \frac{d\theta}{dt}$$

we can rewrite Eq. (1.5.1) as follows:

$$\begin{aligned} \frac{dx^1}{dt} &= x^2 & x^1(0) &= \theta_0 \\ \frac{dx^2}{dt} &= -ax^2 - \omega^2 x^1 + u & x^2(0) &= \theta_0'. \end{aligned} \tag{1.5.3}$$

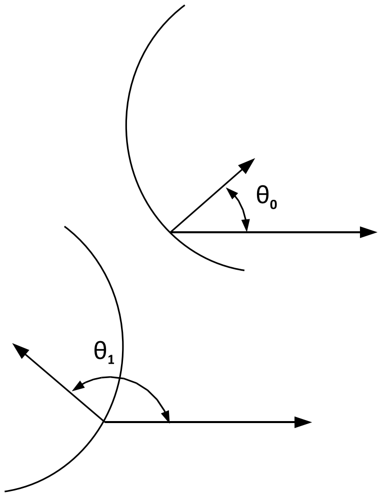

FIGURE 1.1 [From: G. Stephens Jones and Aaron Strauss, An example of optimal control, *SIAM Review*, Vol. 10, 25–55 (1968).]

The problem is the following. A short disturbance has resulted in a deviation $\theta = \theta_0$ from the desired position and a deviation $d\theta/dt = \theta_0'$ from rest. How should the voltage be applied over time so that the control surface is brought back to the set position $\theta = 0$, $d\theta/dt = 0$ in the shortest possible time? In terms of (1.5.3), the problem is to choose a function u from an appropriate class of functions, say piecewise continuous functions, such that u satisfies (1.5.2) at each instant of time and such that the solution (x^1, x^2) of (1.5.3) corresponding to u reaches the origin in (x^1, x^2)-space in minimum time.

Example 1.5.2. Figure 1.1 depicts an antenna free to rotate from any angular position θ_0 to any other angle θ_1. The equation of motion under an applied torque T is given by

$$I\frac{d^2\theta}{dt^2} + \beta\frac{d\theta}{dt} = T \qquad \theta(0) = \theta_0 \quad \theta'(0) = \theta_0', \tag{1.5.4}$$

where β is a damping factor and I is the moment of inertia of the system about the vertical axis.

The objective here is to move from the position and velocity (θ_0, θ_0') at an initial time t_0 to the state and velocity $(\theta_1, 0)$ at some later time t_1 in a way that the following criteria are met.

(a) The transfer of position must take place within a reasonable (but not specified) period of time.

(b) The energy expended in making rotations must be kept within reason-

able (but not specified) bounds in order to avoid excessive wear on components.

(c) The fuel or power expended in carrying out the transfer must be kept within reasonable (but not specified) limits.

Since the energy expended is proportional to $(d\theta/dt)^2$ and the fuel or power expended is proportional to the magnitude of the torque, a reasonable performance criterion would be

$$J = \int_{t_0}^{t_1} (\gamma_1 + \gamma_2 \left(\frac{d\theta}{dt}\right)^2 + \gamma_3|T|)dt,$$

where $\gamma_1 > 0$, $\gamma_2 \geq 0$, $\gamma_3 \geq 0$, and t_1 is free.

The control torque T is constrained in magnitude by a quantity $k > 0$, that is, $|T| \leq k$, and $(d\theta/dt)$ is constrained in magnitude by 1, that is, $|d\theta/dt| \leq 1$.

If as in Example 1.5.1 we set

$$x^1 = \theta \qquad x^2 = \frac{d\theta}{dt},$$

we can write (1.5.4) as the system

$$\begin{aligned} \frac{dx^1}{dt} &= x^2 & x^1(0) &= \theta_0 \\ \frac{dx^2}{dt} &= -\frac{\beta}{I}\, x^1 + \frac{T}{I} & x^2(0) &= \theta_0'. \end{aligned} \tag{1.5.5}$$

The problem then is to choose a torque program (function) T that minimizes

$$J(T) = \int_{t_0}^{t_1} (\gamma_1 + \gamma_2(x^2)^2 + \gamma_3|T|)dt$$

subject to (1.5.5), the terminal conditions $x^1(t_1) = \theta_1$, $x^2(t_1) = 0$, t_1 free and the constraints

$$|T(t)| \leq k \qquad |x^2(t)| \leq 1.$$

This example differs from the preceding examples in that we have a constraint $|x^2(t)| \leq 1$ on the state as well as a constraint on the control.

1.6 The Brachistochrone Problem

We now present a problem from the calculus of variations; the brachistochrone problem, posed by John Bernoulli in 1696. This problem can be regarded as the starting point of the theory of the calculus of variations. Galileo

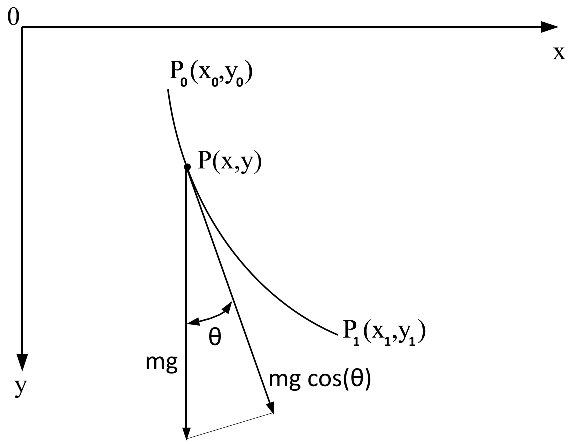

FIGURE 1.2

also seems to have considered this problem in 1630 and 1638, but was not as explicit in his formulation.

Two points P_0 and P_1 that do not lie on the same vertical line are given in a vertical plane with P_0 higher than P_1. A particle, or point mass, acted upon solely by gravity is to move along a curve C joining P_0 and P_1. Furthermore, at P_0 the particle is to have an initial speed v_0 along the curve C. The problem is to choose the curve C so that the time required for the particle to go from P_0 to P_1 is a minimum.

To formulate the problem analytically, we set up a coordinate system in the plane as shown in Fig. 1.2.

Let P_0 have coordinates (x_0, y_0) with $y_0 > 0$, let P_1 have coordinates (x_1, y_1) with $y_1 > 0$, and let C have $y = y(x)$ as its equation. At time t, let $(x(t), y(t))$ denote the coordinates of the particle as it moves along the curve C, let $v(t)$ denote the speed, and let $s(t)$ denote the distance traveled. We shall determine the time required to traverse C from P_0 to P_1.

From the principle of conservation of energy, we have that

$$\frac{1}{2}m(v^2 - v_0^2) = mg(y - y_0). \tag{1.6.1}$$

Also,

$$v = \frac{ds}{dt} = \frac{ds}{dx}\,\frac{dx}{dt} = [1 + (y')^2]^{1/2}\,\frac{dx}{dt}. \tag{1.6.2}$$

Hence, using (1.6.1) and (1.6.2), we get that

$$dt = \frac{[1 + (y')^2]^{1/2}}{v}\,dx = \left[\frac{1 + (y')^2}{2g(y - \alpha)}\right]^{1/2} dx,$$

where

$$\alpha = y_0 - v_0^2/2g.$$

Thus, the time of traverse T along C is given by

$$T = \frac{1}{(2g)^{1/2}} \int_{x_0}^{x_1} \left[\frac{1+(y')^2}{y-\alpha}\right]^{1/2} dx.$$

The problem of finding a curve C that minimizes the time of traverse is that of finding a function $y = y(x)$ that minimizes the integral

$$\int_{x_0}^{x_1} \left[\frac{1+(y')^2}{y-\alpha}\right]^{1/2} dx. \tag{1.6.3}$$

Note that if $v_0 = 0$, then the integral is improper.

We put this problem in a format similar to the previous ones as follows. Change the notation for the independent variable from x to t. Then set

$$y' = u \qquad y(t_0) = y_0. \tag{1.6.4}$$

A continuous function u will be called admissible if it is defined on $[t_0, t_1]$, if the solution of (1.6.4) corresponding to u satisfies $y(t_1) = y_1$, if $y(t) > y_0$ on $[t_0, t_1]$, and if the mapping $t \to [(1 + u^2(t))/(y(t) - \alpha)]^{1/2}$ is integrable on $[t_0, t_1]$. Our problem is to determine the admissible function u that minimizes

$$J(u) = \int_{t_0}^{t_1} \left(\frac{1+u^2}{y-\alpha}\right)^{1/2} dt \tag{1.6.5}$$

in the class of all admissible u.

The brachistochrone problem can be formulated as a control problem in a different fashion. By (1.6.1) and (1.6.2), the speed of the particle along the curve C is given by $(2g(y-\alpha))^{1/2}$. Hence, if θ is the angle that the tangent to C makes with the positive x-axis, then

$$\begin{aligned} \frac{dx}{dt} &= (2g(y-\alpha))^{1/2} \cos\theta \\ \frac{dy}{dt} &= (2g(y-\alpha))^{1/2} \sin\theta. \end{aligned}$$

Let $u = \cos\theta$. Then the equations of motion become

$$\begin{aligned} \frac{dx}{dt} &= (2g(y-\alpha))^{1/2} u & x(t_0) &= x_0 \\ \frac{dy}{dt} &= (2g(y-\alpha))^{1/2}(1-u^2)^{1/2} & y(t_0) &= y_0. \end{aligned} \tag{1.6.6}$$

The problem is to choose a control u satisfying $|u| \le 1$ such that the point (x, y), which at initial time t_0 is at (x_0, y_0), reaches the prescribed point (x_1, y_1) in minimum time. If t_1 is the time at which P_1 is reached, then this is equivalent to minimizing $t_1 - t_0$. This in turn is equivalent to minimizing

$$\int_{t_0}^{t_1} dt \tag{1.6.7}$$

subject to (1.6.6), the terminal condition (x_1, y_1), and the constraint $|u(t)| \leq 1$.

The brachistochrone problem can be modified in the following fashion. One can replace the fixed point P_1 by a curve Γ_1 defined by $y = y_1(x)$ and seek the curve C joining P_0 to Γ_1 along which the mass particle must travel if it is to go from P_0 to Γ_1 in minimum time. We can also replace P_0 by a curve Γ_0 where Γ_0 is at positive distance from Γ_1 and ask for the curve C joining Γ_0 and Γ_1 along which the particle must travel in order to minimize the time of transit.

1.7 An Optimal Harvesting Problem

We present here a population model of McKendric type with crowding effect. The reformulation of the control problem coincides with the reformulation by Gurtin and Murphy [40], [68]. The age-dependent population model is given by

$$\begin{aligned}\frac{\partial p(r,t)}{\partial t} + \frac{\partial p(r,t)}{\partial t} &= -\mu(r)p(r,t) - f(N(t))p(r,t) - u(t)p(r,t) \qquad (1.7.1)\\ p(r,0) &= p_0(r)\\ p(0,t) &= \beta\int_0^\infty k(r)p(r,t)dr, \quad k(r) = \widetilde{k}(r)h(r)\\ N(t) &= \int_0^\infty p(r,t)dr\end{aligned}$$

where $p(r,t)$ denotes the age density distribution at time t and age r, $\mu(r)$ is the mortality rate, $k(r)$ is the female sex ratio at age r, $h(r)$ is the fertility pattern, and β is the specific fertility rate of females. The function $f(N(\cdot))$ indicates decline in the population due to environmental factors such as crowding. The function $u(\cdot) \geq 0$ is the control or harvesting strategy.

We consider the problem of maximizing the harvest

$$J(u) = \int_0^T u(t)N(t)dt \qquad (1.7.2)$$

where $0 \leq u(\cdot) \leq M$ is piecewise continuous and (1.7.1) is satisfied. The upper bound M on $u(\cdot)$ is the maximum effort.

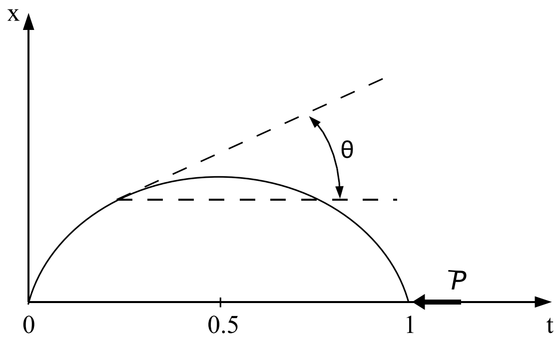

FIGURE 1.3 [From: H. Maurer and H. D. Mittelmann, *Optimal Control Applications and Methods*, 12, 19–31 (1991).]

1.8 Vibration of a Nonlinear Beam

Consider the classical nonlinear Euler beam [56] with deflection limited by an obstacle parallel to the plane of the beam. The beam is axially compressed by a force P, which acts as a branching parameter α.

We assume that the energy of a beam that is compressed by a force P is given by

$$I_\alpha = \frac{1}{2}\int_0^1 \dot{\theta}^2 dt + \alpha \int_0^1 \cos\theta(t) dt.$$

Here $\alpha = P/EJ$, where EJ is the bending stiffness, t denotes the arc length, $\theta(t)$ is the angle between the tangential direction of the beam at t and the reference line (see Fig. 1.3), and the length of the beam is $\ell = 1$.

For the deflection of the beam away from the reference line we have

$$\dot{x} = \sin\theta, \qquad \dot{\theta} = \frac{\ddot{x}}{\sqrt{1-\dot{x}^2}}.$$

Hence, the energy can also be written as

$$I_\alpha = \frac{1}{2}\int_0^1 \frac{\ddot{x}^2}{1-\dot{x}^2}\, dt + \alpha \int_0^1 \sqrt{1-\dot{x}^2}\, dt.$$

We assume that $|\dot{x}(t)| < 1$, that is, $-\pi/2 < \theta(t) < \pi/2$ holds on $[0, 1]$.

The variational problem for the simply supported beam consists of minimizing the energy subject to the boundary conditions

$$x(0) = x(1) = 0$$

and the state constraints

$$-d \le x(t) \le d, \qquad 0 \le t \le 1, \qquad d > 0.$$

In the case of a clamped beam, one replaces the boundary conditions by

$$x(0) = 0, \qquad \theta(0) = 0, \qquad x(1) = \theta(1) = 0.$$

Chapter 2

Formulation of Control Problems

2.1 Introduction

In this chapter we discuss the mathematical structures of the examples in the previous chapter.

We first discuss problems whose dynamics are given by ordinary differential equations. We motivate and give precise mathematical formulations and equivalent mathematical formulations of apparently different problems. We then point out the relationship between optimal control problems and the calculus of variations. Last, we present various formulations of hereditary problems. These problems are also called delay or lag problems.

2.2 Formulation of Problems Governed by Ordinary Differential Equations

Many of the examples in the preceding chapter have the following form. The state of a system at time t is described by a point or vector

$$x(t) = (x^1(t), \ldots, x^n(t))$$

in n-dimensional euclidean space, $n \geq 1$. Initially, at time t_0, the state of the system is

$$x(t_0) = x_0 = (x_0^1, \ldots, x_0^n).$$

More generally, we can require that at the initial time t_0 the initial state x_0 is such that the point (t_0, x_0) belongs to some pre-assigned set $\mathcal{T}_0$ in (t, x)-space. The state of the system varies with time according to the system of differential equations

$$\frac{dx^i}{dt} = f^i(t, x, z) \qquad x^i(t_0) = x_0^i \qquad i = 1, \ldots, n, \tag{2.2.1}$$

where $z = (z^1, \ldots, z^m)$ is a vector in real euclidean space $\mathbb{R}^m$ and the functions f^i are real valued continuous functions of the variables (t, x, z).

By the "system varying according to (2.2.1)" we mean the following. A function u with values in m-dimensional euclidean space is chosen from some prescribed class of functions. In this section we shall take this class to be a subclass $\mathcal{C}$ of the class of piecewise continuous functions. When the substitution $z = u(t)$ is made in the right-hand side of (2.2.1), we obtain a system of ordinary differential equations:

$$\frac{dx^i}{dt} = f^i(t, x, u(t)) = F_u^i(t, x) \qquad i = 1, \ldots, n. \tag{2.2.2}$$

The subscript u on the F_u^i emphasizes that the right-hand side of (2.2.2) depends on the choice of function u. For each u in $\mathcal{C}$ it is assumed that there exists a point (t_0, x_0) in $\mathcal{T}_0$ and a function $\phi = (\phi^1, \ldots, \phi^n)$ defined on an interval $[t_0, t_2]$ with values in $\mathbb{R}^n$ such that (2.2.2) is satisfied. That is, we require that for every t in $[t_0, t_2]$

$$\phi^{'i}(t) = \frac{d\phi^i}{dt} = f^i(t, \phi(t), u(t)) \qquad \phi^i(t_0) = x_0^i \quad i = 1, \ldots, n.$$

At points of discontinuity of u this equation is interpreted as holding for the one-sided limits. The function ϕ describes the evolution of the system with time and will sometimes be called a *trajectory*.

The function u is further required to be such that at some time t_1, where $t_0 < t_1$, the point $(t_1, \phi(t_1))$ belongs to a pre-assigned set $\mathcal{T}_1$ and for $t_0 \leq t < t_1$ the points $(t, \phi(t))$ do not belong to $\mathcal{T}_1$. The set $\mathcal{T}_1$ is called the terminal set for the problem. Examples of terminal sets, taken from Chapter 1, are given in the next paragraph.

In the production planning problem, $\mathcal{T}_1$ is the line $t = T$ in the (t, x) plane. In the first version of the chemical engineering problem, the set $\mathcal{T}_1$ is the hyperplane $t = T$; that is, those points in (t, x)-space with $x = (x^1, \ldots, x^n)$ free and t fixed at T. In the last version of the chemical engineering problem, $\mathcal{T}_1$ is the set of points in (t, x)-space whose coordinates x^i are fixed at x_f^i for $i = 1, \ldots, j$ and whose remaining coordinates are free. In some problems it is required that the solution hit a moving target set $G(t)$. That is, at each time t of some interval $[\tau_0, \tau_1]$ there is a set $G(t)$ of points in x-space, and it is required that the solution ϕ hit $G(t)$ at some time t. Stated analytically, we require the existence of a point t_1 in $[\tau_0, \tau_1]$ such that $\phi(t_1)$ belongs to $G(t_1)$. An example of this type of problem is the rendezvous problem in Section 1.4. The set $\mathcal{T}_1$ in the moving target set problem is the set of all points $(t, x) = (t, z(t), w(t), m)$ with $\tau_0 \leq t \leq \tau_1$ and $m > 0$.

The discussion in the preceding paragraphs is sometimes summarized in less precise but somewhat more graphic language by the statement that the functions u are required to transfer the system from an initial state x_0 at time t_0 to a terminal state x_1 at time t_1, where $(t_0, x_0) \in \mathcal{T}_0$ and $(t_1, x_1) \in \mathcal{T}_1$. Note that to a given u in $\mathcal{C}$ there will generally correspond more than one trajectory ϕ. This results from different choices of initial points (t_0, x_0) in $\mathcal{T}_0$

or from non-uniqueness of solutions of (2.2.2) if no assumptions are made to guarantee the uniqueness of solutions of (2.2.2) with given initial data (t_0, x_0).

It is often further required that a function u in $\mathcal{C}$ and a corresponding solution ϕ satisfy a system of inequality constraints

$$R^i(t, \phi(t), u(t)) \geq 0 \qquad i = 1, 2, \ldots, r, \tag{2.2.3}$$

for all $t_0 \leq t \leq t_1$, where the functions $R^1, \ldots, R^r$ are given functions of (t, x, z). For example, in the production planning problem discussed in Section 1.2 the constraints can be written as $R^i \geq 0$, $i = 1, 2, 3$, where $R^1(t, x, z) = x$, $R^2(t, x, z) = z$, and $R^3(t, x, z) = 1 - z$. In Example 1.5.1, the constraints can be written as $R^i \geq 0$, $i = 1, 2$, where $R^1(t, x, z) = z + A$ and $R^2(t, x, z) = A - z$.

In the examples of Chapter 1, the control u is to be chosen so that certain functionals are minimized or maximized. These functionals have the following form. Let f^0 be a real valued continuous function of (t, x, z), let g_0 be a real valued function defined on $\mathcal{T}_0$, and let g_1 be a real valued function defined on $\mathcal{T}_1$. For each u in $\mathcal{C}$ and each corresponding solution ϕ of (2.2.2), define a *cost* or *payoff* or *performance index* as follows:

$$J(\phi, u) = g_0(t_0, \phi(t_0)) + g_1(t_1, \phi(t_1)) + \int_{t_0}^{t_1} f^0(s, \phi(s), u(s))ds.$$

If the function J is to be minimized, then a u^* in $\mathcal{C}$ and a corresponding solution ϕ^* of (2.2.2) are to be found such that $J(\phi^*, u^*) \leq J(\phi, u)$ for all u in $\mathcal{C}$ and corresponding ϕ. In other problems, the functional J is to be maximized. Examples of J taken from Chapter 1 are given in the next paragraph.

In the examples of Chapter 1, the set $\mathcal{T}_0$ is always a point (t_0, x_0). The differential equations in the examples, except in Section 1.3, are such that the solutions are unique. In Section 1.3 let us assume that the functions G^i are such that the solutions are unique. Thus, in these examples the choice of u completely determines the function ϕ. In the economics example, $J(\phi, u)$ is the total cost $J(u)$ given by (1.2.3). The function f^0 is given by $-U((1-z)F(x))e^{-\gamma t}$ and the functions g_0 and g_1 are identically zero. In the first chemical engineering example of Section 1.3, $J(\phi, u) = V(p, \theta)$, where $V(p, \theta)$ is given by (1.3.3). The functions f^0 and g_0 are identically zero. In the minimum fuel problem of Section 1.4, $J(\phi, u) = P(\beta, \omega)$, where P is given by (1.4.5). Here $f^0 = \beta$ and g_0 and g_1 are identically zero. An equivalent formulation is obtained if one takes $J(\phi, u) = -m_f$. Now $f^0 = 0$, $g_0 = 0$, and $g_1 = -m_f$.

We conclude this section with a discussion of two generalizations that will appear in the mathematical formulation to be given in the next section. The first deals with the initial and terminal data. The initial set $\mathcal{T}_0$ and the terminal set $\mathcal{T}_1$ determine a set $\mathcal{B}$ of points (t_0, x_0, t_1, x_1) in $\mathbb{R}^{2n+2}$ as follows:

$$\mathcal{B} = \{(t_0, x_0, t_1, x_1) : (t_0, x_0) \in \mathcal{T}_0, \quad (t_1, x_1) \in \mathcal{T}_1\}. \tag{2.2.4}$$

Thus, a simple generalization of the requirement that $(t_0, \phi(t_0)) \in \mathcal{T}_0$ and

$(t_1, \phi(t_1)) \in \mathcal{T}_1$ is the following. Let there be given a set $\mathcal{B}$ of points in $\mathbb{R}^{2n+2}$. It is required of a trajectory ϕ that $(t_0, \phi(t_0), t_1, \phi(t_1))$ belong to $\mathcal{B}$. That is, we now permit possible relationships between initial and terminal data. We shall show later that in some sense this situation is really no more general than the situation in which the initial and terminal data are assumed to be unrelated.

The second generalization deals with the description of the constraints on u. For each (t, x), a system of inequalities $R^i(t, x, z) \geq 0$, $i = 1, \ldots, r$ determines a set $U(t, x)$ in the m-dimensional z-space; namely

$$U(t, x) = \{z : R^i(t, x, z) \geq 0, i = 1, \ldots, r\}.$$

The requirement that a function u and a corresponding trajectory satisfy constraints of the form (2.2.3) can therefore be written as follows:

$$u(t) \in U(t, \phi(t)) \qquad t_0 \leq t \leq t_1.$$

Thus, the constraint (2.2.3) is a special case of the following more general constraint condition.

Let Ω be a function that assigns to each point (t, x) of some suitable subset of $\mathbb{R}^{n+1}$ a subset of the z-space $\mathbb{R}^m$. Thus,

$$\Omega : (t, x) \to \Omega(t, x),$$

where $\Omega(t, x)$ is a subset of $\mathbb{R}^m$. The constraint (2.2.3) is replaced by the more general constraint

$$u(t) \in \Omega(t, \phi(t)).$$

2.3 Mathematical Formulation

The formulation will involve the Lebesgue integral. This is essential in the study of solutions to the problem. The reader who wishes to keep the formulation on a more elementary level can replace "measurable controls" by "piecewise continuous controls," replace "absolutely continuous functions" by "piecewise $C^{(1)}$ functions," and interpret the solution of Eq. (2.3.1) as we interpreted the solution of Eq. (2.2.2).

We establish some notation and terminology. Let t denote a real number, which will sometimes be called time. Let x denote a vector in real euclidean space $\mathbb{R}^n$, $n \geq 1$; thus, $x = (x^1, \ldots, x^n)$. The vector x will be called the *state variable*. We shall use superscripts to denote components of vectors and we shall use subscripts to distinguish among vectors. Let z denote a vector in euclidean m-space $\mathbb{R}^m$, $m \geq 1$; thus, $z = (z^1, \ldots, z^m)$. The vector z will be called the *control variable*. Let $\mathcal{R}$ be a region of (t, x)-space and let $\mathcal{U}$ be

a region of z-space, whereby a region we mean an open connected set. Let $\mathcal{G} = \mathcal{R} \times \mathcal{U}$, the cartesian product of $\mathcal{R}$ and $\mathcal{U}$. Let $f^0, f^1, \ldots, f^n$ be real valued functions defined on $\mathcal{G}$. We shall write

$$f = (f^1, \ldots, f^n) \qquad \widehat{f} = (f^0, f^1, \ldots, f^n).$$

Let $\mathcal{B}$ be a set of points

$$(t_0, x_0, t_1, x_1) = (t_0, x_0^1, \ldots, x_0^n, t_1, x_1^1, \ldots, x_1^n)$$

in $\mathbb{R}^{2n+2}$ such that $(t_i, x_i),\ i = 0, 1$ are in $\mathcal{R}$ and $t_1 \geq t_0 + \delta$, for some fixed $\delta > 0$. The set $\mathcal{B}$ will be said to define the *end conditions* for the problem.

Let Ω be a mapping that assigns to each point (t, x) in $\mathcal{R}$ a subset $\Omega(t, x)$ of the region $\mathcal{U}$ in z-space. The mapping Ω will be said to define the *control constraints.* If $\mathcal{U}(t, x) = \mathcal{U}$ for all (t, x) in $\mathcal{R}$, then we say that there are no control constraints.

Henceforth we shall usually use vector-matrix notation. The system of differential equations (2.2.2) will be written simply as

$$\frac{dx}{dt} = f(t, x, u(t)),$$

where we follow the usual convention in the theory of differential equations and take dx/dt and $f(t, x, u(t))$ to be column vectors. We shall not distinguish between a vector and its transpose if it is clear whether a vector is a row vector or a column vector or if it is immaterial whether the vector is a row vector or a column vector. The inner product of two vectors u and v will be written as $\langle u, v \rangle$. We shall use the symbol $|x|$ to denote the ordinary euclidean norm of a vector. Thus,

$$|x| = \left(\sum_{i=1}^{n} |x^i|^2 \right)^{1/2} = \langle x, x \rangle^{1/2}.$$

If A and B are matrices, then we write their product as AB.

If $f = (f^1, \ldots, f^n)$ is a vector valued function from a set Δ in some euclidean space to the euclidean space $\mathbb{R}^n$ such that each of the real value functions $f^1, \ldots, f^n$ is continuous (or $C^{(k)}$, or measurable, etc.) then we shall say that f is continuous (or $C^{(k)}$, or measurable, etc.) on the set Δ. Similarly, if a matrix A has entries that are continuous functions (or $C^{(k)}$, or measurable functions, etc.) defined on a set Δ in some euclidean space, then we shall say that A is continuous (or $C^{(k)}$, or measurable, etc.) on Δ.

Definition 2.3.1. A *control* is measurable function u defined on an interval $[t_0, t_1]$ with range in $\mathfrak{U}$.

Definition 2.3.2. A *trajectory* corresponding to a control u is an absolutely continuous function ϕ defined on $[t_0, t_1]$ with range in $\mathbb{R}^n$ such that:

(i) $(t, \phi(t)) \in \mathcal{R}$ for all t in $[t_0, t_1]$

(ii) ϕ is a solution of the system of differential equations

$$\frac{dx}{dt} = f(t, x, u(t)); \tag{2.3.1}$$

that is,

$$\phi'(t) = f(t, \phi(t), u(t)) \text{ a.e. on } [t_0, t_1].$$

The point $(t_0, \phi(t_0))$ will be called the *initial point* of the trajectory and the point $(t_1, \phi(t_1))$ will be called the *terminal point* of the trajectory. The point $(t_0, \phi(t_0), t_1, \phi(t_1))$ will be called the *end point* of the trajectory.

Note that since ϕ is absolutely continuous, it is the integral of its derivative. Hence (ii) contains the statement that the function $t \to f(t, \phi(t), u(t))$ is Lebesgue integrable on $[t_0, t_1]$.

The system of differential equations (2.3.1) will be called the *state equations*.

We emphasize the following about our notation. We are using the letter z to denote a point of $\mathcal{U}$; we are using the letter u to denote a function with range in $\mathcal{U}$.

Definition 2.3.3. A control u is said to be an *admissible control* if there exists a trajectory ϕ corresponding to u such that

(i) $t \to f^0(t, \phi(t), u(t))$ is in $L_1[t_0, t_1]$.

(ii) $u(t) \in \Omega(t, \phi(t))$ a.e. on $[t_0, t_1]$.

(iii) $(t_0, \phi(t_0), t_1, \phi(t_1)) \in \mathcal{B}$.

A trajectory corresponding to an admissible control as in Definition 2.3.3 will be called an *admissible trajectory.*

Definition 2.3.4. A pair of functions (ϕ, u) such that u is an admissible control and ϕ is an admissible trajectory corresponding to u will be called an *admissible pair.*

Note that to a given admissible control there may correspond more than one admissible trajectory as a result of different choices of permissible end points. Also, even if we fix the endpoint, there may be several trajectories corresponding to a given control because we do not require uniqueness of solutions of (2.3.1) for given initial conditions.

We now state the control problem.

Problem 2.3.1. Let $\mathcal{A}$ denote the set of all admissible pairs (ϕ, u) and let $\mathcal{A}$ be non-empty. Let

$$J(\phi, u) = g(t_0, \phi(t_0), t_1, \phi(t_1)) + \int_{t_0}^{t_1} f^0(t, \phi(t), u(t))dt, \tag{2.3.2}$$

where (ϕ, u) is an admissible pair and g is a given real valued function defined on $\mathcal{B}$. Let $\mathcal{A}_1$ be a non-empty subset of $\mathcal{A}$. Find a pair (ϕ^*, u^*) in $\mathcal{A}_1$ that minimizes (2.3.2) in the class $\mathcal{A}_1$. That is, find an element (ϕ^*, u^*) in $\mathcal{A}_1$ such that

$$J(\phi^*, u^*) \leq J(\phi, u) \quad \text{for all} \quad (\phi, u) \text{ in } \mathcal{A}_1.$$

The precise formulation of Problem 2.3.1 is rather lengthy. Therefore, the following statement, which gives the essential data of the problem, is often used to mean that we are considering Problem 2.3.1.

Minimize (2.3.2) *in the class* $\mathcal{A}_1$ *subject to the state equation* (2.3.1)*, the end conditions* $\mathcal{B}$*, and the control constraints* Ω*.*

We have stated Problem 2.3.1 as a minimization problem. In some applications it is required that the functional J be maximized. There is, however, no need to consider maximum problems separately because the problem of maximizing J is equivalent to the problem of minimizing $-J$. Hence we shall confine our attention to minimum problems.

Definition 2.3.5. A pair (ϕ^*, u^*) that solves Problem 2.3.1 is called an *optimal pair*. The trajectory ϕ^* is called an *optimal trajectory* and the control u^* is called an *optimal control*.

The first term on the right in (2.3.2) is the function g evaluated at the end points of an admissible trajectory. Thus, it assigns a real number to every admissible trajectory and so is a functional G_1 defined on the admissible trajectories. The functional G_1 is defined by the formula

$$G_1(\phi) = g(t_0, \phi(t_0), t_1, \phi(t_1)).$$

Other examples of functionals defined on admissible trajectories are

$$G_2(\phi) = \max\{|\phi(t)| : t_0 \leq t \leq t_1\}$$

and

$$G_3(\phi) = \max\{|\phi(t) - h(t)| : t_0 \leq t \leq t_1\},$$

where h is a given continuous function defined on an interval I containing all the intervals $[t_0, t_1]$ of definition of admissible trajectories. The functionals G_2 and G_3 arise in problems in which in addition to minimizing (2.3.2) it is also desired to keep the state of the system close to some preassigned state.

The preceding discussion justifies the consideration of the following generalization of Problem 2.3.1.

Problem 2.3.2. Let everything be as in Problem 2.3.1, except that (2.3.2) is replaced by

$$\widehat{J}(\phi, u) = G(\phi) + \int_{t_0}^{t_1} f^0(t, \phi(t), u(t))dt, \tag{2.3.3}$$

where G is a functional defined on the admissible trajectories. Find a pair (ϕ^*, u^*) in $\mathcal{A}_1$ that minimizes (2.3.3) in the class $\mathcal{A}_1$.

2.4 Equivalent Formulations

Certain special cases of Problem 2.3.1 are actually equivalent to Problem 2.3.1 in the sense that Problem 2.3.1 can be formally transformed into the special case in question. This information is useful in certain investigations where it is more convenient to study one of the special cases than to study Problem 2.3.1. The reader is warned that in making the transformation to the special case some of the properties of the original problem, such as linearity, continuity, convexity, etc. may be altered. In any particular investigation one must check that the pertinent hypotheses made for the original problem are valid for the transformed problem.

Special cases of Problem 2.3.1 are obtained by taking $f^0 = 0$ or $g = 0$. In keeping with the terminology for related problems in the calculus of variations, we shall call a problem in which $f^0 = 0$ a Mayer problem and we shall call a problem in which $g = 0$ a Lagrange problem. Problem 2.3.1 of Section 2.3 is sometimes called a Bolza problem, also as in the calculus of variations. We shall show that the Mayer formulation and the Lagrange formulation are as general as the Bolza formulation by showing that Problem 2.3.1 can be written either as a Mayer problem or as a Lagrange problem.

We formulate Problem 2.3.1 as a Mayer problem in a higher dimensional euclidean space. Let $\widehat{x} = (x^0, x) = (x^0, x^1, \ldots, x^n)$. Let $\widehat{\mathcal{R}} = \mathbb{R}^1 \times \mathcal{R}$ and let $\widehat{\mathcal{G}} = \widehat{\mathcal{R}} \times \mathcal{U}$. The functions $f^0, f^1, \ldots, f^n$ are defined on $\widehat{\mathcal{G}}$ since they are defined on $\mathcal{G}$ and they are independent of x^0. Let the mapping $\widehat{\Omega}$ be defined on $\widehat{\mathcal{R}}$ by the equation $\widehat{\Omega}(t, \widehat{x}) = \Omega(t, x)$. Let

$$\widehat{\mathcal{B}} = \{(t_0, \widehat{x}_0, t_1, \widehat{x}_1) : (t_0, x_0, t_1, x_1) \in \mathcal{B}, \quad x_0^0 = 0\}.$$

Let (ϕ, u) be an admissible pair for Problem 2.3.1. Let $\widehat{\phi} = (\phi^0, \phi)$, where ϕ^0 is an absolutely continuous function such that

$$\phi^{0'}(t) = f^0(t, \phi(t), u(t)) \qquad \phi^0(t_0) = 0$$

for almost every t in $[t_0, t_1]$. By virtue of (i) of Definition 2.3.3 such a function ϕ^0 exists and is given by

$$\phi^0(t) = \int_{t_0}^{t} f^0(s, \phi(s), u(s)) ds.$$

Then $(\widehat{\phi}, u)$ is an admissible pair for a problem in which $\mathcal{R}, \mathcal{G}, \Omega, \mathcal{B}$, replaced by $\widehat{\mathcal{R}}, \widehat{\mathcal{G}}, \widehat{\Omega}, \widehat{\mathcal{B}}$, respectively, and in which the system of state equations (2.3.1) is replaced by

$$\begin{aligned} \frac{dx^0}{dt} &= f^0(t, x, u(t)) \\ \frac{dx}{dt} &= f(t, x, u, (t)). \end{aligned} \tag{2.4.1}$$

If we set $\widehat{f} = (f^0, f)$, then Eq. (2.4.1) can be written as

$$\frac{d\widehat{x}}{dt} = \widehat{f}(t, x, u(t)).$$

Conversely, to every admissible pair $(\widehat{\phi}, u)$ for a problem involving $\widehat{\mathcal{R}}, \widehat{\mathcal{G}}, \widehat{\Omega}, \widehat{\mathcal{B}}$ and (2.4.1) there corresponds the admissible pair (ϕ, u) for Problem 2.3.1, where ϕ consists of the last n-components of $\widehat{\phi}$. Let

$$\widehat{g}(t_0, \widehat{x}_0, t_1, \widehat{x}_1) = g(t_0, x_0, t_1, x_1) + x_1^0$$

and let

$$\widehat{J}(\widehat{\phi}, u) = \widehat{g}(t_0, \widehat{\phi}(t_0), t_1, \widehat{\phi}(t_1)).$$

Then $\widehat{J}(\widehat{\phi}, u) = J(\phi, u)$, where $\widehat{\phi} = (\phi^0, \phi)$. Hence the Mayer problem of minimizing $\widehat{J}$ subject to state equations (2.4.1), control constraints $\widehat{\Omega}$, and end conditions $\widehat{\mathcal{B}}$ is equivalent to Problem 2.3.1.

We now show that Problem 2.3.1 can be formulated as a Lagrange problem. Let $\widehat{x}, \widehat{\mathcal{R}}, \widehat{\mathcal{G}}, \widehat{\Omega}$ be as in the previous paragraph. Let

$$\widehat{\mathcal{B}} = \{(t_0, \widehat{x}_0, t_1, \widehat{x}_1) : (t_0, x_0, t_1, x_1) \in \mathcal{B}, \quad x_0^0 = g(t_0, x_0, t_1, x_1)/(t_1 - t_0)\}. \tag{2.4.2}$$

(Recall that for all points in $\mathcal{B}$ we have $t_1 > t_0$.) Let (ϕ, u) be an admissible pair for Problem 2.3.1 and let $\widehat{\phi} = (\phi^0, \phi)$ where $\phi^0(t) \equiv g(t_0, x_0, t_1, x_1)/(t_1 - t_0)$. Then $(\widehat{\phi}, u)$ is an admissible pair for a problem in which $\mathcal{R}, \mathcal{G}, \Omega, \mathcal{B}$ are replaced by roofed quantities with $\widehat{\mathcal{B}}$ as in (2.4.2) and with state equations

$$\begin{aligned} \frac{dx^0}{dt} &= 0 \\ \frac{dx}{dt} &= f(t, x, u(t)). \end{aligned} \tag{2.4.3}$$

Conversely, to every admissible pair $(\widehat{\phi}, u)$ for the problem with roofed quantities there corresponds the admissible pair (ϕ, u) for Problem 2.3.1, where ϕ consists of the last n components of $\widehat{\phi}$. If we replace f^0 of Problem 2.3.1 by $f^0 + x^0$ and let

$$\widehat{J}(\widehat{\phi}, u) = \int_{t_0}^{t_1} (f^0(t, \phi(t), u(t)) + \phi^0(t))dt \tag{2.4.4}$$

then $\widehat{J}(\widehat{\phi}, u) = J(\phi, u)$. Hence the Lagrange problem of minimizing (2.4.4) subject to state equations (2.4.3), control constraints $\widehat{\Omega}$, and end conditions $\widehat{\mathcal{B}}$ is equivalent to Problem 2.3.1.

In Problem 2.3.1 the initial time t_0 and the terminal time t_1 need not be fixed. We now show that Problem 2.3.1 can be written as a problem with fixed

initial time and fixed terminal time. We do so by changing the time parameter to s via the equation

$$t = t_0 + s(t_1 - t_0) \qquad 0 \le s \le 1$$

and by introducing new state variables as follows.

Let w be a scalar and consider the problem with state variables (t, x, w), where x is an n-vector and t is a scalar. Let s denote the time variable. Let the state equations be

$$\begin{aligned} \frac{dt}{ds} &= w \qquad\qquad \frac{dw}{ds} = 0 \\ \frac{dx}{ds} &= f(t, x, \bar{u}(s))w \end{aligned} \tag{2.4.5}$$

where $\bar{u}$ is the control and f is as in Problem 2.3.1. Let

$$\begin{aligned} \bar{\mathcal{B}} = \{(s_0, t_0, x_0, w_0, s_1, t_1, x_1, w_1) &: s_0 = 0, s_1 = 1, \\ (t_0, x_0, t_1, x_1) \in \mathcal{B}, &\quad w_0 = t_1 - t_0\}. \end{aligned} \tag{2.4.6}$$

Note that the initial and terminal times are now fixed. Let $\bar{\Omega}(s, t, x, w) = \Omega(t, x)$. Let $\bar{\phi} = (\tau, \xi, \omega)$ be a solution of (2.4.5) corresponding to a control $\bar{u}$, where the Greek-Latin correspondence between (τ, ξ, ω) and (t, x, w) indicates the correspondence between components of $\bar{\phi}$ and the system (2.4.5). Let

$$\bar{J}(\bar{\phi}, \bar{u}) = g(\tau(0), \xi(0), \tau(1), \xi(1)) + \int_0^1 f^0(\tau(s), \xi(s), \bar{u}(s))\omega(s)ds. \tag{2.4.7}$$

Consider the fixed end-time problem of minimizing (2.4.7) subject to the state equations (2.4.5), the control constraints $\bar{\Omega}$, and the end conditions $\bar{\mathcal{B}}$.

Since $t_1 - t_0 > 0$, it follows that for any solution of (2.4.5) satisfying (2.4.6) we have $\omega(s) = t_1 - t_0$, a positive constant, for $0 \le s \le 1$. Let (ϕ, u) be an admissible pair for Problem 2.3.1. It is readily verified that if

$$\begin{aligned} \tau(s) &= t_0 + s(t_1 - t_0) \qquad & \xi(s) &= \phi(t_0 + s(t_1 - t_0)) \\ \bar{u}(s) &= u(t_0 + s(t_1 - t_0)) \qquad & \omega(s) &= t_1 - t_0, \end{aligned}$$

then $(\bar{\phi}, \bar{u}) = (\tau, \xi, \omega, \bar{u})$ is an admissible pair for the fixed end-time problem and $\bar{J}(\bar{\phi}, \bar{u}) = J(\phi, u)$. Conversely, let $(\bar{\phi}, \bar{u})$ be an admissible pair for the fixed end-time problem. If we set

$$\phi(t) = \xi\left(\frac{t - t_0}{t_1 - t_0}\right) \qquad u(t) = \bar{u}\left(t - t_0 t_1 - t_0\right), \qquad t_0 \le t \le t_1,$$

then since $\tau(s) = t_0 + s(t_1 - t_0)$, we have $t = \tau(s)$ for $0 \le s \le 1$. It is readily verified that (ϕ, u) is admissible for Problem 2.3.1 and that $J(\phi, u) = \bar{J}(\bar{\phi}, \bar{u})$. Hence Problem 2.3.1 is equivalent to a fixed end-time problem.

The following observation will be useful in the sequel. Since for any admissible solution of the fixed time problem we have $\omega(s) = t_1 - t_0 > 0$, we can take the set $\bar{\mathcal{R}}$ for the fixed end-time problem to be $[0,1] \times \mathcal{R} \times \mathbb{R}^+$, where $\mathbb{R}^+ = \{w : w > 0\}$.

A special case of the end conditions occurs if the initial and terminal data are separated. In this event, a set $\mathfrak{T}_0$ of points (t_0, x_0) in $\mathbb{R}^{n+1}$ and a set $\mathfrak{T}_1$ of points (t_1, x_1) in $\mathbb{R}^{n+1}$ are given and an admissible trajectory is required to satisfy the conditions

$$(t_i, \phi(t_i)) \in \mathfrak{T}_i, \qquad i = 0, 1. \tag{2.4.8}$$

The set $\mathcal{B}$ in this case is given by (2.2.4). We shall show that the apparently more general requirement (iii) of Definition 2.3.3 can be reduced to the form (2.4.8) by embedding the problem in a space of higher dimension as follows.

Let $y = (y^1, \ldots, y^n)$ and let y^0 be a scalar. Let $\widehat{y} = (y^0, y)$. Let the sets $\mathcal{R}$ and $\mathcal{G}$ of Problem 2.3.1 be replaced by sets $\widetilde{\mathcal{R}} = \mathcal{R} \times \mathbb{R}^{n+1}$ and $\widetilde{\mathcal{G}} = \widetilde{\mathcal{R}} \times \mathcal{U}$. Then the vector function $\widehat{f} = (f^0, f)$ is defined on $\widetilde{\mathcal{G}}$ since it is independent of $\widehat{y}$. Let $\widetilde{\Omega}(t, x, \widehat{y}) = \Omega(t, x)$. Let the state equations be

$$\begin{aligned} \frac{dx}{dt} &= f(t, x, u(t)) \\ \frac{d\widehat{y}}{dt} &= 0. \end{aligned} \tag{2.4.9}$$

Let

$$\begin{aligned} \widetilde{\mathfrak{T}}_0 &= \{(t_0, x_0, y_0^0, y_0) : (t_0, x_0, y_0^0, y_0) \in \mathcal{B}\} \\ \widetilde{\mathfrak{T}}_1 &= \{(t_1, x_1, y_1^0, y_1) : y_1^0 = t_1,\ y_1^i = x_1^i, i = 1, \ldots, n\}. \end{aligned}$$

Replace condition (iii) of Definition 2.3.2 by the condition

$$(t_i, \widetilde{\phi}(t_i)) \in \widetilde{\mathfrak{T}}_i \qquad i = 0, 1, \tag{2.4.10}$$

where $\widetilde{\phi}$ is a solution of (2.4.9). Then it is easily seen that a function u is an admissible control for Problem 2.3.1 if and only if it is an admissible control for the system (2.4.9) subject to control constraints $\widetilde{\Omega}$ and end-condition (2.4.10). Moreover, the admissible trajectories $\widetilde{\phi}$ are of the form $\widetilde{\phi} = (\phi, t_1, x_1)$. Hence if we take the cost functional to be $\widetilde{J}$, where

$$\widetilde{J}(\widetilde{\phi}, u) = J(\phi, u),$$

then Problem 2.3.1 is equivalent to a problem with end conditions of the form (2.4.8).

2.5 Isoperimetric Problems and Parameter Optimization

In some control problems, in addition to the usual constraints there exists constraints of the form

$$\int_{t_0}^{t_1} h^i(t,\phi(t),u(t))dt \le c^i \qquad i=1,\ldots,q \tag{2.5.1}$$
$$\int_{t_0}^{t_1} h^i(t,\phi(t),u(t))dt = c^i \qquad i=q+1,\ldots,p,$$

where the functions h^i are defined on $\mathcal{G}$ and the constants c^i are prescribed. Constraints of the form (2.5.1) are called isoperimetric constraints. A problem with isoperimetric constraints can be reduced to a problem without isoperimetric constraints as follows.

Introduce additional state variables $x^{n+1},\ldots,x^{n+p}$ and let $\widetilde{x}=(x,\bar{x})$, where $\bar{x}=(x^{n+1},\cdots,x^{n+p})$. Let the state equations be

$$\frac{dx^i}{dt} = f^i(t,x,u(t)) \qquad i=1,\ldots,n \tag{2.5.2}$$
$$\frac{dx^{n+i}}{dt} = h^i(t,x,u(t)), \qquad i=1,\ldots,p$$

or

$$\frac{d\widetilde{x}}{dt} = \widetilde{f}(t,x,u,(t)),$$

where $\widetilde{f}=(f,h)$. Let the control constraints be given by the mapping $\widetilde{\Omega}$ defined by the equation $\widetilde{\Omega}(t,\widetilde{x})=\Omega(t,x)$. Let the end conditions be given by the set $\widetilde{\mathcal{B}}$ consisting of all points $(t_0,\widetilde{x}_0,t_1,\widetilde{x}_1)$ such that: (i) $(t_0,x_0,t_1,x_1)\in\mathcal{B}$; (ii) $x_0^i=0$, $i=n+1,\ldots,n+p$; (iii) $x_1^i\le c^i$, $i=n+1,\ldots,n+q$; and (iv) $x_1^i=c^i$, $i=n+q+1,\ldots,n+p$. For the system with state variable $\widetilde{x}$, let $\mathcal{R}$ be replaced by $\widetilde{\mathcal{R}}=\mathcal{R}\times\mathbb{R}^p$ and let $\mathcal{G}$ be replaced by $\widetilde{\mathcal{G}}=\widetilde{\mathcal{R}}\times\mathcal{U}$.

Let (ϕ,u) be an admissible pair for Problem 2.3.1 such that the constraints (2.5.1) are satisfied. Let $\widetilde{\phi}=(\phi,\bar{\phi})$, where

$$\bar{\phi}(t)=\int_0^t h(s,\phi(s),u(s))ds \qquad \bar{\phi}(0)=0.$$

Then $(\widetilde{\phi},u)$ is an admissible pair for the system with state variable $\widetilde{x}$. Conversely, if $(\widetilde{\phi},u)$ is admissible for the $\widetilde{x}$ system then (ϕ,u), where ϕ consists of the first n components of $\widetilde{\phi}$, is admissible for Problem 2.3.1 and satisfies the isoperimetric constraints. Hence by taking the cost functional for the problem in $\widetilde{x}$-space to be $\widetilde{J}$, where $\widetilde{J}(\widetilde{\phi},u)=J(\phi,u)$, we can write the problem with constraints (2.5.1) as an equivalent problem in the format of Problem 2.3.1.

In Problem 2.3.1, the functions $f^0, f^1, \ldots, f^n$ defining the cost functional and the system of differential equations (2.3.1) are regarded as being fixed. In some applications these functions are dependent upon a parameter vector $w = (w^1, \ldots, w^k)$, which is at our disposal. For example, in the rocket problem of Section 1.4 we may be able to vary the effective exhaust velocity over some range $c_0 \leq c \leq c_1$ by proper design changes. The system differential equations (2.3.1) will now read

$$\frac{dx}{dt} = f(t, x, w, u(t)) \qquad w \in W,$$

where W is some preassigned set in $\mathbb{R}^k$. For a given choice of control u a corresponding trajectory ϕ will in general now depend on the choice of parameter value w. Hence, so will the value $J(\phi, u, w)$ of the cost functional. The problem now is to choose a parameter value w^* in W for which there exists an admissible pair (ϕ^*, u^*) such that $J(\phi^*, u^*, w^*) \leq J(\phi, u, w)$ for all w in W and corresponding admissible pairs (ϕ, u).

The problem just posed can be reformulated in the format of Problem 2.3.1 in $(n + k + 1)$-dimensional space as follows. Introduce new state variables $w = (w^1, \ldots, w^k)$ and consider the system

$$\begin{aligned} \frac{dx^i}{dt} &= f^i(t, x, w, u(t)) && i = 1, \ldots, n \\ \frac{dw^i}{dt} &= 0 && i = 1, \ldots, k. \end{aligned} \tag{2.5.3}$$

Let $\widetilde{x} = (x, w)$, let $\widetilde{\mathcal{R}} = \mathcal{R} \times \mathbb{R}^k$, let $\widetilde{\mathcal{G}} = \widetilde{\mathcal{R}} \times \mathcal{U}$, and let $\widetilde{\Omega}(t, x, w) = \Omega(t, x)$. Let the end conditions be given by

$$\widetilde{\mathcal{B}} = \{(t_0, x_0, w_0, t_1, x_1, w_1) : (t_0, x_0, t_1, x_1) \in \mathcal{B},\ w_0 \in W\}.$$

Let $\widetilde{J}(\widetilde{\phi}, u) = J(\phi, w, u)$. It is readily verified that the problem of minimizing J subject to (2.5.3), the control constraints $\widetilde{\Omega}$, and end conditions $\widetilde{\mathcal{B}}$ is equivalent to the problem involving the optimization of parameters.

2.6 Relationship with the Calculus of Variations

The brachistochrone problem in Section 1.6 is an example of the simple problem in the calculus of variations, which can be stated as follows. Let t be a scalar, let x be a vector in $\mathbb{R}^n$, and let x' be a vector in $\mathbb{R}^n$. Let $\mathcal{G}$ be a region in (t, x, x')-space. Let f^0 be a real valued function defined on $\mathcal{G}$. Let $\mathcal{B}$ be a given set of points (t_0, x_0, t_1, x_1) in $\mathbb{R}^{2n+2}$ and let g be a real valued function defined in $\mathcal{B}$. An admissible trajectory is defined to be an absolutely continuous function ϕ defined on an interval $[t_0, t_1]$ such that:

(i) $(t, \phi(t), \phi'(t)) \in \mathcal{G}$ for t in $[t_0, t_1]$

(ii)
$$(t_0, \phi(t_0), t_1, \phi(t_1)) \in \mathcal{B} \tag{2.6.1}$$

(iii) $t \to f^0(t, \phi(t), \phi'(t))$ is integrable on $[t_0, t_1]$.

The problem is to find an admissible trajectory that minimizes

$$g(t_0, \phi(t_0), t_1, \phi(t_1)) + \int_{t_0}^{t_1} f^0(t, \phi(t), \phi'(t))dt.$$

As with the brachistochrone problem, the general simple problem in the calculus of variations can be written as a control problem by relabeling x' as z; that is, we set $u = \phi'$. (Recall that z denotes the control variable and u denotes the control function.) The simple problem in the calculus of variations becomes the following control problem. Minimize

$$g(t_0, \phi(t_0), t_1, \phi(t_1)) + \int_{t_0}^{t_1} f^0(t, \phi(t), u(t))dt$$

subject to the state equations

$$\frac{dx^i}{dt} = u^i(t) \qquad i = 1, \ldots, n,$$

end conditions (ii) of (2.6.1), and control constraints Ω, where

$$\Omega(t, x) = \{z : (t, x, z) \in \mathcal{G}\}.$$

Recall that the region $\mathcal{G}$ is an open set. Thus, Ω is also an open set in case of any simple problem of calculus of variations. The problem of Bolza in the calculus of variations differs from the simple problem in that in addition to (2.6.1) an admissible trajectory is required to satisfy a system of differential equations

$$F^i(t, \phi(t), \phi'(t)) = 0 \qquad i = 1, \ldots, \mu. \tag{2.6.2}$$

The functions $F^1, \ldots, F^\mu$ are defined and continuous on $\mathcal{G}$ and $\mu < n$.

In the development of the necessary conditions in the problem of Bolza, the assumption is usually made that the functions f^0 and $F = (F^1, \ldots, F^\mu)$ are of class $C^{(1)}$ on the region $\mathcal{G}$ of (t, x, x')-space and the matrix of partial derivatives $F_{x'} = (\partial F^i / \partial x'^j)$, $i =, 1 \ldots, \mu$, $j = 1, \ldots, n$, has rank μ everywhere on $\mathcal{G}$. Hence in the neighborhood of any point (t_2, x_2, x_2') at which

$$F^i(t_2, x_2, x_2') = 0 \qquad i = 1, \ldots, \mu, \tag{2.6.3}$$

we can solve for μ components of x' in terms of t, x, and the remaining $n - \mu$ components of x'. Moreover, these μ components of x' will be $C^{(1)}$ functions of their arguments. Let us now suppose that we can solve (2.6.3) globally in

this fashion. Since we can relabel components we can assume that we solve the first μ components in terms of the remaining $n-\mu$, and get

$$x'^{i} = G^i(t, x, \widetilde{x}') \qquad i = 1, \dots, \mu,$$

where $\widetilde{x}' = (x'^{\mu+1}, \dots, x'^{n})$. Thus, Eq. (2.6.2) is equivalent to

$$\phi'^{i}(t) = G^i(t, \phi(t), \widetilde{\phi}'(t)) \qquad i = 1, \dots, \mu,$$

where $\widetilde{\phi}' = (d\phi^{\mu+1}/dt, \dots, d\phi^n/dt)$. Let $m = n-\mu$ and let $z = (z^1, \dots, z^m) = (x'^{\mu+1}, \dots, x'^{n}) = \widetilde{x}'$. It then follows that under the assumptions made here that the Bolza problem is equivalent to the following control problem with control variable $z = (z^1, \dots, z^m)$.

The functional to be minimized is defined by the equation

$$\bar{J}(\phi, u) = g(t_0, \phi(t_0), t_1, \phi(t_1)) + \int_{t_0}^{t_1} \bar{f}^0(t, \phi(t), u(t))dt,$$

where

$$\bar{f}^0(t, x, z) = f^0(t, x, G^1(t, x, z), \dots, G^\mu(t, x, z), z^1, \dots, z^m).$$

The system equations are

$$\begin{aligned} \frac{dx^i}{dt} &= G^i(t, x, u(t)) & i &= 1, \dots, \mu \\ \frac{dx^{\mu+i}}{dt} &= u^i(t) & i &= 1, \dots, m. \end{aligned}$$

The end conditions are defined by the set $\mathcal{B}$ of the Bolza problem and the control constraints Ω are defined as follows:

$$\Omega(t, x) = \{z : (t, x, G^1(t, x, z), \dots, G^\mu(t, x, z), z^1, \dots, z^m) \in \mathcal{G}\}.$$

It is, of course, also required of an admissible (ϕ, u) that the mapping $t \to \bar{f}^0(t, \phi(t), u(t))$ be integrable.

Conversely, under certain conditions the control problem can be written as a problem of Bolza in the calculus of variations. Let us first suppose that $\Omega(t, x) = \mathbb{R}^m$ for all (t, x). That is, there are no constraints on the control. We introduce new coordinates $y^1, \dots, y^m$ and let

$$\frac{dy^i}{dt} = u^i(t) \qquad i = 1, \dots, m.$$

Then Eq. (2.3.1) can be written as

$$\frac{dx^i}{dt} - f^i\left(t, x, \frac{dy}{dt}\right) = 0 \qquad i = 1, \dots, n.$$

If we set

$$F^i(t,x,y,x',y') = x'^i - f^i(t,x,y') \qquad i=1,\dots,n, \tag{2.6.4}$$

then the control problem can be written as the following problem of Bolza in $(n+m+1)$-dimensional (t,x,y)-space. The class of admissible arcs is the set of absolutely continuous functions $\widehat{\phi} = (\phi,\eta) = (\phi^1,\dots,\phi^n,\eta^1,\dots,\eta^m)$ defined on intervals $[t_0,t_1]$ such that:

(i) $(t,\phi(t),\eta'(t))$ is in the domain of definition of the function $\widetilde{f} = (f^0, f)$;

(ii) $(t_0,\phi(t_0),t_1,\phi(t_1))$ is in $\mathcal{B}$ and $\eta(t_0)=0$;

(iii) the function $t \to f^0(t,\phi(t),\eta'(t))$ is integrable; and

(iv)
$$F(t,\widehat{\phi}(t),\widehat{\phi}'(t)) = \phi'(t) - f(t,\phi(t),\eta'(t)) = 0 \tag{2.6.5}$$

a.e. on $[t_0,t_1]$. The problem is to minimize the functional

$$g(t_0,\phi(t_0),t_1,\phi(t_1)) + \int_{t_0}^{t_1} f^0(t,\phi(t),\eta'(t))dt$$

in the class of admissible arcs.

It is clear from (2.6.4) that the ith row of the $n\times(n+m)$ matrix $(F_{x'}F_{y'})$ for the Bolza problem obtained from the control problem has the form

$$(0\dots0\ 1\ 0\dots0 \quad \partial f^i/\partial y'^1 \dots \partial f^i/\partial y'^m),$$

where the entry 1 occurs in the i-th column and all other entries in the first n columns are zero. Thus, the $n\times(n+m)$ matrix $(F_{x'}F_{y'})$ has rank n as usually required in the theory of the necessary conditions for the Bolza problem.

Let us now suppose that control constraints Ω are present and that the sets $\Omega(t,x)$ are defined by a system of inequalities. We suppose that there are r functions $R^1,\dots,R^r$ of class $C^{(1)}$ on $\mathcal{G}$. The set $\Omega(t,x)$ is defined as follows:

$$\Omega(t,x) = \{z : R^i(t,x,z) \geq 0, i=1,\dots,r\}.$$

We impose a further restriction, which we call the *constraint qualification.*

(i) If m, the number of components of z, is less than or equal to r, the number of constraints, then at any point (t,x,z) of $\mathcal{G}$ at most m of the functions $R^1,\dots,R^r$ can vanish at that point.

(ii) At the point (t,x,z) let $i_1,\dots,i_\rho$ denote the set of indices such that $R^i(t,x,z)=0$. Let $R_{z,\rho}(t,x,z)$ denote the matrix formed by taking the rows $i_1,\dots,i_\rho$ of the matrix

$$R_z(t,x,z) = \left(\partial R^i(t,x,z)\partial z^j\right) \qquad i=1,\dots,r \qquad j=1,\dots,m. \tag{2.6.6}$$

Then $R_{z,\rho}(t,x,z)$ has rank ρ.

To formulate the control problem as a Bolza problem we proceed as before and let $y' = z$. The constraints take the form $R^i(t, \phi(t), \eta'(t)) \geq 0$. This restriction is not present in the classical Bolza formulation. We can, however, write the variational problem with constraints as a Bolza problem by introducing a new variable $w = (w^1, \ldots, w^r)$ and r additional state equations

$$R^i(t, x, y') - (w'^i)^2 = 0 \qquad i = 1, \ldots, r. \tag{2.6.7}$$

The Bolza problem now is to minimize

$$g(t_0, \phi(t_0), t_1, \phi(t_1)) + \int_{t_0}^{t_1} f^0(t, \phi(t), \eta'(t))dt$$

subject to the differential equations (2.6.5) and (2.6.7) and the end conditions

$$(t_0, \phi(t_0), t_1, \phi(t_1)) \in \mathcal{B} \qquad \eta(t_0) = 0 \qquad \omega(t_0) = 0,$$

where the function ω is the component of the admissible arc corresponding to the variable w.

Let

$$\begin{aligned} F^i(t, x, y, w, x', y', w') &= x'^i - f^i(t, x, y') \qquad i = 1, \ldots, n, \qquad (2.6.8)\\ F^{n+i}(t, x, y, w, x', y', w') &= R^i(t, x, y') - (w'^i)^2 \qquad i = 1, \ldots, r. \end{aligned}$$

We shall show that the $(n + r) \times (n + m + r)$ matrix

$$M = \left(\frac{\partial F^q}{\partial x'^j} \; \frac{\partial F^q}{\partial y'^k} \; \frac{\partial F^q}{\partial w'^s} \right) = (F_{x'} \; F_{y'} \; F_{w'})$$

has rank $n + r$ as usually required in the theory of the Bolza problem. It is a straightforward calculation using (2.6.4) and (2.6.8) to see that

$$M = \begin{pmatrix} I & -f_{y'} & 0_1 \\ 0_2 & R_{y'} & -W \end{pmatrix}$$

where I is the $n \times n$ identity matrix, $f_{y'}$ is the $n \times m$ matrix with typical entry $\partial f^i / \partial y'^k$, 0_1 is an $n \times r$ zero matrix, 0_2 is an $r \times n$ zero matrix, $R_{y'}$ is the $r \times m$ matrix with typical entry $\partial R^i / \partial y'^k$, and W is an $r \times r$ diagonal matrix with diagonal entries $2w'^i$.

From the form of the matrix M it is clear that to prove that it has rank $n+r$ it suffices to show that the $r \times (m+r)$ matrix $(R_{y'} \, -W)$ has rank r. To do this let us suppose that the indexing is such that the indices $i_1, \ldots, i_\rho$ for which $R^{i_j}(t, x, y') = 0$ are the indices $1, \ldots, \rho$. Let $(R_{y'})_\rho$ denote the submatrix of $R_{y'}$ consisting of the first ρ rows of $R_{y'}$ and let $(R_{y'})_{r-\rho}$ denote the submatrix of $R_{y'}$ consisting of the remaining rows. Thus if $i > \rho$, then $R^i(t, x, y') > 0$; if $i \leq \rho$, then $R^i(t, x, y') = 0$. Hence since

$$(w'^i)^2 = R^i(t, x, y') \qquad i = 1, \ldots, r,$$

it follows that

$$2w'^{i} = 0 \text{ if } i \leq \rho \qquad 2w'^{i} \neq 0 \text{ if } i > \rho.$$

Hence

$$(R_{y'} - W) = \begin{pmatrix} (R_{y'})_\rho & 0_3 & 0_4 \\ (R_{y'})_{r-\rho} & 0_5 & D \end{pmatrix},$$

where D is a diagonal matrix of dimension $(r-\rho)\times(r-\rho)$ with non-zero entries $2w'^{i}$, $i > \rho$, and where 0_3, 0_4, and 0_5 are zero matrices. By the constraint qualification (2.6.6) the matrix $(R_{y'})_\rho$ has rank ρ. Since D has rank $r-\rho$ it follows that $(R_{y'} - W)$ has rank r, as required.

2.7 Hereditary Problems

Hereditary problems, which are also called delay or lag problems, take the history of the system into account in their evolutions and in the measure of their performances. We give a general formulation that takes into account the history of the control as well as the history of the state and then discuss a commonly occuring special case.

The formulation of a hereditary problem requires the introduction of additional notation. Let I_α^t denote the interval $[\alpha, t]$, where $\alpha < t \leq \infty$, let $\mathcal{X}$ denote an open interval in $\mathbb{R}^n$, and let $\mathcal{U}$ denote an open interval in $\mathbb{R}^m$. Let $r > 0$ and let (t, s, x, z) denote a generic point of $I_0^\infty \times I_{-r}^\infty \times \mathcal{X} \times \mathcal{U}$. Let $C(I_{-r}^\tau, \mathcal{X})$, $0 < \tau \leq \infty$ denote the space of continuous functions from I_{-r}^τ to $\mathcal{X}$ with supremum norm, and let $AC(I_{-r}^\tau, \mathcal{X})$ denote the subspace of absolutely continuous functions. Let $\mathcal{M}$ denote the set of measurable functions on I_0^∞ with range in $\mathcal{U}$. We shall denote functions in $\mathcal{M}$ by the letter u. Let

(i) $g^0, \ldots, g^n$ from $I_0^\infty \times I_{-r}^\infty \times \mathfrak{X} \times \mathfrak{U}$ into $\mathbb{R}^n$ be continuous in all variables. Further, $\partial_x g^i(t, s, x, u)$, $0 \leq i \leq n$, are continuous in all arguments. Here, (t, s, x, u) is a generic point of $I_0^\infty \times I_{-r}^\infty \times \mathfrak{X} \times \mathfrak{U}$. We assume that $u(t) \in \Omega$ a.e., where Ω is a fixed compact subset of $\mathfrak{U}$.

(ii) $h^0, h^1, \ldots, h^n$ be functions from $I_0^\infty \times C(I_{-r}^\infty, \mathcal{X}) \times \mathcal{U}$ to $\mathbb{R}$ such that if ϕ_1 and ϕ_2 are in $C(I_{-r}^\infty, \mathcal{X})$ and $\phi_1 = \phi_2$ on I_{-r}^t, $0 < t < \infty$, then

$$h^i(t, \phi_1, u(t)) = h^i(t, \phi_2, u(t)) \qquad i = 0, 1, \ldots, n.$$

The functions $h^i : C(I_{-r}^\infty, \mathfrak{X}) \times \mathfrak{U} \to \mathbb{R}$ are measurable in $t \in I$ and continuous in $u \in \mathfrak{U}$. The functions $h^i(t, \cdot, u)$ are Fréchet differentiable as a map from $C(I_{-r}^\infty, \mathfrak{X})$ into $\mathbb{R}$. Further, the derivatives are continuous in the second and third arguments. Denoting by $dh^i(t, \phi(\cdot), u)$ the Fréchet derivative at ϕ, we assume that there exists $\Lambda \in L_1(\mathbb{R})$ such that

$$|h^i(t, \phi(\cdot), u)| \leq \Lambda(t), \qquad \forall\, \phi \in C(I_{-r}^\infty, \mathfrak{X}),$$

$$|dh^i(t, \phi(\cdot), u)(\psi)| \le \Lambda(t)\|\psi\|_\infty, \qquad \forall\, \psi \in C(I^\infty_{-r}).$$

(iii) $w^0, w^1, \ldots, w^n$ be measurable functions on $I^\infty_0 \times I^\infty_{-r}$ to $\mathbb{R}$ such that for each t in I^∞_0, $w^i(t, \cdot)$ is of bounded variation on I^∞_{-r}, is continuous on the right, and vanishes for $s \ge t$. Let $f^0, f^1, \ldots, f^n$ be functions defined on $I^\infty_0 \times C(I^\infty_{-r}, \mathcal{X}) \times \mathcal{M}$ by the formula

$$\begin{aligned} f^i(t, \phi, u) \quad &= \quad h^i(t, \phi, u(t)) + \int_{-r}^{t} g^i(t, s, \phi(s), u(s)) d_s w^i(t, s) \\ &\qquad i = 0, 1, \ldots, n. \end{aligned} \tag{2.7.1}$$

Recall that ϕ and u denote functions and $\phi(t)$, $\phi(s)$, $u(t)$, $u(s)$ denote the values of functions at the indicated arguments.

A function u in $\mathcal{M}$ is said to be a *control* on the interval $[0, t_1]$ if there exists a function ϕ in $C(I^{t_1}_{-r}, \mathcal{X})$ that is in $AC(I^{t_1}_0, \mathcal{X})$ such that

$$\begin{aligned} \phi'(t) &= f(t, \phi, u) \quad \text{a.e. on } [0, t_1] \\ \phi(t) &= y(t) \quad -r \le t \le 0 \;\; y \in C(I^0_{-r}, \mathcal{X}). \end{aligned} \tag{2.7.2}$$

Here, $f = (f^1, \ldots, f^n)$. The function y is specified and is called the *initial function*. The function ϕ is called a *trajectory* corresponding to u.

Let $t_1 > 0$ be fixed. Let Ω be a mapping from $I^{t_1}_0 \times \mathcal{X}$ to subsets of $\mathcal{U}$ and let $\mathcal{B}$ be a specified set in $\mathbb{R}^{2n+1}$. A control u and a corresponding trajectory ϕ are said to be admissible if

(i) $t \to f^0(t, \phi, u)$ is in $L_1[0, t_1]$

(ii) $u(t) \in \Omega(t, \phi(t))$ a.e. on $[0, t_1]$

(iii) $(\phi(0), t_1, \phi(t_1)) \in \mathcal{B}$.

The control problem is to choose an admissible pair that minimizes

$$g(\phi(0), t_1, \phi(t_1)) + \int_0^{t_1} f^0(t, \phi, u) dt, \tag{2.7.3}$$

where g is a given function defined on $\mathcal{B}$.

In a frequently encountered form of the hereditary problem the dependence on the history of the control is absent and the dependence on the state has a special form. Thus, the integrals in (2.7.1) are absent. For a given ϕ in $C(I^\infty_{-r}, \mathcal{X})$ we define for each t in I^∞_0 a function ϕ_t in $C(I^0_{-r}, \mathcal{X})$ by the formula

$$\phi_t(\theta) = \phi(t + \theta), \quad -r \le \theta \le 0.$$

We now take the function $\widehat{h} = (h^0, h) = (h^0, h^1, \ldots, h^n)$ to be a mapping from $I^\infty_0 \times C(I^0_{-r}, \mathcal{X}) \times \mathcal{U}$ to $\mathbb{R}$ and take the state equations to be

$$\phi'(t) = h(t, \phi_t, u(t)).$$

The expression (2.7.3) becomes

$$g(\phi(0), t_1, \phi(t_1)) + \int_0^{t_1} h^0(t, \phi_t, u(t))dt.$$

As a further specialization we take

$$\widehat{h}(t, \phi_t, u(t)) = \widehat{k}(t, \phi(t-r), u(t)),$$

where $\widehat{k}$ is a function from $I_0^\infty \times \mathcal{X} \times \mathcal{U}$ to $\mathbb{R}^{n+1}$. In this case, the state equations are said to be retarded or delay equations.

Chapter 3

Relaxed Controls

3.1 Introduction

In this chapter we define relaxed controls and the relaxed control problem and determine some of the properties of relaxed controls. For problems with well-behaved compact constraint sets, relaxed controls have a very useful compactness property. Also, at a given point in the subset $\mathcal{R}$ of (t, x) space, the set of directions that the state of a relaxed system may take is convex. This property is needed in existence theorems. We also shall prove an implicit function theorem for measurable functions that permits a definition of relaxed controls alternative to the one given in the next section. This theorem will also be used in our existence theorems. To motivate the definition of relaxed controls, we present two examples.

Example 3.1.1. Let the state equation be

$$\frac{dx^1}{dt} = (x^2)^2 - u^2$$
$$\frac{dx^2}{dt} = u$$

and let $\Omega(t, x) = \{z \colon |z| \leq 1\}$. Note that the constraint sets are constant, compact, and convex. Let the initial set $\mathcal{T}_0$ be given by $(t_0, x_0^1, x_0^2) = (0, 1, 0)$. Let the terminal set $\mathcal{T}_1$ be given by

$$(x_1^1)^2 + (x_1^2)^2 = a^2 \qquad 0 < a < 1,$$

and $t_1 \geq \delta$, where δ is a fixed number satisfying $0 < \delta < 1 - a$. The problem is to minimize the time t_1 at which the terminal set $\mathcal{T}_1$ is attained.

From the state equations it is clear that $t_1 > 1 - a$ for all admissible controls. To attain the terminal time of $1 - a$ we would need to have $\phi^2(t) \equiv 0$ and $(u(t))^2 \equiv 1$ for a trajectory. This is clearly impossible in view of the second state equation. This equation suggests, however, that we can approximate the value $1 - a$ by taking $u(t)$ to be alternately $+1$ and -1 on small intervals. To this end, for each $r = 1, 2, 3, \ldots$ we define a control u_r on the interval $[0, 2r]$ as follows:

$$u_r(t) = \begin{cases} 1 & (i-1)/2r \le t < i/2r \quad i = 1, 3, \ldots, 4r^2 - 1 \\ -1 & (i-1)/2r \le t < i/2r \quad i = 2, 4, \ldots, 4r^2. \end{cases}$$

For $t > 2r$, let $u_r(t) \equiv 0$. Let $\varphi_r = (\varphi_r^1, \varphi_r^2)$ be the trajectory corresponding to u_r. Clearly,

$$0 \le \varphi_r^2(t) \le 1/2r \tag{3.1.1}$$

on $[0, \infty)$, and so $\varphi_r^2(t) \to 0$ uniformly on $[0, \infty)$ as $r \to \infty$. It then follows from the first state equation that for each r, $d\phi_r^1/dt > -1$ except at a finite set of points and that

$$\lim_{r \to \infty} d\phi_r^1/dt = -1$$

uniformly on $[0, \infty]$. Hence for r sufficiently large there exists a point $t_r > 1-a$ such that $(t_r, \phi_r^1(t_r), \phi_r^2(t_r)) \in \mathcal{T}_1$. It therefore follows from $\phi_r^2(t_r) \to 0$ that $\phi_r^1(t_r) \to a$. Now

$$\varphi_r^1(t_r) - 1 = \int_0^{t_r} (\varphi_r^1)' dt \le \int_0^{t_r} \left(\frac{1}{4r^2} - 1 \right) dt = t_r \left(\frac{1}{4r^2} - 1 \right),$$

where the inequality follows from the first state equation and (3.1.1). Hence

$$t_r \left(1 - \frac{1}{4r^2} r \right) \le 1 - \varphi_r^1(t_r),$$

and so

$$\limsup_{r \to \infty} t_r \le 1 - a.$$

Since $t_r > 1 - a$, we get that $\lim_{r \to \infty} t_r = 1 - a$. Recalling that the terminal time for any admissible trajectory exceeds $1 - a$, we have that the infimum of all terminal times is $1 - a$. Thus, the problem has no solution, since as we already noted, the terminal time $1 - a$ cannot be achieved by an admissible trajectory.

The construction of the admissible sequence (φ_r, u_r) suggests that we might attain the terminal time $1 - a$ if we modified the problem to allow controls that are an average in some sense of controls with values in $\Omega \equiv \Omega(t, x) = \{z \colon |z| \le 1\}$. To this end we define a problem with state equations

$$\frac{dx^1}{dt} = (x^2)^2 - \int_\Omega z^2 d\mu_t \tag{3.1.2}$$

$$\frac{dx^2}{dt} = \int_\Omega z d\mu_t,$$

where for each t, $d\mu_t$ is a regular probability measure on Ω. A control u of the original problem is also a control for the relaxed problem. To see this, for each t let $d\mu_t$ be the measure concentrated at the point $u(t)$. We take the

initial and terminal sets to be as before and require that the terminal time be minimized. The problem just formulated is the relaxed version of the original problem. For this problem we also have $t_1 \geq 1-a$, with equality only possible if for each t there exists a $d\mu_t$ such that

$$\int_\Omega z d\mu_t = 0 \quad \text{and} \quad \int_\Omega z^2 d\mu_t = 1. \tag{3.1.3}$$

If we take $d\mu_t$ to be the measure on Ω that assigns the measure $1/2$ to each of the points $z = 1$ and $z = -1$, then (3.1.3) holds. Thus, the relaxed problem has a solution.

It is readily verified that in the relaxed problem at each point (t, x) in $\mathcal{R}$ the set of possible directions is convex. In the original problem the set of admissible directions (v^1, v^2) is the segment of the parabola

$$v^1 = (x^2)^2 - (v^2)^2 \qquad -1 \leq v^2 \leq 1,$$

which is not convex. We shall see later that the set of admissible directions for the relaxed problem is the convex hull of this set.

Example 3.1.2. Let the state equations be

$$\begin{aligned}\frac{dx^1}{dt} &= u^1(t)\\ \frac{dx^2}{dt} &= u^2(t)\\ \frac{dx^3}{dt} &= 1\end{aligned}$$

and let the constraint set be $\Omega = \{z = (z^1, z^2) \colon (z^1)^2 + (z^2)^2 = 1\}$. Let the initial set $\mathcal{T}_0$ be given by $(t_0, x_0^1, x_0^2, x_0^3) = (0, 0, 0, 0)$ and let the terminal set $\mathcal{T}_1$ be given by $(t_1, x_1^1, x_1^2, x_1^3) = (1, 0, 0, 1)$. Let

$$J(\varphi, u) = \int_0^1 [(\phi^1)^2 + (\phi^2)^2] dt,$$

where $\phi = (\phi^1, \phi^2)$ is an admissible trajectory corresponding to an admissible control u. For each $k = 1, 2, 3, \ldots$, let

$$u_k(t) = (u_k^1(t), u_k^2(t)) = (\sin 2\pi kt, \cos 2\pi kt),$$

and let $\phi_k = (\phi_k^1, \phi_k^2, \phi_k^3)$ be defined by

$$\begin{aligned}\phi_k^1(t) &= (1 - \cos 2\pi kt)/2\pi k\\ \varphi_k^2(t) &= \sin 2\pi kt/2\pi k\\ \varphi_k^3(t) &= t.\end{aligned}$$

Then each (φ_k, u_k) is admissible and $0 \leq J(\varphi_k, u_k) \leq (\pi k)^{-2}$. Since $J(\varphi, u) \geq$

0 for all admissible (φ, u), it follows that $\inf\{J(\varphi, u) \colon (\varphi, u) \text{ admissible}\} = 0$. Therefore, if there exists an optimal pair (φ^*, u^*) we must have $J(\varphi^*, u^*) = 0$. But then we must have $u^*(t) \equiv 0$. This control, however, does not satisfy the constraint. Hence an optimal control does not exist.

As in Example 3.1.1, we consider the relaxed problem with state equations

$$\begin{aligned} \frac{dx^1}{dt} &= \int_\Omega z^1 d\mu_t \\ \frac{dx^2}{dt} &= \int_\Omega z^2 d\mu_t \\ \frac{dx^3}{dt} &= 1. \end{aligned} \tag{3.1.4}$$

The sets $\mathcal{T}_0, \mathcal{T}_1$, and Ω are as before, and the functional to be minimized is

$$\int_0^1 [(\psi^1)^2 + (\psi^2)^2] \; dt,$$

where ψ is a solution of (3.1.4). Let $0 \leq \theta < 2\pi$ be arbitrary. For each t in $[0, 1]$ let $d\mu_t$ be the measure such that each of the points $(\cos\theta, \sin\theta)$ and $(-\cos\theta, -\sin\theta)$ have measure $1/2$. Then the system (3.1.4) becomes $dx^1/dt = 0$, $dx^2/dt = 0$, $dx^3/dt = 1$, and the admissible relaxed trajectory $\psi^1(t) \equiv 0$, $\psi^2(t) \equiv 0$, $\psi^3(t) = t$ minimizes.

3.2 The Relaxed Problem; Compact Constraints

In this section we formulate the relaxed problem corresponding to a slightly specialized version of Problem 2.3.1, which we now restate for the reader's convenience.

Minimize

$$g(t_0, x_0, t_1, x_1) + \int_{t_0}^{t_1} f^\circ(t, x, u(t)) dt$$

subject to

$$\begin{gathered} \frac{dx}{dt} = f(t, x, u(t)) \\ (t_0, x_0, t_1, x_1) \in \mathcal{B} \qquad u(t) \in \Omega(t). \end{gathered} \tag{3.2.1}$$

This problem differs from Problem 2.3.1 in that the constraint sets depend only on t and not on (t, x).

The data of the problem are assumed to satisfy the following.

Assumption 3.2.1. (i) The function $\widehat{f} = (f^0, f^1, \ldots, f^n)$, where the f^i are real valued, is defined on a set $\mathcal{R} \times \mathcal{U}$, where $\mathcal{R} = \mathcal{I} \times \mathcal{X}$, $\mathcal{I}$ is a compact interval in $\mathbb{R}^1$, $\mathcal{X}$ is an interval in $\mathbb{R}^n$, and $\mathcal{U}$ is an interval in $\mathbb{R}^m$.

(ii) The function $\widehat{f}$ is continuous on $\mathcal{X} \times \mathcal{U}$ for a.e. t in $\mathcal{I}$ and measurable on $\mathcal{I}$ for each (x, z) in $\mathcal{X} \times \mathcal{U}$.

(iii) For each $t \in \mathcal{I}$, the set $\Omega(t)$ is compact and contained in $\mathcal{U}$.

(iv) There exists at least one measurable function u defined on $\mathcal{I}$ such that the state equation with this u has a solution φ defined on $\mathcal{I}$.

We now recall some definitions from measure theory.

(i) A *probability measure* μ on a compact set K is a positive measure on the Borel sets of K such that $\mu(K) = 1$.

(ii) A positive measure μ defined on the σ-algebra of all Borel sets in a locally compact Hausdorff space X is said to be *regular* if for every Borel set E

$$\begin{aligned}\mu(E) &= \sup\{\mu(K) \colon K \subseteq E,\ K \text{ compact}\} \\ &= \inf\{\mu(O) \colon E \subseteq O,\ O \text{ open}\}.\end{aligned}$$

(iii) If the measure μ only satisfies the first equality, then μ is said to be inner regular. Such measures are also called *Radon measures.*

We shall consider vector valued measures μ of the form $\mu = (\mu^1, \ldots, \mu^n)$, where each of the μ^i is a real valued measure. *We shall say that μ is non-negative, or a probability measure, or is regular, if each of the μ^i has that property.*

Definition 3.2.2. A *relaxed control* on $\mathcal{I}$ is a function

$$\mu \colon t \to \mu_t \qquad \text{a.e.}$$

where μ_t is a regular probability measure on $\Omega(t)$ such that for every function g defined on $\mathcal{I} \times \mathcal{U}$ with range in $\mathbb{R}^n$ that is continuous on $\mathcal{U}$ for a.e. t in $\mathcal{I}$ and measurable on $\mathcal{I}$ for each z in $\mathcal{U}$, the function h defined by

$$h(t) = \int_{\Omega(t)} g(t, z) d\mu_t$$

is Lebesgue measurable.

Remark 3.2.3. Since $\Omega(t)$ is a compact set in a euclidean space, if μ_t is a probability measure on $\Omega(t)$, then μ_t is regular [82, 2.18]. We keep the redundant word "regular" in Definition 3.2.2 to make the definition of relaxed control applicable in situations more general than the one considered here.

Remark 3.2.4. The set of relaxed controls properly contains the set of ordinary controls. To see this, let u be a control defined on $[t_0, t_1]$ with $u(t) \in \Omega(t)$ a.e. Let $\delta_{u(t)}$ be the Dirac measure on $\Omega(t)$ that is equal to one at the point $u(t)$ and equal to zero on any set that does not contain $u(t)$. Then $\delta_{u(t)}$ is a probability measure and for any g as in Definition 3.2.2 with $\mathcal{I} = [t_0, t_1]$, the function h defined by

$$h(t) = g(t, u(t)) = \int_{\Omega(t)} g(t, z) d\delta_{u(t)}$$

is measurable. Thus, the mapping $t \to \delta_{u(t)}$ is a relaxed control. We can therefore consider an ordinary control to be a special type of relaxed control.

We now exhibit relaxed controls that are not ordinary controls. Let $p^1, \ldots, p^k$ be nonnegative measurable functions defined on $\mathcal{I}$ whose sum is one and let $u_1, \ldots, u_k$ be measurable functions defined on $\mathcal{I}$ such that $u_i(t) \in \Omega(t)$. For any Borel set E in $\Omega(t)$ let

$$\mu_t(E) = \sum_{i=1}^{k} p^i(t) \delta_{u_i(t)}(E). \tag{3.2.2}$$

Then μ_t is a probability measure and $\mu \colon t \to \mu_t$ is a relaxed control since

$$h(t) = \int_{\Omega(t)} g(t, z) d\mu_t = \sum_{i=1}^{k} p^i(t) g(t, u_i(t))$$

is Lebesgue measurable.

Let μ be a relaxed control. Then for each x in $\mathcal{X}$ the function F defined by

$$F(t, x) = \int_{\Omega(t)} f(t, x, z) d\mu_t$$

is a measurable function of t and for fixed t is a continuous function of x. Thus, F defines a direction field on $\mathcal{I} \times \mathcal{X}$ and we can consider the differential equation

$$x' = F(t, x).$$

We shall call solutions of this differential equation *relaxed trajectories.*

Definition 3.2.5.

$$\psi'(t) = \int_{\Omega(t)} f(t, \psi(t), z) d\mu_t$$

for a.e. t. We call (ψ, μ) a *relaxed control-trajectory* pair.

Since ordinary controls are also relaxed controls, ordinary trajectories are also relaxed trajectories.

Definition 3.2.6. A relaxed trajectory is said to be *admissible* if (i) $(t_0, \psi(t_0), t_1, \psi(t_1)) \in \mathcal{B}$ and (ii) the function

$$t \to \int_{\Omega(t)} f^{\circ}(t, \psi(t), z) d\mu_t$$

is integrable. The pair (ψ, μ) is said to be a *relaxed admissible* pair.

We now state the relaxed problem corresponding to Problem (3.2.1).

Problem 3.2.1. Find a relaxed admissible pair (ψ^*, μ^*) that minimizes

$$J(\psi, \mu) = g(t_0, \psi(t_0), t_1, \psi(t_1)) + \int_{t_0}^{t_1} \int_{\Omega(t)} f^{\circ}(t, \psi(t), z) d\mu_t dt$$

over some subset $\mathcal{A}_1$ of the set $\mathcal{A}$ of relaxed admissible pairs. That is, find a relaxed admissible pair (ψ^*, μ^*) in $\mathcal{A}_1$ such that $J(\psi^*, \mu^*) \leq J(\psi, \mu)$ for all admissible pairs (ψ, μ) in $\mathcal{A}_1$.

Definition 3.2.7. The pair (ψ^*, μ^*) is called a *relaxed optimal pair.* The function ψ^* is a *relaxed optimal trajectory* and the control μ^* is a *relaxed optimal control.*

The next lemma relates the direction sets of the ordinary problem and the direction sets of the relaxed problem. Readers unfamiliar with the facts about convex sets that we use are referred to [32]. We will, however, state a theorem due to Carathéodory concerning the representation of convey hulls. For a proof see [32]. We denote the convex hull of a set A by co(A).

Theorem 3.2.8 (Theorem (Carathéodory))**.** *Let A be a set in $\mathbb{R}^n$ and let x be a point in $co(A)$. Then there exist $(n+1)$ points $x_1, \ldots, x_{n+1}$ in A and $(n+1)$ nonnegative real numbers $p^1, \ldots, p^{n+1}$ such that $\sum p^i = 1$ and*

$$x = \sum_{i=1}^{n+1} p^i x_i.$$

Lemma 3.2.9. *For each (t, x) in $\mathcal{R}$ let*

$$V(t, x) = \{y \colon y = f(t, x, z),\ z \in \Omega(t)\}$$
$$V_r(t, x) = \{y \colon y = \int_{\Omega(t)} f(t, x, z) d\mu_t,\ \mu \textit{ a relaxed control}\}.$$

Then $V_r(t, x) = coV(t, x)$, where co denotes convex hull.

Proof. Since a convex combination of probability measures is again a probability measure, the sets $V_r(t, x)$ are convex. The sets $V_r(t, x)$ contain the sets $V(t, x)$ because the Dirac measure δ_z concentrated at z is a probability measure. Hence

$$co\ V(t, x) \subseteq V_r(t, x). \tag{3.2.3}$$

$\square$

We now show that equality holds in (3.2.3). Since $\Omega(t)$ is compact and f is continuous in z, each $V(t,x)$ is compact. Therefore, so is $co\, V(t,x)$. If equality did not occur in (3.2.3) there would exist a point

$$w = \int_{\Omega(t)} f(t,x,z)d\mu_t$$

in $V_r(t,x)$ that is not in the compact, convex set $co\, V(t,x)$. Hence there would exist a hyperplane $\langle a, x\rangle = \alpha$ in $\mathbb{R}^n$ such that

$$\langle a, w\rangle > \alpha \text{ and } \langle a, y\rangle < \alpha \text{ for all } y \in co\; V(t,x).$$

In particular,

$$\langle a, f(t,x,z)\rangle < \alpha \text{ for all } z \in \Omega(t).$$

But then, since μ_t is a probability measure,

$$\alpha < \langle a, w\rangle = \int_{\Omega(t)} \langle a, f(t,x,z)\rangle d\mu_t < \int_{\Omega(t)} \alpha d\mu_t = \alpha.$$

This contradiction shows that equality holds in (3.2.3).

We now develop an equivalent formulation of the relaxed problem. Let ψ be a relaxed trajectory on an interval $I = [t_0, t_1]$. Then $\psi'(t) \in V_r(t, \psi(t))$ for a.e t in I, and by Lemma 3.2.9, $\psi'(t) \in co\, V(t, \psi(t))$. Hence by Caratheodory's theorem there exist $(n+1)$ points $z_1(t), \ldots, z_{n+1}(t)$ in $\Omega(t)$ and $(n+1)$ non-negative numbers $\pi^1(t), \ldots, \pi^{n+1}(t)$ whose sum is one such that

$$\psi'(t) = \sum_{i=1}^{n+1} \pi^i(t) f(t, \psi(t), z_i(t)) \tag{3.2.4}$$

for a.e. t in I. The functions $\pi = (\pi^1, \ldots, \pi^{n+1})$ and $z = (z_1, \ldots, z_{n+1})$ are defined pointwise. We assert that there exist nonnegative measurable functions $p^1, \ldots, p^{n+1}$ with sum one and measurable functions $u_1, \ldots, u_{n+1}$ with $u_i(t) \in \Omega(t)$ for a.e. t in I, such that

$$\psi'(t) = \sum_{i=1}^{n+1} p^i(t) f(t, \psi(t), u_i(t)) \tag{3.2.5}$$

for a.e. t in I.

This assertion will follow from Lemma 3.2.10, whose proof will be given in Section 3.4.

Lemma 3.2.10. *Let I denote a real compact interval, U, an interval in $\mathbb{R}^k$ and let h be a map from $I \times U$ into $\mathbb{R}^n$ that is continuous on U for a.e. t in I and is measurable on I for each z in U. Let W be a measurable function from I to $\mathbb{R}^n$ and let $\widetilde{V}$ be a function defined on I with range in U such that*

$$W(t) = h(t, \widetilde{V}(t)) \qquad a.e.$$

Then there exists a measurable function V defined on I with range in U such that

$$W(t) = h(t, V(t)) \qquad a.e.\ on\ I.$$

Moreover, the values $V(t)$ satisfy the same constraints as the values $\widetilde{V}(t)$.

Let $Z = (\pi, \zeta_1, \ldots, \zeta_{n+1})$, where $\pi \in \mathbb{R}^{n+1}$ and each $\zeta_i \in \mathbb{R}^m$, let $W(t) = \psi'(t)$, and let

$$h(t, Z) = \sum_{i=1}^{n+1} \pi^i f(t, \psi(t), \zeta_i).$$

The assertion that there exist measurable functions $p^1, \ldots, p^{n+1}$ and $u_1, \ldots, u_{n+1}$ as in (3.2.5) follows from Lemma 3.2.10 and Eq. (3.2.4).

In summary, we have proved the following theorem.

Theorem 3.2.11. *Every relaxed trajectory ψ is a solution of a differential equation*

$$x' = \sum_{i=1}^{n+1} p^i(t) f(t, x, u_i(t)), \tag{3.2.6}$$

where the real valued measurable functions $p^1, \ldots, p^{n+1}$ are nonnegative and have sum equal to one a.e. and where the functions $u_1, \ldots, u_{n+1}$ are measurable and satisfy $u_i(t) \in \Omega(t)$, a.e., $i = 1, \ldots, n+1$.

Remark 3.2.12. For each t let μ_t be the measure on $\Omega(t)$ defined by (3.2.2). Then we may write (3.2.6) as

$$x' = \int_{\Omega(t)} f(t, x, z) d\mu_t.$$

Since μ_t is a probability measure we see that every solution of (3.2.6) is a relaxed trajectory. Thus, in the case of compact constraints $\Omega(t)$, we could have defined a relaxed trajectory more simply, perhaps, to be a solution of (3.2.6). Thus, in Definition 3.2.2 we could have restricted ourselves to probability measures that are convex combinations of Dirac measures. We did not do so because the larger class of measures in Definition 3.2.2 has a useful compactness property that we shall develop in the next section.

3.3 Weak Compactness of Relaxed Controls

We review some concepts and theorems from functional analysis that we will need. For full discussion and proofs see [82] and [89]. Let I denote a compact interval in $\mathbb{R}^1$ and let Z denote a compact set in $\mathbb{R}^k$. Let $C(I \times Z)$ denote the space of $\mathbb{R}^n$ valued continuous functions on $I \times Z$ with sup norm,

and let $C^*(I \times Z)$ denote the space of continuous linear transformations L from $C(I \times Z)$ to $\mathbb{R}^n$. The space $C^*(I \times Z)$ is a Banach space, with the norm of an element L given by

$$\|L\| = \sup\{|L(g)|\colon \|g\| \leq 1\},$$

where $|\ |$ denotes the euclidean norm in $\mathbb{R}^n$ and $\|g\| = \max\{|g(t,z)|\colon (t,z)$ in $I \times Z\}$. Note that $L = (L^1, \ldots, L^n)$, where each L^i is a continuous linear functional on $I \times Z$.

A sequence $\{L_n\}$ of continuous linear transformations in $C^*(I \times Z)$ is said to *converge weak-star* (written weak-$*$) to an element L in $C^*(I \times Z)$ if for every g in $C(I \times Z)$

$$\lim_{n \to \infty} L_n(g) = L(g).$$

A set Λ in $C^*(I \times Z)$ is said to be *weak-$*$ sequentially compact* if for each sequence $\{L_n\}$ of elements in Λ, there exists an element L in Λ and a subsequence $\{L_{n_k}\}$ such that L_{n_k} converges weak-$*$ to L.

An important set of functions that is weak-$*$ sequentially compact is the following.

A closed ball in $C^*(I \times Z)$ is weak-$*$ sequentially compact.

The Riesz Representation Theorem and its extensions [82] state:

Every continuous linear functional L in $C^*(I \times Z)$ is represented uniquely by a regular Borel measure ν on $I \times Z$ in the sense that

$$L(g) = \int_{I \times Z} g(t,z) d\nu$$

and

$$\|L\| = \|\nu\|_{var} \equiv |\nu|(I \times Z),$$

where $|\nu|$ denotes the total variation measure corresponding to ν. Moreover, if L is positive, then so is ν, and $\|L\| = \nu(I \times Z)$.

We now return to relaxed controls. Let $\Omega(t) = Z$ for each t in I and let μ be a relaxed control on I. From the definition of relaxed control, we have that for g in $C(I \times Z)$, the function h defined by

$$h(t) = \int_Z g(t,z) d\mu_t$$

is measurable. Also,

$$|h(t)| = \left| \int_Z g(t,z) d\mu_t \right| \leq \|g\|, \qquad a.e.$$

Therefore, the formula

$$L_\mu(g) = \int_I \int_Z g(t,z) d\mu_t dt \tag{3.3.1}$$

defines a continuous linear transformation with

$$|L_\mu(g)| \le \|g\||I|,$$

where $|I|$ denotes the length of I. If we take g to be the function identically one on $I \times Z$ we get that

$$\|L_\mu\| = |I|. \tag{3.3.2}$$

Henceforth, to simplify notation we take I to be a generic compact interval with *the origin as left-hand end point.*

We shall also use a theorem from real analysis, which was proved by Urysohn in a more general context than the one we need. See [82].

Lemma 3.3.1 (Urysohn's Lemma)**.** *Let K be a compact set in $\mathbb{R}^k$, let V be an open set in $\mathbb{R}^k$, and let $K \subset V$. Then there exists a continuous nonnegative function f with support contained in V, with $0 \le f(x) \le 1$ for all x and with $f(x) = 1$ for x in K.*

Definition 3.3.2. A sequence $\{\mu_n\}$ of relaxed controls on I is said to *converge weakly* to a relaxed control μ if for each τ in I and each g in $C(I \times Z)$

$$\lim_{n\to\infty} \int_0^\tau \int_Z g(t,z)d\mu_{nt}dt = \int_0^\tau \int_Z g(t,z)d\mu_t dt. \tag{3.3.3}$$

Remark 3.3.3. Let τ be a point in I, let $I_\tau = [0,\tau]$, and take I in (3.3.1) to be I_τ. Then each μ_n defines a continuous linear transformation L_n^τ in $C^*(I_\tau \times Z)$, as does μ. Thus, Definition 3.3.2 states that for each τ in I the sequence L_n^τ converges weak-$*$ to L_μ^τ, the continuous linear transformation defined by (3.3.1). Thus, the weak convergence of μ_n to μ is really a weak-$*$ convergence for each point τ in I. *We are abusing the terminology and calling the convergence weak convergence.* This weak convergence is distinct from the weak convergence of a sequence of elements in a Banach space. The following example illustrates Definition 3.3.2 and Remark 3.3.3.

Example 3.3.4. Let the dimension of the state space be one, let $I = [0,1]$ and let $Z = [-1,1]$. For each positive integer n subdivide I into $2n$ contiguous subintervals, each of length $1/2n$. Let $u_n(t)$ be the function such that $u_n(t) = 1$ for t in $[0,1/2n]$, $u_n(t) = -1$ for t in $[1/2n, 2/2n]$, and $u_n(t)$ then alternates in value between $+1$ and -1 in successive subintervals. Let $\mu_{nt} = \delta_{u_n(t)}$. We assert that $\delta_{u_n(t)}$ converges weakly to $(1/2)\delta_1 + (1/2)\delta_{-1}$, where δ_1 is the probability measure concentrated at $z = 1$ and δ_{-1} is the probability measure concentrated at $z = -1$. In other words, we assert that for every function g in $C[I \times Z]$ and τ in $[0,1]$,

$$\lim_{n\to\infty} \int_0^\tau \int_Z g(t,z)d\mu_{nt}dt = \frac{1}{2}\int_0^\tau g(t,1)dt + \frac{1}{2}\int_0^\tau g(t,-1)dt.$$

Since the set of polynomials in t and z is dense in $C[I \times Z]$ to prove the assertion

it suffices to show that for any τ in $[0,1]$ and any nonnegative integers p and q

$$\lim_{n\to\infty}\int_0^\tau\int_Z t^p z^q d\mu_{nt}dt = \frac{1}{2}\int_0^\tau t^p dt + \frac{1}{2}\int_0^\tau t^p(-1)^q dt$$

If q is even or zero,

$$\int_0^\tau\int_Z t^p z^q d\mu_{nt}dt = \int_0^\tau t^p dt = \tau^{p+1}/(p+1) = \frac{1}{2}\int_0^\tau t^p dt + \frac{1}{2}\int_0^\tau t^p(-1)^q dt.$$

This proves the assertion for q even or zero.

If q is odd and $p=0$, let ℓ be the largest positive integer such that $2\ell/2n \le \tau$. Thus, τ lies in $[2\ell/n,(2\ell+1)/2n)$ or $[(2\ell+1)/2n,(2\ell+2)/2n]$. Then

$$|\int_0^\tau\int_Z z^q d\mu_{nt}dt| = |\int_{2\ell/2n}^\tau\int_Z z^q d\mu_{nt}dt| \le \int_{2\ell/2n}^\tau dt \le \frac{2}{2n}\to 0.$$

If q is odd and $p\ge 1$, let ℓ be as before. Then

$$\int_0^\tau t^p\int_Z z^q d\mu_{nt}dt = \int_0^{2\ell/2n} t^p\int_Z z^q d\mu_{nt}dt + \int_{2\ell/2n}^\tau t^p\int_Z z^q d\mu_{nt}dt.$$

The second integral in absolute value is less than or equal to $2/2n$, and so tends to zero as $n\to\infty$.

We now estimate the first integral.

$$\begin{aligned} I_n \equiv \int_0^{2\ell/2n} t^p\int_Z z^q d\mu_{nt}dt &= \sum_{i=0}^{2\ell-1}(-1)^i\int_{i/2n}^{(i+1)/2n} t^p dt \\ &= \frac{1}{(2n)^{p+1}(p+1)}\sum_{i=0}^{2\ell-1}(-1)^i[(i+1)^{p+1}-i^{p+1}]. \end{aligned}$$

If in the rightmost sum we group the first two terms, then the next two terms, and so on we can rewrite this sum as

$$\frac{1}{(2n)^{p+1}(p+1)}\sum_{j=0}^{\ell-1}\Big\{(-1)^{2j+1}[(2j+2)^{p+1}-(2j+1)^{p+1}] \\ + (-1)^{2j}[(2j+1)^{p+1}-(2j)^{p+1}]\Big\}.$$

This in turn can be written as

$$\frac{1}{(2n)^{p+1}(p+1)}\sum_{j=0}^{\ell-1}[-(2j+2)^{p+1}+2(2j+1)^{p+1}-(2j)^{p+1}].$$

Using the binomial theorem, we find that each term in square brackets is $O(j^{p-1})$. Using the integral

$$\int_0^\ell (x+1)^{p-1}dx = \frac{(\ell+1)^p-1}{p},$$

and using $\ell \leq n\tau$, we get that

$$I_n = \frac{1}{(2n)^{p+1}} O((2n\tau)^p),$$

and so $I_n \to 0$ as $n \to \infty$. Combining this result with previous results gives

$$\lim_{n\to\infty} \int_0^\tau \int_Z t^p z^q d\mu_{nt} dt = 0 = \frac{1}{2}\int_0^\tau t^p dt + \frac{1}{2}\int_0^\tau t^p (-1)^q dt,$$

which proves the assertion for q even, and hence for all q.

On the other hand, we assert that the sequence $\{u_n\}$ as an element of $L_p[I_\tau]$, $1 < p < \infty$, $0 < \tau \leq 1$, converges weakly to zero. That is, we assert that for any function v in $L_q[I_\tau]$, where $1/p + 1/q = 1$, we have

$$\lim_{n\to\infty} \int_0^\tau v(t) u_n(t) dt = 0.$$

Since the set of polynomials in t is dense in $L_q[I_\tau]$, it suffices to show that for any nonnegative integer j,

$$\lim_{n\to\infty} \int_0^\tau t^j u_n(t) dt = 0.$$

Let ℓ again be the largest positive integer such that $2\ell/2n \leq \tau$. If $j = 0$, we have

$$\int_0^\tau u_n(t)dt = \int_{2\ell/2n}^\tau u_n(t)dt \to 0 \text{ as } n \to \infty.$$

For $j \geq 1$, we integrate by parts to get

$$\int_0^\tau t^j u_n(t) dt = \tau^j \int_0^\tau u_n(t)dt - \int_0^\tau (jt^{j-1} \int_0^t u_n(s)ds)dt \to 0$$

Definition 3.3.5. A sequence $\{\mu_n\}$ of relaxed controls on I is *weakly compact* if there exists a subsequence $\{\mu_{n_k}\}$ and a relaxed control μ on I such that μ_{n_k} converges weakly to μ.

The next theorem is the principal result of this section.

Theorem 3.3.6. *A sequence $\{\mu_n\}$ of relaxed controls on a compact interval I is weakly compact.*

Proof. **Step I.** Preliminary observations.

The proof proceeds by induction on n, the dimension of f. Since the proof of the general induction step and the proof for $n = 1$ are the same, we need only present the proof for $n = 1$.

Let I' be an interval contained in I. The sequence $\{\mu_n\}$ defines a sequence $\{L_n\}$ of continuous linear functionals in $C^*(I' \times Z)$ given by (3.3.1) with $I = I'$. By (3.3.2) the sequence $\{L_n\}$ lies in the closed ball of radius $|I'|$ and

hence is weak-$*$ compact. That is, there is a continuous linear functional L in $C^*(I' \times Z)$ with $\|L\| \leq |I'|$ and a subsequence $\{L_{n_k}\}$ such that for every g in $C(I' \times Z)$

$$\lim_{k\to\infty} L_{n_k}(g) = L(g).$$

Henceforth we shall relabel subsequences $\{L_{n_k}\}$ as $\{L_n\}$. It follows from (3.3.1) that the linear functionals L_n are positive, and therefore so is L. From the Riesz Representation Theorem we get that there exists a positive regular Borel measure ν' on $I' \times Z$ such that for each g in $C(I' \times Z)$

$$L(g) = \int_{I'\times Z} g(t,z)d\nu'.$$

Thus,

$$\lim_{n\to\infty} \int_{I'} \left(\int_Z g(t,z)d\mu_{nt} \right) dt = \int_{I'\times Z} g(t,z)d\nu'.$$

□

Step II. Let $\{\tau_i\}$ be a countably dense set of points in I which includes the origin and the right-hand end point of I, and let $I_i = [0,\tau_i]$. Let $\{\mu_n\}$ be a sequence of relaxed controls on I. Then there exists a subsequence (*independent of* i) that we relabel as $\{\mu_n\}$, and for each i a regular positive Borel measure ν_i on $I_i \times Z$ such that for every g in $C(I_i \times Z)$

$$\lim_{n\to\infty} \int_0^{\tau_i} \left(\int_Z g(t,z)d\mu_{nt} \right) dt = \int_{I_i\times Z} g(t,z)d\nu_i.$$

Proof. From Step I it follows that for τ_1 there exists a subsequence $\{\mu_n\}$ and a positive regular Borel measure ν_1 on $I_1 \times Z$ with the requisite properties. At τ_2 we again apply Step I and obtain a subsequence of the subsequence obtained at τ_1 as well as a positive Borel measure ν_2 with the desired properties. Proceeding inductively in this manner we obtain a sequence of subsequences and a sequence of measures. If we take the diagonal elements in the array of subsequences, we obtain the desired subsequence. □

Step III. For each τ in I there exists a positive regular Borel measure ν_τ on $I_\tau = [0,\tau]$ such that for every g in $C(I_\tau \times Z)$

$$\lim_{n\to\infty} \int_0^{\tau} \left(\int_Z g(t,z)d\mu_{nt} \right) dt = \int_{I_\tau\times Z} g(t,z)d\nu_\tau, \tag{3.3.4}$$

where $\{\mu_n\}$ is the subsequence obtained in Step II.

Proof. For points $\{\tau_i\}$ in the dense set this was established in Step II. Now let τ be an arbitrary point in I not in the set $\{\tau_i\}$. For g in $C(I_\tau \times Z)$ let

$$L_n^\tau(g) = \int_0^{\tau} \left(\int_Z g(t,z)d\mu_{nt} \right) dt \qquad n = 1,2,\ldots$$

We shall show that for fixed g the sequence of real numbers $\{L_n^\tau(g)\}$ is Cauchy. $\square$

For every $\tau' < \tau''$ in I, since μ_n is a probability measure, and since $C(I_{\tau''} \times Z) \subseteq C(I_{\tau'} \times Z)$, we have for $g \in C(I_{\tau''} \times Z)$

$$|L_n^{\tau'}(g) - L_n^{\tau''}(g)| = |\int_{\tau'}^{\tau''} \left(\int_Z g(t,z)d\mu_{n,t}\right) dt| \\ \leq \|g\|(\tau'' - \tau'). \tag{3.3.5}$$

Let $\varepsilon > 0$ be arbitrary. Then there exists a point $\tau_j < \tau$ such that $\tau - \tau_j < \varepsilon$. Therefore, for arbitrary positive integers m, n

$$|L_m^\tau(g) - L_n^\tau(g)| \leq |L_m^\tau(g) - L_m^{\tau_j}(g)| + |L_m^{\tau_j}(g) - L_n^{\tau_j}(g)| + |L_n^{\tau_j}(g) - L_n^\tau(g)|.$$

From (3.3.5) and the fact that $\{L_n^{\tau_j}(g)\}$ is Cauchy we get that for m, n sufficiently large

$$|L_m^\tau(g) - L_n^\tau(g)| \leq 2\varepsilon\|g\| + \varepsilon.$$

Hence $\{L_n^\tau(g)\}$ is Cauchy and so converges to a number $L^\tau(g)$.

Since for fixed τ and each n the mapping $g \to L_n^\tau(g)$ is linear, it follows that $g \to L^\tau(g)$ is linear. Also, since each L_n^τ is positive, so is L^τ. Also,

$$|L^\tau(g)| = |\lim_{n\to\infty} \int_0^\tau \left(\int_Z g(t,z)d\mu_{nt}\right) dt| \\ \leq \overline{\lim}_{n\to\infty}\|g\| \int_0^\tau \left(\int_Z d\mu_{nt}\right) dt = \|g\|\tau.$$

If we take g to be the function identically one, we get that

$$\|L^\tau\| = \tau.$$

Hence L^τ is a continuous linear functional on $C(I_\tau \times Z)$. By the Riesz representation theorem there exists a positive regular Borel measure ν_τ on $[I_\tau \times Z]$ such that

$$L^\tau(g) = \int_{I_\tau \times Z} g(t,z)d\nu_\tau.$$

This establishes (3.3.4).

Step IV. There exists a set $T \subseteq I$ with $|T| = |I|$ such that for each g in $C(I \times Z)$ the mapping $\tau \to L^\tau(g)$ is differentiable at all points of T. Here $|T|$ denotes the Lebesgue measure of T.

Proof. Since (3.3.5) holds for arbitrary g in $C(I \times Z)$, we get that for fixed g the mapping $\tau \to L^\tau(g)$ is Lipschitz, and hence is differentiable almost everywhere. Let $\{g_i\}$ be a countable set of functions that is dense in $C(I \times Z)$. Corresponding to each g_i there is a set $T_i \subset I$ with $|T_i| = I$ such that the mapping $\tau \to L^\tau(g_i)$ is differentiable at all points of T_i. Let $T = \bigcap_{i=1}^\infty T_i$.

Then, $|T| = |I|$ and for each i the mapping $\tau \to L^{\tau}(g_i)$ is differentiable at all points of T.

We now show that for each g in $C(I \times Z)$ the mapping $\tau \to L^{\tau}(g)$ is differentiable at all points of T. Let τ_0 be a point in T and for $\tau \neq \tau_0$ define

$$\Delta_{\tau,\tau_0}(g) = \frac{L^{\tau}(g) - L^{\tau_0}(g)}{\tau - \tau_0}.$$

Then to show that $\tau \to L^{\tau}(g)$ is differentiable at τ_0 it suffices to show that for $\tau_1, \tau_2 \to \tau_0$ that

$$\Delta_{\tau_1,\tau_0}(g) - \Delta_{\tau_2,\tau_0}(g) \to 0. \tag{3.3.6}$$

For g_i as in the preceding paragraph,

$$\begin{aligned}|\Delta_{\tau_1,\tau_0}(g) - \Delta_{\tau_2,\tau_0}(g)| \leq |\Delta_{\tau_1,\tau_0}(g - g_i)| + |\Delta_{\tau_2,\tau_0}(g - g_i)| \\ + |\Delta_{\tau_1,\tau_0}(g_i) - \Delta_{\tau_2,\tau_0}(g_i)|.\end{aligned}$$

From the definition of $\Delta_{\tau,\tau_0}(g)$ and (3.3.5) we get that each of the first two terms on the right do not exceed $\|g - g_i\|$. Hence by appropriate choice of g_i they can be made as small as we wish. Since $\tau \to L^{\tau}(g_i)$ is differentiable at τ_0, the last term tends to zero as $\tau_1, \tau_2 \to \tau_0$, and (3.3.6) is established. □

Step V. For each t in T there exists a positive regular Borel measure μ_t on Z such that for every g in $C(I \times Z)$ the mapping

$$t \to \int_Z g(t, z) d\mu_t \tag{3.3.7}$$

is Lebesgue measurable and for every τ in I

$$L^{\tau}(g) = \int_0^{\tau} \left(\int_Z g(t, z) d\mu_t \right) dt. \tag{3.3.8}$$

Proof. For each t in T we have a functional λ_t on $C(I \times Z)$ defined by

$$\lambda_t(g) = [dL^{\tau}(g)/d\tau]_{\tau=t}.$$

Clearly λ_t is linear. We next show that λ_t is bounded, and hence continuous.

$$\begin{aligned}|\lambda_t(g)| &= \lim_{\tau \to t} \left| \frac{L^{\tau}(g) - L^{t}(g)}{\tau - t} \right| \\ &\leq \overline{\lim_{\tau \to t}} \{\max |g(s, z)| \colon t \leq s \leq \tau,\ z \in Z\} \\ &= \max\{|g(t, z)| \colon z \in Z\} \leq \|g\|,\end{aligned} \tag{3.3.9}$$

where the first inequality follows from (3.3.5). □

We now show that λ_t is positive. Let g in $C(I\times Z)$ be nonnegative. Then for each n, $\tau \to L_n^\tau(g)$ is a nondecreasing function, and therefore so is $\tau \to L^\tau(g)$. Hence

$$\lambda_t(g) = [dL^\tau(g)/d\tau]_{\tau=t} \geq 0.$$

From the Riesz Representation Theorem we get that there is a positive regular Borel measure μ_t on $I\times Z$ such that $\|\lambda_t\| = \|\mu_t\|$ and for g in $C(I\times Z)$

$$\lambda_t(g) = \int_{I\times Z} g(s,z)d\mu_t.$$

We assert that μ_t is concentrated on $\{t\}\times Z$. If the assertion were false, then since μ_t is regular and positive, there would exist a compact set K in $I\times Z$ disjoint from $\{t\}\times Z$ such that $\mu_t(K) > 0$. By Urysohn's Lemma there exists a nonnegative function g_0 in $C(I\times Z)$ that is one on K and zero on $\{t\}\times I$. Hence, $\lambda_t(g_0) > 0$. But from (3.3.9) we have

$$|\lambda_t(g_0)| \leq \max\{|g_0(t,z)| \colon z\in Z\} = 0.$$

Thus, μ_t is concentrated on $\{t\}\times Z$ and we may write

$$\lambda_t(g) = \int_Z g(t,z)d\mu_t \qquad t\in T \tag{3.3.10}$$

and

$$\|\lambda_t\| = \|\mu_t\|_{\text{var}} = \mu_t(Z). \tag{3.3.11}$$

Since for fixed g, the mapping $t\to\lambda_t(g)$ is the derivative of a Lipschitz function, it is measurable. From this and from (3.3.10) the measurability of (3.3.7) follows.

It follows from (3.3.5) that the function $\tau\to L^\tau(g)$ is Lipschitz and hence absolutely continuous. It is therefore the integral of its derivative, and so

$$L^\tau(g) = \int_0^\tau [dL^t(g)/dt]dt = \int_0^\tau \lambda_t(g)dt = \int_0^\tau (\int_Z g(t,z)d\mu_t)dt,$$

which establishes (3.3.8).

The integrals are actually taken over the set $[0,\tau]\cap T$, where T is the same for *all* g.

Step VI. For all t in T, μ_t is a regular probability measure.

Proof. Since μ_t is a positive regular measure, to prove this statement it suffices, in view of (3.3.11), to show that $\|\lambda_t\| = 1$. □

Let t be an arbitrary point in T. Let g_1 be a nonnegative function in $I\times Z$ with range in $[0,1]$ such that $g_1\leq 1$ and $g_1\equiv 1$ on a set $\{(\tau,z)\colon |\tau - t|\leq \varepsilon,\ z\in Z\}$ for ε sufficiently small. Because Z is compact, such a function exists. Hence for τ sufficiently close to t we have by (3.3.8) that

$$L^\tau(g_1) = \int_0^t \left(\int_Z g_1(s,z)d\mu_s\right)ds + \int_t^\tau \left(\int_Z g_1(s,z)d\mu_s\right)ds$$

$$= L^t(g_1) + \lim_{n\to\infty} \int_t^\tau \left(\int_Z g_1(s,z) d\mu_{ns} \right) dt = L^t(g_1) + \lim_{n\to\infty} (\tau - t),$$

since for each n and s $d\mu_{ns}$ is a probability measure. Thus,

$$L^\tau(g_1) = L^t(g_1) + (\tau - t),$$

and so

$$\lambda_t(g_1) = dL^\tau(g_1)/d\tau]_{\tau=t} = 1 = \|g_1\|.$$

But from (3.3.9) we have that $\|\lambda_t(g)\| \leq \|g\|$ for all g in $C(I \times Z)$. Hence $\|\lambda_t\| = 1$.

Step VII. Completion of proof.

The mapping (3.3.7) is Lebesgue measurable and μ_t is a regular probability measure on Z for almost all t in I. Hence

$$\mu \colon t \to \mu_t$$

is a relaxed control. Since for every g in $C(I \times Z)$ and every τ in I

$$\lim_{n\to\infty} L_n^\tau(g) = L^\tau(g),$$

it follows from (3.3.8) that the subsequence $\{\mu_n\}$ converges weakly to μ.

Remark 3.3.7. The condition that Z is compact cannot be replaced by the condition that Z is closed. To see this let Z be an unbounded closed set and let I be a compact interval in $\mathbb{R}$. Let $\{z_i\}$ be a sequence of points in Z whose norms tend to infinity. Let g be a function in $C(I \times Z)$ with compact support and independent of t. Thus, $g(t,z) = g(z)$. Let $\{\mu_n\}$ be a sequence of relaxed controls defined as follows:

$$\mu_n(t) = \delta_{z_n} \qquad t \in I,$$

where δ_{z_n} is the Dirac measure equal to one at the point z_n and equal to zero on any set that does not contain z_n. Then

$$L_n(g) = \int_I (\int_Z g(z) d\mu_n) dt = g(z_n)|I|.$$

For n sufficiently large, $g(z_n) = 0$, and so $L_n(g) \to 0$.

On the other hand, we cannot find a regular probability measure μ such that for all g in $C(Z)$ with compact support

$$\int_Z g(z) d\mu = 0.$$

For if μ is a regular probability measure, there is a compact set $K \subset Z$ such that $\mu(K) > 0$. Let g_K be a nonnegative continuous function with compact support such that $g_K(z) = 1$ for $z \in K$. Then $L_n(g_K) \to 0$ as $n \to \infty$, but

$$\int_Z g_K(z) d\mu \geq \int_K g_K(z) d\mu = \mu(K) > 0.$$

The compactness of Z was used in the proof to show that for $t \in T$, $\|\mu_t\| = 1$; it was not used elsewhere.

In establishing existence theorems and necessary conditions we shall be considering sequences of the form

$$\int_I \int_{\Omega(t)} g(t,z) d\mu_{nt} dt,$$

where for each t the measure $\mu_{n,t}$ is concentrated on a compact set $\Omega(t)$. This is in contrast to the sequences in Theorem 3.3.6, where for each t the measure $\mu_{n,t}$ is concentrated on a fixed compact set Z. We shall impose a regularity condition on the mapping $\Omega \colon t \to \Omega(t)$ which guarantees that all of the sets $\Omega(t)$ lie in some fixed compact set Z. We can then consider the measures μ_{nt} to be measures on Z that are concentrated on $\Omega(t)$. Thus, we can write

$$\int_I \int_{\Omega(t)} g(t,z) d\mu_{nt} dt = \int_I \int_Z g(t,z) d\mu_{nt} dt.$$

Theorem 3.3.6 gives the existence of a subsequence $\{\mu_n\}$ that converges weakly to a relaxed control μ such that for almost every t, the probability measure μ_t is concentrated on Z. We would like to conclude, however, that for almost every t the measure μ_t is concentrated on $\Omega(t)$. We shall prove that this is indeed the case, and begin by introducing some notation and definitions.

Let X be a subset of $\mathbb{R}^p$ and let $|\xi - \eta|$ denote the euclidean distance between points ξ and η in $\mathbb{R}^p$. Let Λ be a mapping from X to subsets of a euclidean space $\mathbb{R}^q$. For ξ_0 in X let $N_\delta(\xi_0)$ denote the delta neighborhood of ξ_0 relative to X; that is,

$$N_\delta(\xi_0) = \{x \colon x \in X,\ |x - \xi_0| < \delta\}.$$

Let $\Lambda(N_\delta(\xi_0))$ denote the image of $N_\delta(\xi_0)$ under Λ. If U is a set in $\mathbb{R}^q$, let $\operatorname{dist}(y, U)$ denote the euclidean distance between a point y in $\mathbb{R}^q$ and U. Thus,

$$\operatorname{dist}(y, U) = \inf\{|y - z| \colon z \in U\}.$$

Let

$$[U]_\varepsilon = \{y \colon \ \operatorname{dist}(y, U) \le \varepsilon\}.$$

We shall call $[U]_\varepsilon$ a *closed ε-neighborhood of U*.

Definition 3.3.8. A mapping Λ is said to be *upper semi-continuous with respect to inclusion at a point* x_0, written u.s.c.i, if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x in $N_\delta(x_0)$ the inclusion $\Lambda(x) \subseteq [\Lambda(x_0)]_\varepsilon$ holds. The mapping Λ is u.s.c.i. on X if it is u.s.c.i. at every point of X.

Remark 3.3.9. In Section 5.2 we shall define the notion of upper semi-continuity of a mapping. Some authors define *upper semi-continuity to be* the notion that we call u.s.c.i. Readers of the literature should check which definition of upper semi-continuity is being used.

Example 3.3.10. Let Λ_1 and Λ_2 be maps from $[0,1]$ to subsets of $\mathbb{R}^1$ defined by

$$\Lambda_1(t) = \begin{cases} z \; |z| \le \frac{1}{t} & 0 < t \le 1 \\ \mathbb{R}^1 & t = 0 \end{cases}$$

$$\Lambda_2(t) = \begin{cases} z \; |z| \le \frac{1}{t} & 0 < t \le 1 \\ 0 & t = 0. \end{cases}$$

Then Λ_1 is u.s.c.i on $[0,1]$, while Λ_2 is u.s.c.i on $(0,1]$ but is not u.s.c.i at $t=0$. Note that Λ_1 is u.s.c.i on I but that $G_{\Lambda_1} = \{(\xi,\lambda)\, \xi \in [0,1],\, \lambda \in \Lambda(\xi)\}$ is not compact. This does not contradict Lemma 3.3.11 because $\Lambda_1(0)$ is not compact and one of the hypotheses of the lemma is that all sets $\Lambda(\xi)$ are compact.

Let Λ be a constant map; that is, a map such that for all ξ in X, $\Lambda(\xi) = K$, K a fixed set. Then Λ is u.s.c.i.

The next lemma shows that if the mapping Ω is u.s.c.i. on I, then all of the sets $\Omega(t)$ lie in a fixed compact set.

Lemma 3.3.11. *Let Λ be a mapping from a compact set X in $\mathbb{R}^p$ to subsets of $\mathbb{R}^q$ such that for each ξ in X, the set $\Lambda(\xi)$ is compact. A necessary and sufficient condition that Λ be u.s.c.i. on X is that the set*

$$G_\Lambda = \{(\xi,\lambda) \colon \xi \in X,\ \lambda \in \Lambda(\xi)\}$$

is compact.

Proof. Suppose that Λ is u.s.c.i. on X. Let $\{(\xi_n,\lambda_n)\}$ be a sequence of points in G_Λ. Since X is compact, there exists a subsequence $\{(\xi_n,\lambda_n)\}$ and a point ξ_0 in X such that $\lim \xi_n = \xi_0$. Let $\varepsilon > 0$ be given. Since $\lambda_n \in \Lambda(\xi_n)$ and Λ is u.s.c.i. at ξ_0, there exists a positive integer N such that for $n > N$, $\lambda_n \in [\Lambda(\xi_0)]_\varepsilon$. Thus, $\{\lambda_n\}$ is bounded and $\text{dist}(\lambda_n, \Lambda(\xi_0)) \to 0$. Hence there exists a subsequence that we again label as $\{(\xi_n,\lambda_n)\}$ and a point λ_0 such that $\lim \xi_n = \xi_0$ and $\lim \lambda_n = \lambda_0$. Letting $n \to \infty$ in the relation

$$\text{dist }(\lambda_0, \Lambda(\xi_0)) \le |\lambda_n - \lambda_0| + \text{ dist}(\lambda_n, \Lambda(\xi_0))$$

and recalling that $\Lambda(\xi_0)$ is closed gives $\lambda_0 \in \Lambda(\xi_0)$. Hence $(\xi_0,\lambda_0) \in G_\Lambda$, and so G_Λ is compact.

Now suppose that G_Λ is compact and that Λ is not u.s.c.i at some point ξ_0 in X. Then there exists an $\varepsilon_0 > 0$ and a sequence $\{(\xi_n,\lambda_n)\}$ in G_Λ such that $\lim \xi_n = \xi_0$ and $\text{dist}(\lambda_n, \Lambda(\xi_0)) > \varepsilon_0$. Since G_Λ is compact there exists a subsequence that we relabel as $\{(\xi_n,\lambda_n)\}$ and a point (ξ_0', λ_0) in G_Λ such that $\lim(\xi_n,\lambda_n) = (\xi_0',\lambda_0)$. Hence $\xi_0' = \xi_0$, and so $\lambda_0 \in \Lambda(\xi_0)$. This contradicts $\text{dist}(\lambda_n, \Lambda(\xi_0)) > \varepsilon_0$, and so Λ is u.s.c.i at ξ_0. $\square$

Theorem 3.3.12. *Let I be a compact interval in $\mathbb{R}^1$. Let $\Omega\colon t \to \Omega(t)$ be a mapping from I to subsets of $\mathbb{R}^k$ that is upper semi-continuous with respect to inclusion on I and such that for each t in I the set $\Omega(t)$ is compact. Let $\{\mu_n\}$ be a sequence of relaxed controls such that for each n the measure μ_{nt} is for almost every t concentrated on $\Omega(t)$. Then there exists a subsequence of the sequence $\{\mu_n\}$ that converges weakly to a relaxed control μ such that for almost all t the measure μ_t is concentrated on $\Omega(t)$.*

Proof. It follows from the upper semi-continuity with respect to inclusion of Ω and Lemma 3.3.11 that there exists a compact set Z such that all the sets $\Omega(t)$ are contained in Z. It then follows from Theorem 3.3.6 that there exists a subsequence of $\{\mu_n\}$ that converges weakly to a relaxed control $\mu\colon t \to \mu_t$, where for all t in a set T of full measure in I the measure μ_t is concentrated on $\{t\} \times Z$. We shall show that for all t in T the measure μ_t is concentrated on $\{t\} \times \Omega(t)$. □

Suppose that at a point t in T, the set $\Omega(t)$ is a proper subset of Z and that μ_t is not concentrated on $\{t\} \times \Omega(t)$. Then there exists a compact set K in $\{t\} \times Z$ such that K is disjoint from $\Omega(t)$ and $\mu_t(K) > 0$. Let $\varepsilon_0 > 0$ denote the distance between the compact sets K and $\Omega(t)$. Since Ω is u.s.c.i, there exists a $\delta > 0$ such that for $|\tau - t| \le \delta$, the sets $\Omega(\tau)$ are contained in $[\Omega(t)]_{\varepsilon_0/2}$. Moreover, the set

$$\Gamma_\delta = \{(\tau, z)\colon |\tau - t| \le \delta,\ z \in \Omega(\tau)\}$$

is compact by virtue of Lemma 3.3.11. Thus, the compact sets K and Γ_δ are disjoint. By Urysohn's Lemma there exists a nonnegative continuous function g_0 on $I \times Z$ that takes on the value one on K and the value zero on Γ_δ.

Let $\lambda_t(g_0)$ be as in Step V in the proof of Theorem 3.3.6. Then from (3.3.10) we get

$$\lambda_t(g_0) = \int_Z g_0(t, z) d\mu_t = \mu_t(K) > 0. \tag{3.3.12}$$

On the other hand, from the definition

$$\lambda_t(g) = [dL^\tau(g)/d\tau|_{\tau=t}$$

in Step V of the proof of Theorem 3.3.6 we get

$$\begin{aligned} \lambda_t(g_0) &= \lim_{\tau\to t} \frac{L^\tau(g_0) - L^t(g_0)}{\tau - t} \\ &= \lim_{\tau\to t} \frac{1}{\tau - t} \left[\lim_{n\to\infty} \int_t^\tau \left(\int_Z g_0(s, z) d\mu_{ns} \right) dt \right], \end{aligned} \tag{3.3.13}$$

where the notation is as in the proof of Theorem 3.3.6. Since μ_{ns} is concentrated on $\Omega(s)$, we have

$$\int_t^\tau \int_Z g_0(s, z) d\mu_{ns} = \int_t^\tau \int_{\Omega(s)} g_0(s, z) d\mu_{ns}. \tag{3.3.14}$$

For $|t-\tau| \leq \delta$, the integrand on the right in (3.3.14) has domain Γ_δ, and so is zero. From (3.3.13) we then get that $\lambda_t(g_0) = 0$, which contradicts (3.3.12). Thus, μ_t is concentrated on $\Omega(t)$ for all t in T.

Remark 3.3.13. It follows from (3.3.1) and Definition 3.3.2 that if $\{\mu_n\}$ is a sequence of relaxed controls that converges weakly to a relaxed control μ, and if $\{L_n\}$ and L are the corresponding continuous linear transformations, then L_n converges weak-$*$ to L. In proving Theorems 3.3.6 and 3.3.12 we have shown that the converse is also true. Let $\{L_n\}$ be a sequence of continuous linear transformations on $C(I \times Z)$ corresponding to a sequence of relaxed controls μ_n such that for each n and a.e. t in I, the measure μ_{nt} is concentrated on $\Omega(t)$, and let L_n converge weak-$*$ to a continuous linear transformation L from $C(I \times Z)$ to $\mathbb{R}^n$. Then there exists a relaxed control μ such that μ_t is concentrated on $\Omega(t)$ for a.e. t in I and L is given by (3.3.1). Moreover, μ_n converges weakly to μ.

3.4 Filippov's Lemma

In this section we shall prove a general implict function theorem for measurable functions. Originally, this theorem was given in a less general form by A. F. Filippov [33]. The form given here is due to McShane and Warfield [64]. This theorem has several important applications in optimal control theory. One was already given in proving Theorem 3.2.11, where we employed a corollary of the theorem.

Theorem 3.4.1. *Let T be a measure space, let Z be a Hausdorff space, and let D be a topological space that is the countable union of compact metric spaces. Let Γ be a measurable map from T to Z and let φ be a continuous map from D to Z such that $\Gamma(T) \subseteq \varphi(D)$. Then there exists a measurable map m from T to D such that the composite map $\varphi\, *\, m$ from T to Z is equal to Γ.*

Remark 3.4.2. In our applications, $T = I$, where I is a real interval with Lebesgue measure, $Z = \mathbb{R}^p$ and $D = \mathbb{R}^q$, real euclidean spaces. Recall that a mapping Γ is measurable if for every compact set K in Z, the set $\Gamma^{-1}(K)$ in T is measurable.

Corollary 3.4.3 (Lemma 3.2.10)**.** *Let I denote a real compact interval, U an interval in $\mathbb{R}^k$, and let h be a map from $I \times U$ into $\mathbb{R}^n$ that is continuous on U for a.e. t in I and is measurable on I for each z in U. Let W be a measurable function on I with range in $\mathbb{R}^n$, and let $\widetilde{V}$ be a function from I to U such that*

$$W(t) = h(t, \widetilde{V}(t)) \qquad a.e. \tag{3.4.1}$$

Then there exists a measurable function $V \colon I \to U$ such that

$$W(t) = h(t, V(t)) \quad a.e. \tag{3.4.2}$$

Proof of Theorem. Let $\{C_i\}_{i=1}^{\infty}$ denote the compact metric spaces whose union equals D. By a theorem of Urysohn, every compact metric space is the continuous image of a Cantor set. (For a proof of this theorem see [45]). Let L_i, $i = 1, 2, 3, \ldots$ be the translate of the Cantor set in $[0, 1]$ to the interval $[2i - 1, 2i]$. Let ψ_i be the continuous map of L_i onto the compact metric space C_i. Let $L = \bigcup_{i=1}^{\infty} L_i$. Since the complement of L is open, L is closed. Define a mapping ψ from L to D as follows. If $t \in L_i$, then $\psi(t) = \psi_i(t)$. The mapping ψ is clearly continuous. Let $\theta = \varphi * \psi$. Then θ is a continuous mapping from L onto $\varphi(D)$ in Z.

Let $t \in T$. Since $\Gamma(T) \subseteq \varphi(D) = \theta(L)$, it follows that $\theta^{-1}(\Gamma(t))$ is a nonempty set in L. Since θ is continuous and $\Gamma(t)$ is a point, the set $\theta^{-1}(\Gamma(t))$ is closed and is bounded below. Hence $\inf\{\theta^{-1}(\Gamma(t))\}$ is finite and is in the set $\theta^{-1}(\Gamma(t))$, which is contained in L. Define a mapping γ from T to L as follows:

$$\gamma(t) = \inf\{\theta^{-1}(\Gamma(t))\}.$$

Hence

$$\theta(\gamma(t)) = \Gamma(t). \tag{3.4.3}$$

From the definition of L, it follows that $\gamma(t) \geq 1$. From (3.4.3) and the definition of θ, we get that

$$(\varphi * \psi * \gamma)(t) = \Gamma(t).$$

Let $m = (\psi * \gamma)$. Then m is a mapping from T to D, and $(\varphi * m)(t) = \Gamma(t)$ for t in T. To complete the proof we must show that m is measurable. Since ψ is continuous, it suffices to show that γ is measurable.

To show that γ is measurable, we need to show that for every real number c, the set

$$T_c = \{t \colon \gamma(t) \leq c\}$$

is measurable. If $c < 1$, then T_c is empty, so we need only consider $c \geq 1$. If $t \in T_c$, then $\theta(\gamma(t)) \subseteq \theta(L \cap [0, c])$. Thus, by (3.4.3), $\Gamma(t) \in \theta(L \cap [0, c])$. Now suppose that $\Gamma(t) \in \theta(L \cap [0, c])$. Then by (3.4.3), $\theta(\gamma(t)) \in \theta(L \cap [0, c])$. Hence $\gamma(t) \leq c$. Thus,

$$T_c = \{t \colon \Gamma(t) \in \theta(L \cap [0, c])\}$$

or

$$T_c = \Gamma^{-1}(\theta(L \cap [0, c])).$$

Now, $L \cap [0, c]$ is compact and therefore so is $\theta(L \cap [0, c])$. Since Γ is measurable, $\Gamma^{-1}(K)$ is measurable for any compact set. Hence, T_c is measurable for every $c \geq 1$. $\square$

Proof of Corollary. We first prove the corollary under the assumption that h is continuous on $\mathcal{I} \times \mathcal{U}$. In reading the proof, the reader should note that the corollary will hold if we replace $\mathcal{I}$ by a measurable set T and assume that h is continuous on $T \times U$.

Let $T = I$, let $Z = I \times \mathbb{R}^n$, and let $D = I \times \mathbb{R}^k$. Let $(\tau, \widetilde{V})$, where $\tau \in I$

and $\widetilde{V} \in \mathbb{R}^k$, denote a generic point in D. Define a mapping $\varphi\colon D \to Z$ by the formula

$$\varphi(\tau, \widetilde{V}) = (\tau, h(\tau, \widetilde{V})).$$

Since h is continuous, so is φ. Let $\Gamma\colon I \to Z$ be defined by

$$\Gamma(t) = (t, W(t)).$$

Thus, Γ is measurable. Equation (3.4.1) implies that $\Gamma(I) \subseteq \varphi(D)$. Thus, all the hypotheses of the theorem are fulfilled. Hence there exists a measurable map m from I to $D = I \times \mathbb{R}^k$ with

$$m(t) = (\tau(t), V(t))$$

such that $(\varphi * m)(t) = \Gamma(t)$. Thus, for all t in I

$$(\varphi * m)(t) = \varphi(\tau(t), V(t)) = (\tau(t), h(\tau(t), V(t))) = (t, W(t)).$$

Since m is measurable, so is V. From the rightmost equality we first get that $\tau(t) = t$ and then (3.4.2). $\square$

The completion of the proof of the corollary (Lemma 3.2.10) utilizes the following result, whose proof we give at the end of this section. For many control questions this result is used to validate one of Littlewood's principles of analysis, which states that generally what is true for continuous functions is true for measurable functions.

Lemma 3.4.4. *Let T be a compact interval in $\mathbb{R}^s$ and let U be an interval in $\mathbb{R}^n$. Let h be a function from $T \times U$ to $\mathbb{R}^n$ such that for almost all t in T, $h(t, \cdot)$ is continuous in U and for all z in U, $h(\cdot, z)$ is measurable on T. Then for each $\rho > 0$ there exists a closed set F with meas $(T - F) < \rho$ such that h is continuous on $F \times U$. If U is closed, then there exists a continuous function H from $T \times U$ to $\mathbb{R}^n$ such that $H(t, z) = h(t, z)$ for all t in F and all z in U.*

By Lemma 3.4.4, for each positive integer j, there exists a closed set $F_j \subseteq I$ such that meas $(I - F_j) < 2^{-j}$ and such that h is continuous on $F_j \times U$. By what was just proved there exists a function V_j defined and measurable on F_j, with range in U such that

$$W(t) = h(t, V_j(t)) \quad \text{a.e. in } F_j. \tag{3.4.4}$$

Let E_j denote the set of measure zero on which (3.4.4) fails. Let E denote the union of the sets $E_j, j = 1, 2, \ldots$. Then meas $E = 0$.

We now define a sequence of closed sets F'_j and measurable functions V'_j inductively. Define

$$F'_1 = F_1 \quad V'_1(t) = V_1(t) \quad t \notin E$$

and $V'_1(t) =$ any point in U if $t \in E$. Then meas $(I - F'_1) < 2^{-j}$ and $W(t) =$

$h(t, V_1'(t))$ for $t \in F_1'$, $t \notin E$. Suppose now that for each $j = 1, \dots, k$ there is defined a closed set F_j', and a measurable function V_j' defined on F_j' with range in U such that

$$\begin{aligned} \text{meas } (I - F_j') < 2^{-j} \quad & F_j' \supset F_{j-1}' \\ W(t) = h(t, V_j'(t)) \quad & t \in F_j' \quad t \notin E. \\ V_j'(t) = V_{j-1}'(t) \quad & t \in F_{j-1}' \quad t \notin E. \end{aligned} \tag{3.4.5}$$

Define $F_{k+1}' = F_k' \cup F_{k+1}$. Since meas$(I - F_k') < 2^{-k}$ and meas $(I - F_{k+1}) < 2^{-k+1}$ neither of the sets F_k' or F_{k+1} is contained in the other. Thus, $F_{k+1}' \supset F_k'$. For any sets A and B, $c(A \cup B) = (cA) \cap (cB)$, where cA denotes the complement of A. It therefore follows that meas $(I - F_{k+1}') < 2^{-k+1}$. Define, for $t \notin E$,

$$V_{k+1}'(t) = \begin{cases} V_k'(t) & \text{if } t \in F_k' \\ V_{k+1}(t) & \text{if } t \in F_{k+1} - F_k'. \end{cases}$$

Then V_{k+1}' is measurable on I, $V_{k+1}'(t) = V_k'(t)$, and $W(t) = h(t, V_{k+1}(t))$ for $t \in (F_{k+1}' - E)$. We have thus defined a sequence of measurable sets $\{F_j'\}$ and measurable functions $\{V_j'\}$ such that V_j' is defined on F_j' and (3.4.5) holds. We extend the definition of each V_j' to I by setting $V_j'(t)$ equal to an arbitrary element of U if $t \notin F_j'$.

Let

$$G = \bigcup_{j=1}^{\infty} F_j'.$$

Then $G \subseteq I$ and meas $(I - G) = 0$. For each $t \in G$, there exists a positive integer $j_0(t)$ such that for $j \geq j_0(t)$, $V_j'(t) = V_{j_0}'(t)$. Hence there exists a measurable function V defined in G such that for $t \in G$, $V(t) = \lim_{j \to \infty} V_j'(t)$. If for $t \in I - G$ we define $V(t) = z$, where z is an arbitrary element of U, we have that

$$V(t) = \lim_{j \to \infty} V_j'(t) \quad \text{a.e. in } I,$$

V is measurable and

$$W(t) = h(t, V(t)) \quad \text{a.e.}$$

We conclude this section with a proof of Lemma 3.4.4.

Proof of Lemma 3.4.4. It will be convenient for us to define the norm of a vector $z = (z^1, \dots, z^n)$ in $\mathbb{R}^n$ by

$$\|z\| = \max\{|z^i| \colon i = 1, \dots, n\}.$$

Since all norms in $\mathbb{R}^n$ are equivalent there is no loss of generality by taking this definition. The set

$$C(0, a) = \{z \colon |z^i| < a, \ i = 1, \dots, n\}$$

will be called the open cube of radius a centered at the origin. If we denote the closure of $C(0,a)$ by $\overline{C(0,a)}$, then $\overline{C(0,a)} = \{z \colon \|z\| \le a\}$ and $C(0,a) = \{z \colon \|z\| < a\}$. Let

$$T_c = \{t \colon h(t,\cdot) \text{ is continuous on } U\},$$

and let $A = T - T_c$. Then by hypothesis, meas $A = 0$.

For each positive integer a, let U_a denote the closure of the intersection of U and $C(0,a)$. We first consider h on the cartesian product $T \times U_a$ and assume that T is bounded.

Let $\varepsilon > 0$ be given. For each positive integer m define

$$\begin{aligned} E_{\varepsilon m} = \{t \colon \ &\text{If } z_1, z_2 \text{ in } U_a, \|z_1 - z_2\| \\ &\le \sqrt{n}/m, \text{ then } |h(t,z_1) - h(t,z_2)| \le \varepsilon/4\}. \end{aligned}$$

We assert that each $E_{\varepsilon m}$ is measurable. If $E_{\varepsilon m}$ is empty, there is nothing to prove. We shall show the set $cE_{\varepsilon m} = T - E_{\varepsilon m}$ is measurable, from which it follows that $E_{\varepsilon m}$ is measurable.

A point t_0 is in $cE_{\varepsilon m} - A$ if and only if there exist points z_1, z_2 in U_a such that $\|z_1 - z_2\| \le \sqrt{n}/m$ and $|h(t_0,z_1) - h(t_0,z_2)| > \varepsilon/4$. We can assume that z_1 and z_2 have rational coordinates. For each pair of points z_1, z_2 in U_a with rational coordinates and satisfying $\|z_1 - z_2\| \le \sqrt{n}/m$ let

$$E_{z_1 z_2} = \{t \colon |h(t,z_1) - h(t,z_2)| > \varepsilon/4\}.$$

The point t_0 belongs to one of the sets $E_{z_1 z_2}$. Also, each set $E_{z_1 z_2}$ is in $cE_{\varepsilon m}$. Since the functions $h(\cdot, z_1)$ and $h(\cdot, z_2)$ are measurable, the set $E_{z_1 z_2}$ is measurable. Let D denote the union of the sets $E_{z_1 z_2}$. Then D is the countable union of measurable sets and so is measurable. From $t_0 \in E_{z_1 z_2}$ for some z_1, z_2 and $E_{z_1 z_2} \subseteq cE_{\varepsilon m}$, it follows that

$$cE_{\varepsilon m} - A \subseteq D \subseteq cE_{\varepsilon m}.$$

Since A has measure zero, it follows that $cE_{\varepsilon m}$ is measurable.

From the definition it is clear that $E_{\varepsilon 1} \subseteq E_{\varepsilon 2} \subseteq \ldots$ Let

$$E_{\varepsilon 0} = \bigcup_{m=1}^{\infty} E_{\varepsilon m} \qquad F_0 = T - E_{\varepsilon 0}.$$

If $t \notin A$, then $h(t,\cdot)$ is uniformly continuous on U_a, and $t \in E_{\varepsilon m}$ for sufficiently large m. Hence $t \in E_{\varepsilon 0}$. If $t \in F_0$, than $t \notin E_{\varepsilon m}$ for all m. Hence $t \in A$, and so $F_0 \subset A$. Since A has measure zero, F_0 has measure zero. Thus, meas $E_{\varepsilon 0} =$ meas T.

From the definition of $E_{\varepsilon 0}$ it follows that given an $\eta > 0$, there exists a positive integer $m_0(\varepsilon, \eta)$ such that meas $E_{\varepsilon m_0} >$ meas $E_{\varepsilon 0} - (\varepsilon\eta)/2$. In the last paragraph we showed that meas $T =$ meas $E_{\varepsilon 0}$, so

$$\text{meas } E_{\varepsilon m_0} > \ \text{meas } T - (\varepsilon\eta)/2. \tag{3.4.6}$$

By the definition of $E_{\varepsilon m_0}$, for every pair of points z_1, z_2 in U_a with $\|z_1 - z_2\| \le \sqrt{n}/m_0$ we have

$$|h(t, z_1) - h(t, z_2)| \le \varepsilon/4 \text{ for all } t \in E_{\varepsilon m_0}. \tag{3.4.7}$$

Set $p = 1 + [a]$, where $[a]$ is the largest integer $\le a$, and divide the cube $\overline{C(0, a)}$ into $q = p^n m_0^n$ congruent closed cubes $d_1, \ldots, d_q$. The side of each cube d_j will have length $a/pm_0 \le 1/m_0$. We shall say that cubes d_j and d_k are *contiguous* if d_j and d_k have non-empty intersection. Thus, a cube d_j such that no point of d_j is a boundary point of $\overline{C(0, a)}$ will have $3^n - 1$ contiguous cubes. Let z_j denote the center of the cube d_j. For any pair of contiguous cubes d_j and d_k we either have $\|z_j - z_k\| = m_0^{-1}$ or $\|z_j - z_k\| = \sqrt{n}/m_0$. In either case we have

$$\|z_j - z_k\| \le \sqrt{n}/m_0.$$

By hypothesis, for each $j = 1, \ldots, q$ the function $h(\cdot, z_j)$ is measurable on T. Hence, by Lusin's theorem, for each j there exists a closed set $F_j \subset T$ with

$$\text{meas } F_j > \text{ meas } T - (\varepsilon\eta)/2q \tag{3.4.8}$$

such that $h(\cdot, z_j)$ is continuous on F_j. Since T is compact and F_j is closed, the continuity is uniform. Let

$$V = \bigcap_{j=1}^{q} F_j.$$

Then by (3.4.8)

$$\text{meas } V > \text{ meas } T - (\varepsilon\eta)/2. \tag{3.4.9}$$

Each function $h(\cdot, z_j)$ will be uniformly continuous on V. Hence there exists a $\delta_1(\varepsilon)$ such that if t and t' are in V and both lie in some cube in $\mathbb{R}^s$ of radius $\delta_1(\varepsilon)$

$$|h(t, z_j) - h(t', z_j)| < \varepsilon/4 \tag{3.4.10}$$

for all $j = 1, \ldots, q$. Let $E_\varepsilon = V \cap E_{\varepsilon m_0}$. Then E_ε is measurable and by (3.4.6) and (3.4.9),

$$\text{meas } E_\varepsilon > \text{ meas } T - \varepsilon\eta. \tag{3.4.11}$$

Note that E_ε depends on η as well as ε.

Let $\delta(\varepsilon) = \min(\delta_1(\varepsilon), m_0^{-1})$. Let (t, z) and (t', z') be two points in $E_\varepsilon \times U_a$ such that $\|(t, z) - (t', z')\| \le \delta(\varepsilon)$. Then

$$\|t - t'\| \le \delta_1(\varepsilon) \quad \text{and} \quad \|z - z'\| \le m_0^{-1}. \tag{3.4.12}$$

It follows from (3.4.12) that z and z' are in the same cube d_j or in contiguous cubes d_j and d_k. If $z \in d_j$ and $z' \in d_k$ where d_j and d_k are contiguous, then (recall that z_i is the center of cube $d_i, i = 1, \ldots, q$)

$$|h(t', z') - h(t, z)| \le |h(t', z') - h(t', z_k)| + |h(t', z_k) - h(t', z_j)|$$

$$+|h(t', z_j) - h(t, z_j)| + |h(t, z_j) - h(t, z)|.$$

Since $t' \in E_{\varepsilon m_0}$ and $\|z' - z_k\| \leq \sqrt{n}m_0^{-1}$, from (3.4.7) we see that the first term on the right is $\leq \varepsilon/4$. Since $\|z_k - z_j\| = \sqrt{n}m_0^{-1}$, $\|z - z_j\| < \sqrt{n}m_0^{-1}$, and $t \in E_{\varepsilon m_0}$, it again follows from (3.4.7) that the second and last terms on the right are each $\leq \varepsilon/4$. It follows from (3.4.10) that the second term on the right is $\leq \varepsilon/4$. Hence

$$|h(t', z') - h(t, z)| \leq \varepsilon \tag{3.4.13}$$

whenever $\|(t', z') - (t, z)\| \leq \delta(\varepsilon)$.

If z and z' are in the same cube d_j

$$|h(t', z') - h(t, z)| \leq |h(t', z') - h(t', z_j)| + |h(t', z_j) - h(t, z_j)| \\ +|h(t, z_j) - h(t, z|.$$

Using the argument of the preceding paragraph, we again conclude that (3.4.13) holds whenever $\|(t', z') - (t, z)\| \leq \delta(\varepsilon)$.

Let $\varepsilon_i = 2^{-i}$, let $E_i = E_{\varepsilon_i}$ and let

$$E = \bigcap_{i=1}^{\infty} E_i.$$

From (3.4.11) we see that

$$\text{meas } E > \text{ meas } T - (\sum_{i=1}^{\infty} \varepsilon_i)\eta = T - \eta. \tag{3.4.14}$$

We assert that h is continuous on the cartesian product $E \times U_a$. To prove this, let $\gamma > 0$ be given. Let $i = i(\gamma)$ be such that $\varepsilon_i = 2^{-i} < \gamma$. Let (t, z) and (t', z') be in $E \times U_a$ and satisfy $\|(t', z') - (t, z)\| \leq \delta(\varepsilon_i)$. Since t and t' are in E_i we have by (3.4.13) that $|h(t', z') - h(t, z)| \leq 2^{-i} < \gamma$.

The sequence $\{U_k\}$ is an increasing sequence of compact sets whose union is U. We have just proved that for each $\rho > 0$, there exists a measurable set $E_i \subset T$ such that meas $E_i >$ meas $T - \rho 2^{-(i+1)}$ and such that h is continuous on the cartesian product $T \times E_i$. Let

$$G = \bigcap_{i=1}^{\infty} E_i.$$

Then h is continuous on $U \times G$, and

$$\text{meas } (G) > \text{ meas } T - \rho \sum_{i=1}^{\infty} 2^{-(i+1)} = \text{ meas } T - \rho/2.$$

Since G has positive finite measure, there exists a closed set $F \subset G$ such that

$$\text{meas } (F) > \text{ meas } G - \rho/2 > \text{ meas } T - \rho.$$

Moreover h is continuous on $F \times U$. If U is closed, the existence of h follows from Tietze's Extension Theorem. $\square$

In Assumption 3.2.1(iv) we postulated the existence of a measurable function u defined on $\mathcal{I}$ with range in $\mathcal{U}$ such that the state equation with this u has a solution ϕ defined on $\mathcal{I}$. Lemma 3.4.5 shows that if Ω is u.s.c.i and each $\Omega(t)$ is compact, then there exists a measurable function u with $u(t) \in \Omega(t)$ a.e.

Lemma 3.4.5. *Let Ω be a mapping form $\mathcal{I}$ to compact sets $\Omega(t)$ in $\mathbb{R}^m$ that is u.s.c.i. Then there exists a measurable function u such that $u(t) \in \Omega(t)$ for all t in $\mathcal{I}$.*

Proof. Let

$$\Delta = \{(t, z) \colon t \in \mathcal{I},\ z \in \Omega(t)\}.$$

Then by Lemma 3.3.11, since Ω is u.s.c.i, the set Δ is compact. Let

$$d(t) = \inf\{|z| \colon z \in \Omega(t)\}.$$

Then $d(t) \geq 0$ and is finite; since $\Omega(t)$ is compact, there exists a $z(t) \in \Omega(t)$ such that $d(t) = |z(t)|$.

We assert that the function d is lower semicontinuous and hence measurable. To show this we shall show that for each real α, the set $E_\alpha = \{t \colon d(t) \leq \alpha\}$ is closed. Let t_0 be a limit point of E_α. Then there exists a sequence $\{t_n\}$ in E_α such that $t_n \to t_0$. The points $(t_n, z(t_n))$ are in the compact set Δ. Hence there exists a subsequence $\{(t_n, z(t_n)\}$ and a point $(t_0, z(t_0))$ such that $(t_n, z(t_n)) \to (t_0, z(t_0)) \in \Delta$. Hence $z(t_0) \in \Omega(t_0)$. Now $\alpha \geq d(t_n) = |z|(t_n)$, so $\alpha \geq |z_0| \geq d(t_0)$. Thus, $t_0 \in E_\alpha$, and so E_α is closed.

Since the norm function is continuous, it follows from the measurability of d, Lemma 3.2.10, and the relation $d(t) = |z(t)|$ that there exists a measurable function u with $u(t) \in \Omega(t)$ such that $d(t) = |u(t)|$. □

Remark 3.4.6. Let Ω be as above and let the state equations be

$$\frac{dx}{dt} = A(t)x + h(t, z),$$

where A is measurable in $\mathcal{I}$ and h is measurable in t for fixed z and continuous in z for fixed t. Then the system

$$\frac{dx}{dt} = A(t)x + h(t, u(t))$$

has a solution ϕ defined on all of $\mathcal{I}$. This follows from standard theorems on differential equations.

3.5 The Relaxed Problem; Non-Compact Constraints

In Definition 3.2.2 we defined a relaxed control to be a mapping that assigned a regular probability measure μ_t to each compact constraint set $\Omega(t)$.

We then showed that if we restrict the measures μ_t to be convex combinations of Dirac measures, then we obtain the same set of relaxed trajectories as we do using Definition 3.2.2. We introduced Definition 3.2.2 rather than the simpler one involving convex combinations of Dirac measures because the proof of the weak compactness of relaxed controls in the case of compact constraints required the consideration of general probability measures.

In Remark 3.3.7 we showed that weak compactness of relaxed controls may fail if the sets $\Omega(t)$ are not contained in some compact set. Thus, if the sets $\Omega(t)$ are not compact one of the advantages of using Definition 3.2.2 is lost. Moreover, if $\Omega(t)$ is not compact, for the integral

$$\int_{\Omega(t)} g(t,z)d\mu_t$$

to exist, we must place conditions on the behavior of the function $g(t,\cdot)$. Hence in the case of non-compact constraint sets we shall define a relaxed control to be a convex combination of Dirac measures.

Assumption 3.5.1. Let $\widehat{f} = (f^0, f^1, \ldots, f^n)$ be as in Assumption 3.2.1. Let Ω be a mapping from I to subsets of U; that is, $\Omega\colon t \to \Omega(t)$, where $\Omega(t) \subseteq U$.

Definition 3.5.2. A *relaxed control* is a function v of the form $v = (u_1, \ldots, u_{n+2}, p^1, \ldots, p^{n+2})$, where each u_i is a measurable function on I with range in $\mathbb{R}^m$ satisfying the relation $u_i(t) \in \Omega(t)$ and each p^i is a measurable real valued nonnegative function on I such that

$$\sum_{i=1}^{n+2} p^i(t) = 1 \quad \text{a.e.}$$

Definition 3.5.3. A *discrete measure control* on I is a mapping μ on I to probability measures

$$\mu_t = \sum_{i=1}^{n+2} p^i(t)\delta_{u_i(t)}, \tag{3.5.1}$$

where each p^i is a nonnegative measurable function, $\Sigma p^i(t) = 1$, and each u_i is a measurable function with $u_i(t) \in \Omega(t)$ a.e.

Let g be a mapping from $I \times U$ to $\mathbb{R}^n$ that is continuous on U for a.e. t in I and measurable on I for all z in U. Then for a discrete measure control if

$$h(t) = \int_{\Omega(t)} g(t,z)d\mu_t = \sum_{i=1}^{n+2} p^i(t)g(t,u_i(t)),$$

then h is measurable. Thus, a discrete measure control is a relaxed control.

Remark 3.5.4. If we *define a relaxed control to be a discrete measure control*, then this definition is equivalent to Definition 3.5.3.

Definition 3.5.5. An absolutely continuous function $\psi = (\psi^1, \ldots, \psi^n)$ defined in interval $[t_0, t_1] \subseteq I$ is a *relaxed trajectory* corresponding to a relaxed control v if

(i) $(t, \psi(t)) \in \mathcal{R}$ for all $t \in [t_0, t_1]$.

(ii) ψ is a solution of the differential equation

$$\frac{dx}{dt} = \sum_{i=1}^{n+2} p^i(t) f(t, x, u_i(t)) = \int_{\Omega(t)} f(t, x, z) d\mu_t, \tag{3.5.2}$$

where μ_t is as in (3.5.1).

The differential equation (3.5.2) is called the *relaxed differential* equation.

Definition 3.5.6. A relaxed trajectory ψ corresponding to a relaxed control v is said to be *admissible* if

(i) $(t_0, \psi(t_0), t_1, \psi(t_1)) \in \mathcal{B}$ and the function

$$t \to \sum_{i=1}^{n+2} p^i(t) f^0(t, \psi(t), u_i(t)) = \int_{\Omega(t)} f^0(t, \psi(t), z) d\mu_t$$

is integrable. The pair (ψ, v) or (ψ, μ) is said to be an *admissible pair.*

The relaxed problem in the case of non-compact constraints is as follows:

Problem 3.5.1. Find a relaxed admissible pair (ψ, v) that minimizes

$$J(\psi, v) = g(t_0, \psi(t_0), t_1, \psi(t_1)) + \int_{t_0}^{t_1} \left[\sum_{i=1}^{n+2} p^i(t) f^0(t, \psi(t), u_i(t)) \right] dt. \tag{3.5.3}$$

Remark 3.5.7. In view of Theorem 3.2.11, the relaxed problem in the case of compact constraints can also be formulated as Problem 3.5.1.

Remark 3.5.8. The relaxed problem can also be viewed as an ordinary problem with state variable x in $\mathbb{R}^n$, with control variable

$$\overline{z} \equiv (\pi, \zeta) = (\pi^1, \ldots, \pi^{n+2}, z_1, \ldots, z_{n+2}) \quad \pi^i \in \mathbb{R}, \; z_i \in \mathbb{R}^m,$$

with state equations

$$\frac{dx}{dt} \equiv f_r(t, x, \overline{z}) = \sum_{i=1}^{n+2} \pi^i f(t, x, z_i) \quad \pi^i \in R, \; z_i \in \mathbb{R}^m.$$

The integrand in (3.5.3) can also be written as

$$\int_{\Omega(t)} f^0(t, \psi(t), z) d\mu_t,$$

with $d\mu_t$ as in (3.5.1), or as

$$f^0_r(t,x,\overline{z}) \equiv \sum_{i=1}^{n+2} \pi^i f^0(t,x,z_i) \quad \pi^i \in \mathbb{R},\ z_i \in \mathbb{R}^m.$$

The constraints are now

$$z_i \in \Omega(t) \quad \pi^i \geq 0 \quad \sum_{i=1}^{n+2} \pi^i = 1. \tag{3.5.4}$$

The end set $\mathcal{B}$ and the terminal payoff function g are as before.

Remark 3.5.9. Let

$$\widehat{f} = (f^0, f) \quad \widehat{y} = (y^0, y) \quad y^0 \in \mathbb{R},\ y \in \mathbb{R}^n$$

and let

$$\begin{aligned} \widetilde{V}_r(t,x) &= \left\{ \widehat{y} \colon \widehat{y} = \sum_{i=1}^{n+2} \pi^i \widehat{f}(t,x,z_i), \quad \pi^i,\ z_i \text{ as in } (3.5.3) \right\} \\ V(t,x) &= \left\{ \widehat{y} \colon \widehat{y} = \widehat{f}(t,x,z) \quad z \in \Omega(t) \right\}. \end{aligned}$$

Then

$$\widetilde{V}_r(t,x) = co\ V(t,x).$$

This follows from Carathéodory's theorem and the fact that for any set A, the set co(A) consists of all convex combinations of A.

3.6 The Chattering Lemma; Approximation to Relaxed Controls

In this section we show that, under reasonable hypotheses, ordinary trajectories of a control system are dense in the relaxed trajectories of the system. The essence of this result is the following theorem, which is sometimes called the "Chattering Lemma" for reasons to be given in Remark 3.6.9.

Theorem 3.6.1. *Let $\mathcal{I}$ be a compact real interval and let $\mathcal{X}$ be a compact set in $\mathbb{R}^n$. Let $f_1, \ldots, f_q$ be functions defined on $\mathcal{I} \times \mathcal{X}$ with range in $\mathbb{R}^n$ and possessing the following properties:*

(i) Each f_i is a measurable function on $\mathcal{I}$ for each x in $\mathcal{X}$.

(ii) Each f_i is continuous on $\mathcal{X}$ for each t in $\mathcal{I}$.

(iii) There exists an integrable function μ defined on $\mathcal{I}$ such that for all (t,x) and (t,x') in $\mathcal{I} \times \mathcal{X}$ and $i = 1,\ldots,q$:

$$|f_i(t,x)| \le \mu(t) \tag{3.6.1}$$
$$|f_i(t,x) - f_i(t,x')| \le \mu(t)|x - x'|.$$

Let p^i, $i = 1,\ldots,q$, be real valued nonnegative measurable functions defined on $\mathcal{I}$ and satisfying

$$\sum_{i=1}^{q} p^i(t) = 1 \ \ a.e. \tag{3.6.2}$$

Then for every $\overline{\epsilon} > 0$ there exists a subdivision of $\mathcal{I}$ into a finite collection of non-overlapping intervals E_j, $j = 1,\ldots,k$ and an assignment of one of the functions $f_1,\ldots,f_q$ to each E_j such that the following holds. If f_{E_j} denotes the function assigned to E_j and if f is a function that agrees with f_{E_j} on the interior E_j^0 of each E_j, that is,

$$f(t,x) = f_{E_j}(t,x) \qquad \text{if } t \in E_j^0 \quad j = 1,\ldots,k,$$

then for every t', t'' in $\mathcal{I}$ and all x in $\mathcal{X}$

$$\left| \int_{t'}^{t''} \left(\sum_{i=1}^{q} p^i(t) f_i(t,x) - f(t,x) \right) dt \right| < \overline{\epsilon}. \tag{3.6.3}$$

Remark 3.6.2. Let $E_j = [\tau_j, \tau_{j+1}]$, $j = 1,\ldots,k$. If we set $f(\tau_j, x) = f_{E_j}(\tau_j, x)$, $j = 1,\ldots,k$, and set $f(\tau_{k+1}, x) = f_{E_k}(\tau_{k+1}, x)$, then (3.6.3) still holds. Moreover the function f satisfies (3.6.1), and the functions f_i are of class $C^{(r)}$ on $\mathcal{X}$ for some values of t, then f is of class $C^{(r)}$ for the same values of t.

The first step in our proof is to establish the following lemma.

Lemma 3.6.3. *Let $\mathcal{I}$ and $\mathcal{X}$ be as in the theorem and let f be a function from $\mathcal{I} \times \mathcal{X}$ to $\mathbb{R}^n$ having the same properties as the functions $f_1,\ldots,f_q$ of the theorem. Then for every $\epsilon > 0$ there exists a continuous function g, depending on ϵ, from $\mathcal{I} \times \mathcal{X}$ to $\mathbb{R}^n$ such that for every x in $\mathcal{X}$*

$$\int_{\mathcal{I}} |f(t,x) - g(t,x)| dt < \epsilon. \tag{3.6.4}$$

Proof. It follows from (3.6.1) that for x and x' in $\mathcal{X}$

$$\int_{\mathcal{I}} |f(t,x) - f(t,x')| dt \le |x - x'| \int_{\mathcal{I}} \mu(t) dt.$$

Hence for arbitrary $\epsilon > 0$, we have

$$\int_{\mathcal{I}} |f(t,x) - f(t,x')| dt < \epsilon/2 \tag{3.6.5}$$

whenever $|x - x'| < \epsilon/2 \int_{\mathcal{I}} \mu(t)dt$. Since $\mathcal{X}$ is compact, there exists a finite open cover $\mathcal{O}_1, \ldots, \mathcal{O}_k$ of $\mathcal{X}$ such that if x and x' are in the same $\mathcal{O}_i$, then (3.6.5) holds.

Let $x_1, \ldots, x_k$ be a finite set of points such that $x_i \in \mathcal{O}_i$. For each $i = 1, \ldots, k$ there exists a continuous function h_i defined on $\mathcal{I}$ such that

$$\int_{\mathcal{I}} |f(t, x_i) - h_i(t)|dt < \epsilon/2. \tag{3.6.6}$$

Let $\gamma_1, \ldots, \gamma_k$ be a partition of unity corresponding to the finite open cover $\mathcal{O}_1, \ldots, \mathcal{O}_k$ of $\mathcal{X}$. That is, let $\gamma_1, \ldots, \gamma_k$ be continuous real valued functions on $\mathcal{X}$ such that

$$\begin{aligned} &\text{(i)} \quad \gamma_i(x) \geq 0 \text{ for all } x \in \mathcal{X} \\ &\text{(ii)} \quad \gamma_i(x) = 0 \quad \text{if } x \notin \mathcal{O}_i \\ &\text{(iii)} \quad \textstyle\sum_{i=1}^{k} \gamma_i(x) = 1 \end{aligned} \tag{3.6.7}$$

For a proof of the existence of partitions of unity corresponding to finite open covers of compact subsets of locally compact Hausdorff spaces, see Rudin [82, p. 40].

Define

$$g(t, x) = \sum_{i=1}^{k} \gamma_i(x) h_i(t).$$

Then g is continuous on $\mathcal{I} \times \mathcal{X}$. We now show that g satisfies (3.6.4) and therefore is the desired function.

$$\begin{aligned} \int_{\mathcal{I}} |g(t,x) - f(t,x)|dt &\leq \int_{\mathcal{I}} \left| \sum_{i=1}^{k} \gamma_i(x) h_i(t) - \sum_{i=1}^{k} \gamma_i(x) f(t, x_i) \right| dt \\ &\quad + \int_{\mathcal{I}} \left| \sum_{i=1}^{k} \gamma_i(x) f(t, x_i) - \sum_{i=1}^{k} \gamma_i(x) f(t, x) \right| dt \\ &\leq \sum_{i=1}^{k} \gamma_i(x) \int_{\mathcal{I}} |h_i(t) - f(t, x_i)|dt \\ &\quad + \sum_{i=1}^{k} \gamma_i(x) \int_{\mathcal{I}} |f(t, x_i) - f(t, x)|dt. \end{aligned}$$

By virtue of (3.6.6) each of the integrals in the first sum on the right is less than $\epsilon/2$. From this and from (3.6.7)(iii) it follows that the first sum on the right is less than $\epsilon/2$. We now examine the i-th summand in the second sum on the right. If $x \notin \mathcal{O}_i$ then by (3.6.7)(ii), $\gamma_i(x) = 0$ and so the summand is zero. If $x \in \mathcal{O}_i$, then by (3.6.5) the integral is less than $\epsilon/2$ and therefore by (3.6.7)(i) the summand is less than $\epsilon\gamma_i(x)/2$. Therefore, each summand in the second sum is less than $\epsilon\gamma_i(x)/2$. It now follows from (3.6.7)(iii) that

the second sum is less than $\epsilon/2$. Hence g satisfies (3.6.4) and the lemma is proved. $\square$

We now return to the proof of Theorem 3.6.1. Let $\bar{\epsilon} > 0$ be given and let

$$\epsilon = \bar{\epsilon}/2(2 + q + |\mathcal{I}|), \tag{3.6.8}$$

where $|\mathcal{I}|$ denotes the length of $\mathcal{I}$. Henceforth if A is a measurable set we shall use $|A|$ to denote the measure of A. From Lemma 3.6.3 we get that for each $i = 1, \ldots, q$ there is a continuous function g_i defined on $\mathcal{I} \times \mathcal{X}$ with range in $\mathbb{R}^n$ such that

$$\int_{\mathcal{I}} |f_i(t,x) - g_i(t,x)| dt < \epsilon. \tag{3.6.9}$$

Since each g_i is continuous on $\mathcal{I} \times \mathcal{X}$ and $\mathcal{I}$ and $\mathcal{X}$ are compact, each g_i is uniformly continuous on $\mathcal{I} \times \mathcal{X}$. Therefore, there exists a $\delta > 0$ such that for all $i = 1, \ldots, q$ if $|t - t'| < \delta$ then

$$|g_i(t,x) - g_i(t',x)| < \epsilon. \tag{3.6.10}$$

Moreover, we may suppose that δ is such that if E is a measurable subset of $\mathcal{I}$ with $|E| < \delta$, then

$$\int_E \mu(t) dt < \epsilon. \tag{3.6.11}$$

Let $\{I_k\}$ be a subdivision of $\mathcal{I}$ into a finite number of non-overlapping intervals with $|I_k| < \delta$ for each interval I_k. Moreover, suppose that $I_k = [t_k, t_{k+1}]$ and that $\ldots < t_{k-1} < t_k < t_{k+1} < t_{k+2} < \ldots$. For each I_k we can construct a subdivision of I_k into non-overlapping subintervals $E_{k1}, \ldots, E_{kq}$ such that

$$|E_{ki}| = \int_{I_k} p^i(t) dt. \tag{3.6.12}$$

This is possible since

$$\sum_{i=1}^{q} |E_{ki}| = \sum_{i=1}^{q} \int_{I_k} p^i(t) dt = \int_{I_k} \left(\sum_{i=1}^{q} p^i(t) \right) dt = |I_k|,$$

the last equality following from (3.6.2).

Define

$$f(t,x) = f_i(t,x) \qquad t \in E_{ki}^0, \tag{3.6.13}$$

where E_{ki}^0 denotes the interior of E_{ki}. Thus, f is defined at all points of $\mathcal{I}$ except the end points of the intervals E_{ki}. At these points f can be defined as in Remark 3.6.2 or in any arbitrary manner. Let

$$\lambda(t,x) = \sum_{i=1}^{q} p^i(t) f_i(t,x) - f(t,x). \tag{3.6.14}$$

The collection of intervals $\{E_{ki}\}$ where k ranges over the same index set as do the intervals I_k and i ranges over the set $1, \ldots, q$, constitutes a subdivision of $\mathcal{I}$ into a finite number of non-overlapping subintervals. This subdivision, relabeled as $\{E_j\}$, is the subdivision whose existence is asserted in the theorem. If an interval E_j was originally the interval E_{ki}, then the function f_{E_j} assigned to E_j is f_i. If we now compare the definition of λ in (3.6.14) with (3.6.3) and we see that to prove the theorem we must show that for arbitrary t' and t'' in $\mathcal{I}$ and all x in $\mathcal{X}$

$$\left| \int_{t'}^{t''} \lambda(t, x) dt \right| < \bar{\epsilon}. \tag{3.6.15}$$

There is no loss of generality in assuming that $t' < t''$. The point t' will belong to some interval I_α of the subdivision $\{I_k\}$ and the point t'' will belong to some interval I_β. If $I_\alpha \neq I_\beta$, let s_1 denote the right-hand end point $t_{\alpha+1}$ of I_α and let s_2 denote the left-hand end point t_β of I_β. Then if J denotes the set of indices $\{\alpha + 1, \alpha + 2, \ldots, \beta - 1\}$, we have

$$[s_1, s_2] \equiv [t_{\alpha+1}, t_\beta] = \bigcup_{j \in J} I_j.$$

See Figure 3.1.

Hence we have

$$\left| \int_{t'}^{t''} \lambda dt \right| \leq \left| \int_{t'}^{s_1} \lambda dt \right| + \left| \int_{s_1}^{s_2} \lambda dt \right| + \left| \int_{s_2}^{t''} \lambda dt \right| \equiv A + B + C.$$

It follows from (3.6.14), (3.6.1), (3.6.2), (3.6.11), and the fact that t' and s_1 are in an interval I_α with $|I_\alpha| < \delta$ that:

$$\begin{aligned} A &\leq \int_{t'}^{s_1} \left(\sum_{i=1}^{q} |p^i f_i| + |f| \right) dt = \int_{t'}^{s_1} \sum_{i=1}^{q} p^i |f_i| dt + \int_{t'}^{s_1} |f| dt \\ &\leq \int_{t'}^{s_1} \left(\sum_{i=1}^{q} p^i \right) \mu dt + \int_{t'}^{s_1} \mu dt = 2 \int_{t'}^{s_1} \mu dt < 2\epsilon. \end{aligned}$$

Note that if t' and t'' are in the same interval I_α, then the preceding estimate and (3.6.8) combine to give (3.6.15).

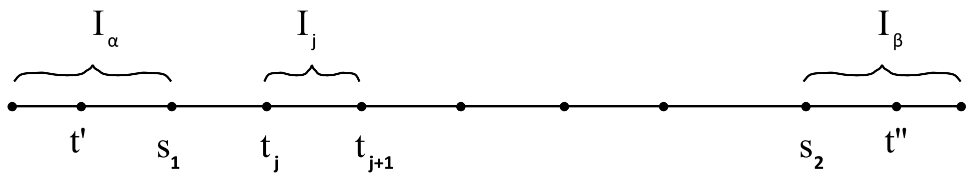

FIGURE 3.1

An argument similar to the preceding one gives $C < 2\epsilon$.

We now estimate B. Recall that $I_k = [t_k, t_{k+1}]$. Then

$$B = \left| \int_{s_1}^{s_2} \lambda dt \right| \leq \sum_{j \in J} \left| \int_{t_j}^{t_{j+1}} \lambda dt \right|.$$

Let $g(t,x) = g_i(t,x)$ for $t \in E_{ji}$, where $i = 1, \ldots, q$ and $j \in J$; then we can estimate each summand on the right as follows

$$\begin{aligned}\left| \int_{t_j}^{t_{j+1}} \lambda dt \right| &\leq \left| \int_{t_j}^{t_{j+1}} \left(\sum_{i=1}^{q} p^i (f_i - g_i) \right) dt \right| \\ &\quad + \left| \int_{t_j}^{t_{j+1}} \left(\sum_{i=1}^{q} p^i g_i - g \right) dt \right| + \left| \int_{t_j}^{t_{j+1}} (g - f) dt \right| \\ &\equiv A_j + B_j + C_j.\end{aligned}$$

Hence

$$B \leq \sum_{j \in J} (A_j + B_j + C_j). \tag{3.6.16}$$

From $p^i \geq 0$ and (3.6.2) we get that

$$A_j \leq \int_{t_j}^{t_{j+1}} \left(\sum_{i=1}^{q} p^i |f_i - g_i| \right) dt \leq \sum_{i=1}^{q} \int_{t_j}^{t_{j+1}} |f_i - g_i| dt.$$

From the definitions of f and g we get that

$$C_j \leq \int_{t_j}^{t_{j+1}} |g - f| dt \leq \sum_{i=1}^{q} \int_{t_j}^{t_{j+1}} |f_i - g_i| dt.$$

Therefore,

$$\sum_{j \in J} (A_j + C_j) \leq 2 \sum_{i=1}^{q} \int_{s_1}^{s_2} |f_i - g_i| dt \leq 2 \sum_{i=1}^{q} \int_{\mathcal{I}} |f_i - g_i| dt < 2q\epsilon, \tag{3.6.17}$$

where the last inequality follows from (3.6.9).

We now consider B_j.

$$B_j = \left| \int_{t_j}^{t_{j+1}} \left(\sum_{i=1}^{q} p^i g_i - g \right) dt \right| = \left| \sum_{i=1}^{q} \int_{t_j}^{t_{j+1}} p^i g_i dt - \sum_{i=1}^{q} \int_{E_{ji}} g_i dt \right|.$$

In each set E_{ji} select a point t_{ji}. Since $E_{ji} \subset I_j$ and $|I_j| < \delta$ it follows from (3.6.10) that for all t in I_j and all x in $\mathcal{X}$ and all $i = 1, \ldots, q$

$$g_i(t,x) = g_i(t_{ji}, x) + \eta_i(t,x),$$

where $|\eta_i(t,x)| < \epsilon$. Therefore, using (3.6.12), we get

$$\begin{aligned}
B_j &= \Big|\sum_{i=1}^{q}\Big(\int_{t_j}^{t_{j+1}} (p^i(t)g_i(t_{ji},x) + p^i(t)\eta_i(t,x))dt \\
&\qquad - \int_{E_{ji}} (g_i(t_{ji},x) + \eta_i(t,x))dt\Big)\Big| \\
&= \Big|\sum_{i=1}^{q}\Big(g_i(t_{ji},x)|E_{ji}| - g_i(t_{ji},x)|E_{ji}| \\
&\qquad + \int_{t_j}^{t_{j+1}} p^i(t)\eta_i(t,x)dt - \int_{E_{ji}} \eta_i(t,x)dt\Big)\Big| \\
&< \sum_{i=1}^{q}\left(\epsilon\int_{t_j}^{t_{j+1}} p^i dt + \epsilon|E_{ji}|\right) = 2\epsilon|I_j|.
\end{aligned} \tag{3.6.18}$$

Hence

$$\sum_{j\in J} B_j < 2\epsilon|s_2 - s_1| \le 2\epsilon|\mathcal{I}|.$$

Combining this with (3.6.17) gives $B < 2\epsilon(q + |\mathcal{I}|)$. If we now combine this estimate with the estimates on A and C and use (3.6.8) we get that

$$\left|\int_{t'}^{t''} \lambda dt\right| < 2\epsilon(2 + q + |\mathcal{I}|) = \bar{\epsilon},$$

which is (3.6.15), as required. This completes the proof of Theorem 3.6.1.

Remark 3.6.4. Recall that a family Ψ of functions ψ defined on a set X in $\mathbb{R}^n$ with range in $\mathbb{R}^m$ is said to be *equicontinuous at a point* x_0 in X if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \text{ implies } |\psi(x) - \psi(x_0)| < \epsilon \tag{3.6.19}$$

for all ψ in Ψ. The family is *equicontinuous on* X if it is equicontinuous at all points of X. If X is compact, then each function is uniformly continuous on X. Thus, (3.6.19) holds for all x, x_0 in X. Since Ψ is equicontinuous on X, (3.6.19) holds for all x, x_0 in X and all ψ in Ψ. Thus, the functions ψ in Ψ are uniformly equicontinuous.

The next theorem states that (3.6.3) remains true if we replace the vectors x in $\mathcal{X}$ by functions ψ from an equicontinuous family.

Theorem 3.6.5. *Let $f_1, \ldots, f_q$ and $p^1, \ldots, p^q$ be as in Theorem 3.6.1. Let Ψ be a family of equicontinuous functions on $\mathcal{I}$ with range in $\mathcal{X}$. Then for every $\epsilon > 0$ there exists a subdivision of $\mathcal{I}$ into a finite number of disjoint intervals E_j and an assignment of one of the functions $f_1, \ldots, f_q$ to each interval E_j*

such that the following holds. If f_{E_j} denotes the function assigned to E_j and if f is a function that agrees with f_{E_j} on E_j^0, the interior of E_j, that is,

$$f(t,x) = f_{E_j}(t,x) \qquad t \in E_j^0,$$

then for every t' and t'' in $\mathcal{I}$ and every function ψ in Ψ

$$\left| \int_{t'}^{t''} \left(\sum_{i=1}^{q} p^i(t) f_i(t,\psi(t)) - f(t,\psi(t)) \right) dt \right| < \epsilon. \tag{3.6.20}$$

Proof. Let $\epsilon > 0$ be given. Since the functions in Ψ are equicontinuous on $\mathcal{I}$ and $\mathcal{I}$ is compact, there exists a partition of $\mathcal{I}$ into a finite number of non-overlapping subintervals $\{I_j\} = \{[t_j, t_{j+1}]\}$, $j = 1, \ldots, k$ such that $\cdots < t_{j-1} < t_j < t_{j+1} < t_{j+2} < \ldots$ and such that for all ψ in Ψ and all t in I_j, $j = 1, \ldots, k$.

$$|\psi(t) - \psi(t_j)| < \epsilon \left(4 \int_{\mathcal{I}} \mu dt \right)^{-1} \equiv \epsilon'. \tag{3.6.21}$$

□

We now apply Theorem 3.6.1 to $f_1, \ldots, f_q$ and $p^1, \ldots, p^q$ with $\overline{\epsilon}$ replaced by $\epsilon/2k$. Then there exists a function f as described in Theorem 3.6.1 such that for all x in $\mathcal{X}$ and all t', t'' in $\mathcal{I}$,

$$\left| \int_{t'}^{t''} \lambda(t,x) dt \right| < \epsilon/2k, \tag{3.6.22}$$

where λ is defined in (3.6.14). We must show that for all ψ in Ψ and t', t'' in $\mathcal{I}$,

$$\left| \int_{t'}^{t''} \lambda(t,\psi(t)) dt \right| < \epsilon.$$

Define

$$\widehat{\lambda}(t) = \lambda(t, \psi(t_j)) \qquad t_j \le t < t_{j+1}, \quad j = 1, \ldots, k$$

and let $\widehat{\lambda}(t_{k+1}) = \lambda(t_{k+1}, \psi(t_k))$. Then

$$\begin{aligned} \left| \int_{t'}^{t''} \lambda(t,\psi(t)) dt \right| &\le \left| \int_{t'}^{t''} (\lambda(t,\psi(t)) - \widehat{\lambda}(t)) dt \right| \\ &\quad + \left| \int_{t'}^{t''} \widehat{\lambda}(t) dt \right| \equiv A + B. \end{aligned} \tag{3.6.23}$$

Let $t' < t''$, let $t' \in I_\alpha = [t_\alpha, t_{\alpha+1}]$, and let $t'' \in I_\beta = [t_\beta, t_{\beta+1}]$. Let J now denote the index set $\{\alpha, \alpha+1, \alpha+2, \ldots, \beta\}$. Then

$$A \le \int_{t'}^{t''} \left| \lambda(t,\psi(t)) - \widehat{\lambda}(t) \right| \le \int_{t_\alpha}^{t_{\beta+1}} |\lambda(t,\psi(t)) - \widehat{\lambda}(t)| dt$$

$$
\begin{aligned}
&= \sum_{j\in J}\int_{t_j}^{t_{j+1}} |\lambda(t,\psi(t)) - \lambda(t,\psi(t_j))|dt \\
&= \sum_{j\in J}\int_{t_j}^{t_{j+1}} \Big|\sum_{i=1}^{q} p^i(t)\{f_i(t,\psi(t)) - f_i(t,\psi(t_j))\} \\
&\qquad + f(t,\psi(t_j)) - f(t,\psi(t))\Big| dt \\
&\leq \sum_{j\in J}\int_{t_j}^{t_{j+1}} \left(\sum_{i=1}^{q} p^i(t)\mu(t)|\psi(t)-\psi(t_j)| + \mu(t)|\psi(t)-\psi(t_j)|\right) dt,
\end{aligned}
$$

where the last inequality follows from (3.6.1) and Remark 3.6.2. From (3.6.2) and from (3.6.21) we see that the preceding sum in turn is less than

$$
\sum_{j\in J}\epsilon'\int_{t_j}^{t_{j+1}} 2\mu dt \leq 2\epsilon'\int_{\mathcal{I}}\mu dt = \epsilon/2.
$$

We have thus shown that $A < \epsilon/2$.

To estimate B we write

$$
\begin{aligned}
B \leq & \left|\int_{t'}^{t_{\alpha+1}} \lambda(t,\psi(t_\alpha))dt\right| + \sum_{j=\alpha+1}^{\beta-1}\left|\int_{t_j}^{t_{j+1}} \lambda(t,\psi(t_j))dt\right| \\
& + \left|\int_{t_\beta}^{t''} \lambda(t,\psi(t_\beta))dt\right|.
\end{aligned}
$$

By (3.6.22) each summand on the right is $< \epsilon/2k$. Since there are at most k summands (the number of intervals I_j), it follows that $B < \epsilon/2$. If we combine this estimate with the estimate for A and substitute into (3.6.23), then we get the desired result.

The proof of our next theorem requires an inequality that is very useful in the study of differential equations, and is known as Gronwall's Inequality.

Lemma 3.6.6. *Let ρ and μ be nonnegative real valued functions continuous on $[0,\infty)$ such that*

$$
\rho(t) \leq \alpha + \int_{t_0}^{t} \mu(s)\rho(s)ds \qquad \alpha \geq 0 \tag{3.6.24}
$$

for all t_0, t in $[0,\infty)$. Then

$$
\rho(t) \leq \alpha\exp\left(\int_{t_0}^{t}\mu(s)ds\right). \tag{3.6.25}
$$

Proof. Suppose that $\alpha > 0$. Then the right-hand side of (3.6.24) is strictly positive and we get that

$$
\rho(\tau)\mu(\tau)\left[\alpha + \int_{t_0}^{\tau}\mu(s)\rho(s)ds\right]^{-1} \leq \mu(\tau).
$$

Integrating both sides of this inequality from t_0 to t and using (3.6.24) gives

$$\log \rho(t) \leq \log \left[\alpha + \int_{t_0}^{t} \mu\rho ds\right] \leq \log \alpha + \int_{t_0}^{t} \mu ds.$$

From this we get (3.6.25). □

If $\alpha = 0$, then (3.6.24) holds for all $\alpha_1 > 0$. Hence (3.6.25) holds for all $\alpha_1 > 0$. Letting $\alpha_1 \to 0$ now yields $\rho(t) \equiv 0$. Hence (3.6.25) is trivially true.

Remark 3.6.7. The proof shows that if $\alpha > 0$ and strict inequality holds in (3.6.24), then strict inequality holds in (3.6.25).

Theorem 3.6.8. *Let $\mathcal{I}$ be a compact interval in $\mathbb{R}^1$, let $\mathcal{X}$ be a compact interval in $\mathbb{R}^n$, and let $\mathcal{R} = \mathcal{I} \times \mathcal{X}$. Let $\mathcal{G} = \mathcal{R} \times \mathcal{U}$, where $\mathcal{U}$ is a region of $\mathbb{R}^m$, and let f be a continuous mapping from $\mathcal{G}$ to $\mathbb{R}^n$. Let Ω be a mapping from $\mathcal{R}$ to subsets of $\mathcal{U}$ that is independent of x; that is, $\Omega(t, x') = \Omega(t, x) \equiv \Omega(t)$ for all x and x' in $\mathcal{X}$. Let there exist an integrable function μ defined on $\mathcal{I}$ such that for all (t, x, z) in $\mathcal{G}$*

$$|f(t, x, z)| \leq \mu(t)$$

and for all (t, x, z) and (t, x', z) in $\mathcal{G}$

$$|f(t, x, z) - f(t, x', z)| \leq \mu(t)|x - x'|. \tag{3.6.26}$$

Let $\mathcal{I}_1 = [t_0, t_1]$ be a compact interval contained in the interior of $\mathcal{I}$ and $\mathcal{X}_1$ be a compact interval in the interior of $\mathcal{X}$. Let $\mathcal{R}_1 = \mathcal{I}_1 \times \mathcal{X}_1$. Let $v = (u_1, \ldots, u_{n+2}, p^1, \ldots, p^{n+2})$ be a relaxed control on $\mathcal{I}_1$ for the relaxed system

$$\frac{dx}{dt} = \sum_{i=1}^{n+2} p^i(t) f(t, x, u_i(t))$$

corresponding to the control system

$$\frac{dx}{dt} = f(t, x, u(t)).$$

Let both systems have initial point (t_0, x_0) in $\mathcal{I}_1 \times \mathcal{X}_1$. Let ψ be a relaxed trajectory corresponding to v on $\mathcal{I}_1$ and let $\psi(t) \in \mathcal{X}_1$ for all t in $[t_0, t_1]$. Then there exists an $\epsilon_0 > 0$ such that for each ϵ satisfying $0 < \epsilon < \epsilon_0$ there is a control u_ϵ defined on $\mathcal{I}_1$ with the following properties.

(i) The control $u_\epsilon(t) \in \Omega(t)$ for a.e. t in $\mathcal{I}_1$,

(ii) the trajectory ϕ_ϵ corresponding to u_ϵ lies in $\mathcal{I}_1 \times \mathcal{X}$, and

(iii) $|\phi_\epsilon(t) - \psi(t)| < \epsilon$, for all t in $\mathcal{I}_1$.

Remark 3.6.9. Theorem 3.6.8 states that under appropriate hypotheses the ordinary trajectories of a system are dense in the set of relaxed trajectories in the uniform topology on $[t_0, t_1]$. Thus, for any relaxed trajectory ψ on $[t_0, t_1]$ there is a sequence of controls $\{u_k\}$ and a sequence of corresponding trajectories $\{\phi_k\}$ such that $u_k(t) \in \Omega(t)$ a.e. and $\phi_k \to \psi$ uniformly on $[t_0, t_1]$. We caution the reader that with reference to a specific control problem, if ψ is an admissible relaxed trajectory the pairs (ϕ_k, u_k) need not be admissible for the original problem in that either $t \to f^0(t, \phi_k(t), u_k(t))$ may not be integrable or the end points of the ϕ_k may not satisfy the end condition. Recall the distinction between a control (Definition 2.3.1) and an admissible control (Definition 2.3.2). We will return to this point in Chapter 4. Under a "local controllability" condition, the approximating relaxed trajectories ϕ_k can be slightly modified to satisfy the required end condition. See Theorem 4.4.6.

Note that no assumption is made concerning the nature of the constraint sets $\Omega(t)$.

Proof. Let ϵ_0 denote the distance between $\partial\mathcal{X}$ and $\partial\mathcal{X}_1$, where for any set A the symbol ∂A denotes the boundary of A. Then $\epsilon_0 > 0$. Let

$$K = \int_{t_0}^{t_1} \mu dt \tag{3.6.27}$$

and let ϵ be any number satisfying $0 < \epsilon < \epsilon_0$. For (t, x) in $\mathcal{I}_1 \times \mathcal{X}$ and $i = 1, \ldots, n+2$ let

$$f_i(t, x) = f(t, x, u_i(t)). \tag{3.6.28}$$

The hypotheses of the present theorem imply that the functions f_i satisfy the hypotheses of Theorems 3.6.1 and 3.6.5. In particular note that since f is continuous on $\mathcal{R}$ and each u_i is measurable, the functions f_i are measurable on $\mathcal{I}_1$ for each fixed x in $\mathcal{X}$.

Let $\epsilon' = \epsilon e^{-K}$. We next apply Theorem 3.6.5 to the functions $f_1, \ldots, f_{n+2}$ just defined, the functions $p^1, \ldots, p^{n+2}$ in the relaxed control, the family Ψ consisting of one element — the relaxed trajectory ψ, and the value of epsilon equal to ϵ'. We obtain the existence of a function $\widehat{f}$ such that for $x \in \mathcal{X}_1$ and $t \in \mathcal{I}_1$

$$\widehat{f}(t, x) = f_{E_j}(t, x) \qquad t \in E_j^0 \tag{3.6.29}$$

and

$$\left| \int_{t'}^{t''} \left(\sum_{i=1}^{n+2} p^i(t) f_i(t, \psi(t)) - \widehat{f}(t, \psi(t)) \right) dt \right| < \epsilon' \tag{3.6.30}$$

for arbitrary t' and t'' in $\mathcal{I}_1$.

It follows from the definition of f_i and from (3.6.29) that

$$\widehat{f}(t, x) = f_{E_j}(t, x) = f(t, x, u_{E_j}(t)) \qquad t \in E_j^0. \tag{3.6.31}$$

Define

$$u_\epsilon(t) = u_{E_j}(t) \qquad \text{if } t \in E_j^0.$$

Then since u_{E_j} is one of the $u_1, \ldots, u_{n+2}$ and each u_i satisfies $u_i(t) \in \Omega(t)$ a.e. on $\mathcal{I}_1$ it follows that $u_\epsilon(t) \in \Omega(t)$ on $\mathcal{I}_1$ a.e. From the definition of u_ϵ and (3.6.31) we get

$$\widehat{f}(t,x) = f(t,x,u_\epsilon(t)).$$

Consider the system

$$\frac{dx}{dt} = f(t,x,u_\epsilon(t)) = \widehat{f}(t,x) \tag{3.6.32}$$

with initial point (t_0, x_0). Since f satisfies (3.6.26) it follows that through each point (t_2, x_2) in the interior of $\mathcal{I} \times \mathcal{X}$, there passes a unique solution of (3.6.32), provided we extend u_ϵ to be defined and measurable on $\mathcal{I}$. In particular there exists a unique solution ϕ_ϵ of (3.6.32) with initial point (t_0, x_0). This solution will be defined on some open interval containing t_0 in its interior. Let $\mathcal{I}_{\max} = (a,b)$ denote the maximal interval on which ϕ_ϵ is defined. If $[a,b] \subset \mathcal{I}_1$, then $\limsup_{t\to b} \phi_\epsilon(t)$ must be a boundary point of $\mathcal{X}$; otherwise we could extend the solution ϕ_ϵ to an interval containing $\mathcal{I}_{\max}$ in its interior. This would contradict the maximality of $\mathcal{I}_{\max}$. We shall show that for all t in $\mathcal{I}_{\max}$, the inequality $|\phi_\epsilon(t) - \psi(t)| < \epsilon$ holds. Since $\psi(t) \in \mathcal{X}_1$ for all t in $[t_0, t_1]$ and since $\epsilon < \epsilon_0 = \text{dist } (\partial\mathcal{X}_1, \partial\mathcal{X})$ it will follow that $[a,b] \supset \mathcal{I}_1$ and ϕ_ϵ is defined in all of $\mathcal{I}_1$. Moreover, we shall have $|\phi_\epsilon(t) - \psi(t)| < \epsilon$ for all of $\mathcal{I}_1$.

Since ψ is defined on all of $\mathcal{I}_1$ and $\psi(t_0) = \phi_\epsilon(t_0) = x_0$, we have for all t in $[t_0, b]$

$$\begin{aligned}
|\psi(t) - \phi_\epsilon(t)| &= \left| \int_{t_0}^{t} (\psi'(s) - \phi'_\epsilon(s))ds \right| \\
&= \left| \int_{t_0}^{t} \left(\sum_{i=1}^{n+2} p^i(s) f_i(s,\psi(s)) - \widehat{f}(s,\phi_\epsilon(s)) \right) ds \right| \\
&\le \left| \int_{t_0}^{t} \left(\sum_{i=1}^{n+2} p^i(s) f_i(s,\psi(s)) - \widehat{f}(s,\psi(s)) \right) ds \right| \\
&\quad + \left| \int_{t_0}^{t} (\widehat{f}(s,\psi(s)) - \widehat{f}(s,\phi_\epsilon(s)))ds \right| \\
&< \epsilon' + \int_{t_0}^{t} |\widehat{f}(s,\psi(s)) - \widehat{f}(s,\phi_\epsilon(s))|ds,
\end{aligned}$$

where the last inequality follows from (3.6.30). It now follows from (3.6.31) and (3.6.26) that

$$\int_{t_0}^{t} |\widehat{f}(s,\psi(s)) - \widehat{f}(s,\phi_\epsilon(s))|ds \le \int_{t_0}^{t} \mu(s)|\psi(s) - \phi_\epsilon(s)|ds.$$

Combining this with the preceding inequality gives

$$|\psi(t) - \phi_\epsilon(t)| < \epsilon' + \int_{t_0}^{t} \mu(s)|\psi(s) - \phi_\epsilon(s)|ds.$$

From Lemma 3.6.6, from (3.6.27), and the definition of ϵ' we now conclude that

$$|\psi(t) - \phi_\epsilon(t)| < \epsilon' \exp\left(\int_{t_0}^{t} \mu ds\right) \leq \epsilon' e^K = \epsilon,$$

and the theorem is proved. □

Remark 3.6.10. From the proof of Theorem 3.6.8 we see why we must assume that the functions f_i of Theorem 3.6.1 are measurable in t and continuous in x, rather than continuous in (t, x). Since controls u are only assumed to be measurable, we can only guarantee that the functions f_i defined in (3.6.28) will be measurable in t, no matter how regular we assume the behavior of f to be.

The reason for calling Theorem 3.6.1 the "Chattering Lemma" can now be given. In most applications the functions $f_1, \ldots, f_q$ are obtained as in Theorem 3.6.8. That is, we have a system with state equations $dx/dt = f(t, x, u(t))$, we choose q controls $u_1, \ldots, u_q$, and define functions $f_1, \ldots, f_q$ by means of Eq. (3.6.28). The function f of Theorem 3.6.1 is obtained in the same fashion as the function $\widehat{f}$ of the present theorem. That is, the basic interval $\mathcal{I}$ is divided up into a large number of small intervals and on each subinterval we choose one of the controls $u_1, \ldots, u_q$ to build the control u_ϵ. In a physical system, the control u_ϵ corresponds to a rapid switching back and forth among the various controls $u_1, \ldots, u_q$. In the engineering vernacular, the system is said to "chatter." The control u_ϵ is therefore sometimes called a *chattering control.* In Example 3.1.1, the controls u_r are chattering controls.

From the proof of Theorem 3.6.8 we learn more than just the fact that a relaxed trajectory can be approximated as close as we please by an ordinary trajectory. We learn that the approximation can be effected through the use of a chattering control built from the controls used to define the relaxed control in question.

Remark 3.6.11. The theorem remains valid if we take $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{X}_1$ any compact set.

Chapter 4

Existence Theorems; Compact Constraints

4.1 Introduction

Examples 3.1.1 and 3.1.2 in Section 3.1 of Chapter 3 showed that if the set of admissible directions is not convex, then existence may fail, even though the dynamics and the constraints exhibit regular behavior. Relaxed controls were introduced to provide convexity of the set of admissible directions. If the constraint sets are compact and exhibit a certain regular behavior, then the relaxed controls were shown to have a compactness property that will be used to prove the existence theorems in Section 4.3 and the necessary conditions of later chapters. In the next section we shall present examples of non-existence that illustrate the need for conditions on the behavior of the constraint sets, terminal sets, and dynamics. In Section 4.4 we introduce a convexity condition implying that an optimal relaxed trajectory is an ordinary trajectory, and thus is a solution of the ordinary problem. In Section 4.5 we give examples from applied areas that are covered by the existence theorems of Section 4.4.

In Section 4.6 we present an existence theorem for problems with inertial controllers. Section 4.7 is devoted to problems with system equations linear in the state. For such problems we obtain the deep result that relaxed attainable sets and ordinary attainable sets are equal. One consequence of this is that if the integrand in the payoff function is also linear in the state, then an ordinary optimal solution exists without the requirement that the direction set be convex. Another consequence is the "bang-bang" principle, which holds for problems that are also linear in the control and have compact, convex constraint sets. The "bang-bang" principle states that if the system can reach a point x_0 in time t_1, then this point can also be reached in time t_1 using a control that assumes values at the extreme points of the constraint set.

4.2 Non-Existence and Non-Uniqueness of Optimal Controls

Given a system of equations together with end conditions and control constraints there is no guarantee that admissible pairs exist. The following simple example emphasizes this point.

Example 4.2.1. Let x be one-dimensional. Let the state equation be

$$\frac{dx}{dt} = u(t). \tag{4.2.1}$$

Let $\mathcal{B}$ consist of the single point $(t_0, x_0, t_1, x_1) = (0, 0, 1, 2)$ and let

$$\Omega(t, x) = \{z : |z| \leq 1\}.$$

Thus, an admissible control satisfies the inequality $|u(t)| \leq 1$ for almost every t in $[0, 1]$ and transfers the system from $x_0 = 0$ at time $t_0 = 0$ to the state $x_1 = 2$ at time $t_1 = 1$. From (4.2.1) it is clear that $|\phi(1)| \leq 1$. Thus, the set of admissible pairs is empty.

If the class of admissible controls is not void, it does not necessarily follow that an optimal control exists. The following examples illustrate this point.

Example 4.2.2. Let x be one-dimensional. Let the state equation be $dx/dt = u(t)$. Let $\mathcal{B}$ consist of the single point $(t_0, x_0, t_1, x_1) = (0, 1, 1, 0)$ and let $\Omega(t, x) = \mathbb{R}$. Let

$$J(\phi, u) = \int_0^1 t^2 u^2(t)dt. \tag{4.2.2}$$

The set of controls is the set of functions in $L_1([0, 1])$. To each control u there corresponds a unique trajectory ϕ satisfying $\phi(0) = 1$, namely the trajectory given by

$$\phi(t) = 1 + \int_0^t u(s)ds.$$

For each $0 < \epsilon < 1$ define a control u_ϵ as follows:

$$u_\epsilon(t) = \begin{cases} 0, & \epsilon \leq t \leq 1 \\ -\epsilon^{-1}, & 0 \leq t < \epsilon. \end{cases}$$

Let ϕ_ϵ denote the unique trajectory corresponding to u_ϵ and satisfying $\phi_\epsilon(0) = 1$. Clearly, $(\phi_\epsilon, u_\epsilon)$ is an admissible pair. The class $\mathcal{A}$ of admissible pairs is not void. Moreover,

$$J(\phi_\epsilon, u_\epsilon) = \int_0^\epsilon t^2 \epsilon^{-2} dt = \frac{1}{3}\epsilon.$$

Since $J(\phi, u) \geq 0$ for all admissible pairs (ϕ, u), it follows that $0 = \inf\{J(\phi, u) \mid (\phi, u) \in \mathcal{A}\}$. From (4.2.2) it is clear that $J(\phi, u) = 0$ if and only if $u(t) = 0$ a.e. on $[0, 1]$. However, $u^* = 0$ is not admissible because the corresponding trajectory ϕ^* is identically one and thus does not satisfy the terminal constraint.

In this example, the set of admissible directions is convex, but the constraint set, while constant, is not compact.

Example 4.2.3. Let everything be as in Example 4.2.2 except that the control set is given as follows:

$$\begin{aligned}
&\Omega(t, x) = \{z : |z| \leq 1/t\} \qquad \text{if } 0 < t \leq 1 \\
&\Omega(0, x) = \mathbb{R}.
\end{aligned}$$

The arguments of Example 4.2.2 are still valid and an optimal control fails to exist.

The set of admissible directions is convex. The constraint sets depend on t alone and fail to be compact at the *single point* $t = 0$.

Example 4.2.4. Let everything be as in Example 4.2.2 except that the control set is given as follows:

$$\begin{aligned}
&\Omega(t, x) = \{z : |z| \leq 1/t\} \qquad \text{if } 0 < t \leq 1 \\
&\Omega(0, x) = \{0\}.
\end{aligned}$$

If we now define

$$u_\epsilon(t) = \begin{cases} 0, & \epsilon \leq t \leq 1 \\ -\epsilon^{-1}, & 0 < t < \epsilon \\ 0, & t = 0 \end{cases}$$

and proceed as in Example 4.2.2, we again find that an optimal control fails to exist.

The set of admissible directions is convex, the constraint sets depend on t alone, and for each t the set $\Omega(t)$ is compact. The mapping Ω fails to be u.s.c.i at the *single point* $t = 0$.

In the next example, the terminal set $\mathcal{B}$ is not compact and the time interval is unbounded.

Example 4.2.5. The state equation is $dx/dt = u(t)$, and the control set is $\Omega(t, x) = \{z : |z| \leq 1\}$. Let $\mathcal{B} = \{(t_0, x_0, t_1, x_1) \colon t_0 = 0,\ x_0 = 0,\ x_1 = 1/t_1,\ t_1 > 0\}$. The functional $J(\phi, u)$ is given by $J(\phi, u) = \phi(t_1)$. Let $0 < \widehat{t}$ be arbitrary. Let $u(t) = 0$ for $0 \leq t \leq \widehat{t}$ and then let $u(t) = 1$ until time t_1 defined by $\phi(t_1) = 1/t_1$. Clearly $1/t_1 \leq 1/\widehat{t}$. Since $\widehat{t}$ can be taken arbitrarily large it follows $\inf\{J(\phi, u) \colon (\phi, u) \in \mathcal{A}\} = 0$. However, for no admissible pair (ϕ^*, u^*) is $J(\phi^*, u^*) = 0$.

In the next example an optimal pair fails to exist because the trajectories are not confined to a compact set in (t, x) space. The constraint set is constant and compact and the sets of admissible directions are convex. The dynamics, however, grow as x^2, suggesting that some restriction on the rate of growth of the dynamics is needed.

Example 4.2.6. Let x be a scalar and let the state equation be

$$\frac{dx}{dt} = 2x^2(1-t) - 1 + u(t). \tag{4.2.3}$$

Let $\Omega(t,x) = \{z : |z| \leq 1\}$. Let the end conditions be given by $\mathcal{T}_0 = \{(0, x_0) : 0 \leq x_0 \leq 1\}$ and $\mathcal{T}_1 = \{(a, x_1) : 0 \leq x_1 \leq (1-a)^{-2}\}$, with $a > 1$. Let $J(\phi, u) = -\phi(a)$. Hence, if u is an admissible control and ϕ is a corresponding trajectory it is required to maximize $\phi(a)$ over all admissible pairs (ϕ, u).

The set of admissible controls for this problem is a subset of the measurable functions u on $[0, a]$ such that $|u(t)| \leq 1$ a.e. Since dx/dt is maximized when $u(t) = 1$, we substitute $u(t) = 1$ into the right-hand side of (4.2.3) and we get

$$\frac{dx}{dt} = 2x^2(1-t). \tag{4.2.4}$$

The solution of this differential equation satisfying the initial condition $\phi(0) = x_0$, $x_0 \neq 0$ is

$$\phi(t) = [(1-t)^2 + c]^{-1}, \tag{4.2.5}$$

where $c = (1 - x_0)/x_0$. The solution of (4.2.4) satisfying the initial condition $\phi(0) = 0$ is $\phi(t) \equiv 0$. The field of trajectories corresponding to $u = 1$ is indicated in Figure 4.1. Values of $c \geq 0$ correspond to initial points x_0 in the interval $0 < x_0 \leq 1$. Note that if $x_0 = 1$, then $c = 0$ and $u = 1$ is not admissible.

Let $\mathcal{F}$ denote the field of trajectories corresponding to $u(t) = 1$ and initial conditions $0 \leq x_0 < 1$. Note that $\mathcal{F}$ does not include the trajectory starting from $x_0 = 1$ at $t_0 = 0$. It is clear from (4.2.3) and properties of the field of trajectories $\mathcal{F}$ that if an optimal pair (ϕ^*, u^*) exists and if $\phi^*(0) = x_0 < 1$, then we must have $u^*(t) = 1$ a.e. It then follows from (4.2.5) (see Fig. 4.1) that $u^*(t) = 1$ and $0 \leq x_0 < 1$ cannot be optimal. For if we take a new initial state x_0', where $x_0 < x_0' < 1$, then the solution ϕ of (4.2.4) corresponding to x_0' will give $\phi(a) > \phi^*(a)$. On the other hand, an optimal trajectory cannot have $x_0 = 1$ as initial point. For if $x_0 = 1$, then $u(t) \equiv 1$ is not admissible. Moreover, once we take $u(t) < 1$ on a set E of positive measure the trajectory goes into the interior of $\mathcal{F}$. It is then possible to modify the control so as to increase the value $\phi(a)$. We leave the rigorous formulation of this argument to the reader.

The next example shows that an optimal pair need not be unique.

Example 4.2.7. Let x be one-dimensional. Let the state equation be $dx/dt =$

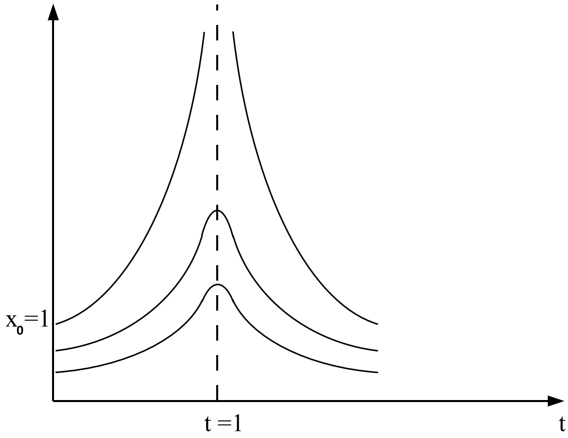

FIGURE 4.1

u. Let $\mathcal{B}$ consist of the single point $(t_0, x_0, t_1, x_1) = (0, 0, 1, 0)$. Let $\Omega(t, x) = \{z : |z| \leq 1\}$, and let

$$J(\phi, u) = \int_0^1 (1 - u^2(t))dt.$$

Clearly, $J(\phi, u) \geq 0$. Define a control u_1^* as follows: $u_1^*(t) = 1$ if $0 \leq t < 1/2$ and $u_1^*(t) = -1$ if $1/2 \leq t \leq 1$. Then, u_1^* is admissible and $J(\phi_1^*, u_1^*) = 0$, where ϕ_1^* is the unique trajectory corresponding to u_1^*. Hence u_1^* is optimal. We now show that there are infinitely many optimal controls. For each integer $n = 1, 2, 3, \ldots$ define a control u_n^* as follows:

$$u_n^*(t) = (-1)^k \quad \text{if } \frac{k}{2^n} \leq t \leq \frac{k+1}{2^n}, \qquad k = 0, 1, 2, \ldots, 2^n - 1.$$

Then, $J(\phi_n^*, u_n^*) = 0$, $n = 1, 2, 3, \ldots$, where ϕ_n^* is the unique trajectory corresponding to u_n^*. Thus, we have nonuniqueness.

4.3 Existence of Relaxed Optimal Controls

In Chapter 3, Definition 3.2.5, we defined a relaxed trajectory corresponding to a relaxed control μ to be an absolutely continuous solution of the differential equation

$$x' = \int_{\Omega(t)} f(t, x, z)d\mu_t.$$

Henceforth we shall simplify our notation by writing the preceding integral as follows:

$$f(t, x, \mu_t) \equiv \int_{\Omega(t)} f(t, x, z) d\mu_t. \tag{4.3.1}$$

Thus, a relaxed trajectory corresponding to a control μ will be a solution of the differential equation

$$x' = f(t, x, \mu_t). \tag{4.3.2}$$

We shall use Greek letters to denote relaxed controls. The subscript t denotes the probability measure μ_t on $\Omega(t)$, that is, the value of μ at t. The subscript notation is used to emphasize that $f(t, x, \mu_t)$ is defined by (4.3.1).

In this section we shall be concerned with functions $f = (f^1, \ldots, f^n)$, where each of the f^i is real valued and defined on a set $\mathcal{I} \times \mathcal{X} \times \mathcal{U}$, where $\mathcal{I}$ is a real compact interval, $\mathcal{X}$ is an open interval in $\mathbb{R}^n$, and $\mathcal{U}$ is an open interval in $\mathbb{R}^m$. These functions are assumed to satisfy the following:

Assumption 4.3.1. (i) Each f^i is measurable on $\mathcal{I}$ for each fixed (x, z) in $\mathcal{X} \times \mathcal{U}$ and is continuous on $\mathcal{X} \times \mathcal{U}$ for each fixed t in $\mathcal{I}$.

(ii) For each compact set K in $\mathcal{I} \times \mathcal{X} \times \mathcal{U}$ there exists a function M_K in $L_2[\mathcal{I}]$ such that

$$|f(t, x, z)| \leq M_K(t)$$

$$|f(t, x, z) - f(t, x', z)| \leq M_K(t)|x - x'|.$$

for all (t, x, z) and (t, x', z) in $\mathcal{I} \times \mathcal{X} \times \mathcal{U}$.

Remark 4.3.2. If f is continuous on $\mathcal{I} \times \mathcal{X} \times \mathcal{U}$, it follows that for each compact set K in $\mathcal{I} \times \mathcal{X} \times \mathcal{U}$, there exists a constant $A_K > 0$ such that

$$|f(t, x, z)| \leq A_K$$

for all (t, x, z) in K.

The weak compactness of relaxed controls will enable us to prove the existence of a relaxed optimal control by following the pattern of the proof of the fact that a real valued continuous function defined on a compact set in a metric space attains a minimum. We begin with a result that will also be used in other contexts.

Lemma 4.3.3. *Let f either (i) be continuous on $\mathcal{I} \times \mathcal{X} \times \mathcal{U}$ or (ii) be a function as in Assumption 4.3.1. Let $\{\varphi_n\}$ be a sequence of continuous functions defined on $\mathcal{I}$ and converging uniformly to a continuous function φ on $\mathcal{I}$. Let $\{\mu_n\}$ be a sequence of relaxed controls with measures μ_{nt} all concentrated on a fixed compact set Z and converging weakly to a relaxed control μ on $\mathcal{I}$. Then for any function g in $L_2[\mathcal{I}]$*

$$\lim_{n \to \infty} \int_{\mathcal{I}} g(t) f(t, \varphi_n(t), \mu_{nt}) dt = \int_{\mathcal{I}} g(t) f(t, \varphi(t), \mu_t) dt.$$

Moreover, for any measurable set $\Delta \subseteq \mathcal{I}$

$$\lim_{n\to\infty} \int_\Delta g(t) f(t, \varphi_n(t), \mu_{nt}) dt = \int_\Delta g(t) f(t, \varphi(t), \mu_t) dt.$$

Proof. We first suppose that f is continuous on $\mathcal{I} \times \mathcal{X} \times \mathcal{U}$. Since the sequence $\{\varphi_n\}$ converges uniformly to φ, all of the points $(t, \varphi_n(t), z)$ and $(t, \varphi(t), z)$, where $t \in \mathcal{I}$ and $z \in Z$, lie in a compact set $K \subseteq \mathcal{I} \times \mathcal{X} \times \mathcal{U}$. Hence since f is uniformly continuous on K, since φ_n converges uniformly to φ, and since each μ_{nt} and μ_t are probability measures, there exists a constant A_K such that

$$|f(t, \varphi_n(t), \mu_{nt})| \leq A_K \text{ and } |f(t, \varphi(t), \mu_t)| \leq A_K \tag{4.3.3}$$

for all n and all t in I. Let

$$\delta_n = \int_{\mathcal{I}} g(t)[f(t, \varphi_n(t), \mu_{nt}) - f(t, \varphi(t), \mu_{nt})]\, dt. \tag{4.3.4}$$

Then

$$\begin{aligned} &\int_{\mathcal{I}} g(t) f(t, \varphi_n(t), \mu_{nt}) dt - \int_{\mathcal{I}} g(t) f(t, \varphi(t), \mu_t) dt \\ &\quad = \delta_n + \int_{\mathcal{I}} g(t)[f(t, \varphi(t), \mu_{nt}) - f(t, \varphi(t), \mu_t)]\, dt \end{aligned} \tag{4.3.5}$$

Since f is uniformly continuous on K and each μ_{nt} is a probability measure on Z, it follows that for each $\varepsilon > 0$ there exists a positive integer $n(\varepsilon)$ such that for $n > n(\varepsilon)$

$$|\delta_n| \leq \varepsilon \|g\|,$$

where $\| \ \|$ denotes the L-norm. Thus, $\delta_n \to 0$ as $n \to \infty$. Also, for each $\varepsilon > 0$ there is a continuous function h_ε on $\mathcal{I}$ such that $\|h_\varepsilon - g\| < \varepsilon$. The integral on the right in (4.3.5) is equal to

$$\begin{aligned} &\int_{\mathcal{I}} [g(t) - h_\varepsilon(t)][f(t, \varphi(t), \mu_{nt}) - f(t, \varphi(t), \mu_t)] dt \\ &+ \int_{\mathcal{I}} h_\varepsilon(t)[f(t, \varphi(t), \mu_{nt}) - f(t, \varphi(t), \mu_t)] dt \end{aligned} \tag{4.3.6}$$

The second integral in (4.3.6) tends to zero as $n \to \infty$ because μ_{nt} converges weakly to μ_t. It follows from (4.3.3) that the absolute value first integral in (4.3.6) does not exceed

$$2A_K \int_{\mathcal{I}} |g(t) - h_\varepsilon(t)| dt \leq 2A_K \|g - h_\varepsilon\| < \varepsilon 2A_K.$$

The first conclusion now follows if (i) holds. We obtain the second conclusion by replacing g by $g\chi_\Delta$, where χ_Δ is the characteristic function of Δ. □

We now suppose that f satisfies Assumption 4.3.1. All of the points $(t, \phi_n(t), z)$ and $(t, \phi(t), z)$, where $t \in I$ and $z \in Z$, lie in a compact set $K \subseteq \mathcal{I} \times \mathcal{X} \times \mathcal{U}$. Hence in place of (4.3.3) we have

$$\begin{aligned} |f(t,x,z)| &\leq M_K(t) \\ |(f(t,x,z) - f(t,x',z)| &\leq M_K(t)|x - x'| \end{aligned} \tag{4.3.7}$$

for all (t, x, z) and (t, x', z) in K. Let δ_n be as in (4.3.4). Then (4.3.5) holds.

Since μ_{nt} is a probability measure, we get from (4.3.7) that for each $\varepsilon > 0$ there exists a positive integer $n(\varepsilon)$ such that for $n > n(\varepsilon)$

$$|\delta_n| \leq \int_{\mathcal{I}} |g(t)| M_K(t) |\phi_n(t) - \phi(t)| dt \leq \varepsilon \|g\| \|M_K\|,$$

where $\| \ \|$ denotes the L_2 norm. Thus, again, $\delta_n \to 0$.

Let I_n denote the integral on the right in (4.3.5). Let

$$h(t,z) = g(t) f(t, \phi(t), z).$$

The function h is defined on an interval $T \times U$, where $T = \mathcal{I}$, a compact interval, and U is a compact interval in $\mathbb{R}^m$ with $Z \subset U \subset \mathcal{U}$. Then

$$I_n = \int_{\mathcal{I}} \left\{ \int_Z h(t,z) d\mu_{nt} - \int_Z h(t,z) d\mu_t \right\} dt.$$

The function h is measurable on $\mathcal{I}$ for fixed z in U and is continuous on U for almost all t in $\mathcal{I}$. By Lemma 3.4.4, for each $\varepsilon > 0$ there exists a closed set $F \subset \mathcal{I}$ with meas$(\mathcal{I} - \mathcal{F}) < \varepsilon$ and a continuous function H on $\mathcal{I} \times \mathcal{U}$ such that $H(t,z) = h(t,z)$ for all t in F and all z in U. Hence

$$\begin{aligned} I_n = &\int_{\mathcal{I}} [H(t,\mu_n) - H(t,\mu)] dt - \int_{\mathcal{I}-\mathcal{F}} \left\{ \int_Z H(t,z) d\mu_{nt} - \int_Z H(t,z) d\mu_t \right\} dt \\ &+ \int_{\mathcal{I}-\mathcal{F}} \left\{ \int_Z h(t,z) d\mu_{nt} - \int_Z h(t,z) d\mu_t \right\}. \end{aligned}$$

Since μ_n converges weakly to μ, the first integral on the right tends to zero as $n \to \infty$. The function H is continuous on the compact set $I \times Z$ and μ and μ_{nt} are probability measures, so the second term on the right is bounded in absolute value, independent of n, by A meas $(\mathcal{I} - \mathcal{F})$, for some constant $A > 0$. Since $|h(t,z)| \leq |g(t)| M_K(t)$, the absolute value of the third integral on the right can be made arbitrarily small, independent of n, by taking ε sufficiently small. Hence for arbitrary $\eta > 0$, $\limsup |I_N| < \eta$, and so $\lim I_n = 0$.

The first conclusion now follows if (ii) holds. To obtain the second conclusion, replace g by $\chi_{\Delta} g$.

Remark 4.3.4. The definition of a sequence of relaxed controls $\{\mu_n\}$ converging weakly to a relaxed control requires that for all t in $\mathcal{I}$, and g in $C(I_t \times Z)$, where $I_t = [0, t]$ that

$$\lim_{n \to \infty} \int_0^t \int_Z g(s,z) d\mu_s ds = \int_0^t \int_Z g(s,z) d\mu_s ds.$$

The proof of Lemma 4.3.3, however, only requires that the preceding holds at $t = 1$. We shall make use of this observation in the sequel.

Theorem 4.3.5. *Let $\widehat{f} = (f^0, f^1, \ldots, f^n)$ be defined on $\mathcal{I} \times \mathcal{X} \times \mathcal{U}$, where $\mathcal{I}$ is a compact interval in $\mathbb{R}^1$, $\mathcal{X}$ is an open interval in $\mathbb{R}^n$, and $\mathcal{U}$ is an open interval in $\mathbb{R}^m$. Let $\widehat{f}$ either satisfy Assumption 4.3.1 or be continuous on $\mathcal{I} \times \mathcal{X} \times \mathcal{U}$. Let $\mathcal{B}$ be a closed set of points (t_0, x_0, t_1, x_1) in $\mathbb{R}^{n+2}$ with $t_0 < t_1$ and with both (t_0, x_0) and (t_1, x_1) in $\mathcal{R} = \mathcal{I} \times \mathcal{X}$. Let g be a real valued lower semi-continuous function defined on $\mathcal{B}$. Let Ω be a mapping from $\mathcal{I}$ to compact sets $\Omega(t)$ contained in $\mathcal{U}$ that is u.s.c.i. on $\mathcal{I}$. Let the set of relaxed admissible pairs be non-empty and be such that all admissible trajectories are defined on all of $\mathcal{I}$ and such that the graphs of these trajectories are contained in a compact subset $\mathcal{R}_0$ of $\mathcal{R} = \mathcal{I} \times \mathcal{X}$. Then there exists an admissible relaxed pair (ψ^*, μ^*) that minimizes*

$$J(\psi, \mu) = g(t_0, \psi(t_0), t_1, \psi(t_1)) + \int_{t_0}^{t_1} f^0(t, \psi(t), \mu_t)dt \tag{4.3.8}$$

over all relaxed admissible pairs.

Remark 4.3.6. In Lemma 4.3.14, which immediately follows the proof of this theorem, we give a sufficient condition for trajectories to be defined on all of I and to lie in a compact set. This condition is not necessary.

Remark 4.3.7. Henceforth, to simplify notation we define

$$e(\psi) \equiv (t_0, \psi(t_0), t_1, \psi(t_1))$$

and call $e(\psi)$ the *end point* of ψ.

Remark 4.3.8. The assumptions that the graphs of admissible trajectories lie in a compact set and that $\mathcal{B}$ is closed imply that the end points $e(\psi)$ of admissible trajectories lie in a compact subset of $\mathcal{B}$. Hence, we may assume $\mathcal{B}$ to be compact to start with.

Proof of Theorem. By virtue of Remark 4.3.8 we take $\mathcal{B}$ to be compact. Since $\mathcal{B}$ is compact and g is lower semi-continuous on $\mathcal{B}$, there exists a constant B such that

$$g(e(\psi)) \geq B \tag{4.3.9}$$

for all admissible ψ. Since Ω is u.s.c.i, it follows from Lemma 3.3.11 that all the sets $\Omega(t)$, where $t \in \mathcal{I}$, are contained in a compact set Z. By hypotheses all points $(t, \psi(t))$ where $t \in I$ and ψ is an admissible relaxed trajectory lie in a compact set $\mathcal{R}_0$ in $\mathcal{I} \times \mathcal{X}$. It then follows that there exists a nonnegative function M in $L_2[\mathcal{I}]$ such that

$$|\widehat{f}(t, \psi(t), z)| \leq M(t) \tag{4.3.10}$$

for all $t \in \mathcal{I}$, ψ admissible and $z \in \Omega(t)$. If f is continuous, then M is a constant.

Let

$$m = \inf\{J(\psi,\mu) \colon (\psi,\mu) \text{ an admissible relaxed pair}\},$$

where $J(\psi,\mu)$ is defined in (4.3.8). From (4.3.9) and (4.3.10) we get that m is finite. Let $\{(\psi_n,\mu_n)\}$ be a sequence of admissible pairs such that

$$\lim_{n\to\infty} J(\psi_n,\mu_n) = m. \tag{4.3.11}$$

Each admissible pair (ψ_n,μ_n) is defined on $\mathcal{I}$ and has a restriction to $I_n = [t_{0n}, t_{1n}]$ such that $e(\psi_n) = (t_{0n}, \psi_n(t_{0n}), t_{1n}, \psi_n(t_{1n}))$ is in $\mathcal{B}$. Since the points $e(\psi_n)$ all lie in the compact set $\mathcal{B}$, there is a subsequence of $\{\psi_n\}$, that we relabel as $\{\psi_n\}$, and a point (t_0, x_0, t_1, x_1) in $\mathcal{B}$ such that the $e(\psi_n) \to (t_0, x_0, t_1, x_1)$. In particular, $t_{0n} \to t_0$ and $t_{1n} \to t_1$.

From

$$\psi_n'(t) = f(t, \psi_n(t), \mu_{nt}) \quad \text{a.e. in } \mathcal{I},$$

(4.3.10) and the fact that μ_{nt} is a probability measure on $\Omega(t)$ we get that for all n and a.e. t in $\mathcal{I}$,

$$|\psi_n'(t)| = |f(t, \psi_n(t),\ \mu_{nt})| \le M(t) \tag{4.3.12}$$

where M is as in (4.3.10). From this and the relation

$$\psi_n(t) - \psi_n(t') = \int_{t'}^{t} \psi_n'(s)ds$$

it follows that the functions $\{\psi_n\}$ are equi-continuous on the compact interval I. Since all trajectories lie in a compact set in $\mathcal{R}$, the functions $\{\psi_n\}$ are uniformly bounded. Hence, by Ascoli's theorem, there is a subsequence of the $\{\psi_n\}$, that we relabel as $\{\psi_n\}$, and a continuous function ψ^* on $\mathcal{I}$ such that

$$\lim_{n\to\infty} \psi_n(t) = \psi^*(t)$$

uniformly on $\mathcal{I}$. By Theorem 3.3.12 there exists a subsequence of the relaxed controls corresponding to $\{\psi_n\}$ that converges weakly to a relaxed control μ^* on $\mathcal{I}$ such that μ_t is concentrated on $\Omega(t)$. Let $\{\mu_n\}$ denote the subsequence. Corresponding to $\{\mu_n\}$ there is a subsequence of the relaxed trajectories that we relabel as $\{\psi_n\}$. In summary, we have a subsequence of admissible pairs $\{(\psi_n,\mu_n)\}$ such that ψ_n converges uniformly on $\mathcal{I}$ to a continuous function ψ^* and μ_n converges weakly to a relaxed control μ^* such that μ_t^* is concentrated on $\Omega(t)$.

From

$$|\psi_n(t_{in}) - \psi^*(t_i)| \le |\psi_n(t_{in}) - \psi^*(t_{in})| + |\psi^*(t_{in}) - \psi^*(t_i)| \quad i = 0, 1,$$

from $t_{in} \to t_i,\ i = 0, 1$, from the uniform convergence of ψ_n to ψ^*, and from the continuity of ψ^* we get that

$$x_i = \lim_{n\to\infty} \psi(t_{in}) = \psi^*(t_i) \quad i = 0, 1. \tag{4.3.13}$$

For each n, the admissible pair (ψ_n, μ_n) in the subsequence satisfies

$$\psi_n(t) = \psi_n(t_{0n}) + \int_{t_{0n}}^{t} f(s, \psi_n(s), \mu_{ns}) ds.$$

If we let $n \to \infty$ and use (4.3.12), (4.3.13), and Lemma 4.3.3, we get that

$$\psi^*(t) = \psi^*(t_0) + \int_{t_0}^{t} f(s, \psi^*(s), \mu_s^*) ds \tag{4.3.14}$$

for $t \in \mathcal{I}$. A similar argument gives

$$\lim_{n\to\infty} \int_{t_{0n}}^{t_{1n}} f^0(s, \psi_n(s), \mu_{n,s}) ds = \int_{t_0}^{t_1} f^0(s, \psi^*(s), \mu_s^*) ds. \tag{4.3.15}$$

From $(t_0, x_0, t_1, x_1) \in \mathcal{B}$, (4.3.13), (4.3.14), and the fact that μ_t^* is concentrated on $\Omega(t)$ we get that (ψ^*, μ^*) is an admissible pair.

Since g is lower semicontinuous

$$\liminf_{n\to\infty} g(e(\psi_n)) \geq g(e(\psi^*)).$$

From this and (4.3.11) and (4.3.15) we get that

$$m = \lim_{n\to\infty} J(\psi_n, \mu_n) \geq J(\psi^*, \mu^*) \geq m.$$

Hence $J(\psi^*, \mu^*) = m$, and the theorem is proved. □

Remark 4.3.9. A sequence $\{(\psi_n, \mu_n)\}$ as in (4.3.11) is called a *minimizing sequence*. From the proof it is clear that we only need to assume that there exists a minimizing sequence that lies in a compact set $\mathcal{R}_0$.

Remark 4.3.10. Theorem 4.3.5 holds if we replace the function g in (4.3.8) by a functional G, where G is a lower semicontinuous functional defined on $C[t_0, t_1]$, the space of continuous functions on $[t_0, t_1]$ with the uniform topology. Clearly the function g is a special case of G. We leave the minor modifications in the proof in the case of G to the reader.

Remark 4.3.11. In some problems the admissible trajectories ψ are required to satisfy the additional constraint $\psi(t) \in C(t)$ for each t, where each $C(t)$ is a closed set. Since an optimal trajectory is the uniform limit of a sequence of admissible trajectories, it follows that an optimal trajectory will also satisfy this constraint.

Remark 4.3.12. In some problems it is possible to show that there is a compact set $\mathcal{R}_0$ such that those trajectories that do not lie in $\mathcal{R}_0$ give larger values to the cost J than do those that lie in $\mathcal{R}_0$. In that event one can ignore the trajectories that do not lie in $\mathcal{R}_0$. One simply redefines $\mathcal{R}$ to be $\mathcal{R}_0$ and redefines the set of admissible pairs to be those pairs whose trajectories lie in $\mathcal{R}_0$.

The proof of Theorem 4.3.5 shows that the following statement is true.

Corollary 4.3.13. *Let the hypotheses of Theorem 4.3.5 hold with the assumption that all trajectories lie in a compact set $\mathcal{R}_0 \subset \mathcal{R}$ replaced by the assumption that there exists a minimizing sequence all of whose trajectories lie in a compact set $\mathcal{R}_0 \subset \mathcal{R}$. Then there exists a relaxed optimal pair (ψ^*, μ^*).*

We next give a sufficient condition for the graphs of ordinary and relaxed trajectories to lie in a compact set. Note that we allow the constraint sets to depend on (t, x) and do not assume that the sets $\Omega(t, x)$ are compact.

Lemma 4.3.14. *Let $\mathcal{R} = \mathcal{I} \times \mathbb{R}^n$. Let $\Delta = \{(t, x, z) : (t, x) \in \mathcal{R}, z \in \Omega(t, x)\}$. Let the function $f = (f^1, \ldots, f^n)$ satisfy*

$$|\langle x, f(t, x, z)\rangle| \leq \Lambda(t)(|x|^2 + 1), \tag{4.3.16}$$

for all (t, x, z) in Δ, where $\Lambda \in L_1(\mathcal{I})$. Let each admissible trajectory ϕ contain at least one point $(\widehat{t}, \phi(\widehat{t}))$ that belongs to a given compact set C in $\mathcal{R}$. Then, there exists a compact set $\mathcal{R}_0$ contained in $\mathcal{R}$ such that each admissible trajectory lies in $\mathcal{R}_0$. If we require that all initial points of admissible trajectories lie in C, then we can omit the absolute value in the left-hand side of (4.3.16).

Proof. For any trajectory ϕ, let $\Phi(t) = |\phi(t)|^2 + 1$. Then, $\Phi'(t) = 2\langle\phi(t), f(t, \phi(t), u(t))\rangle$, and by virtue of (4.3.16)

$$|\Phi'(t)| \leq 2\Lambda(t)(|\phi(t)|^2 + 1) = 2\Lambda(t)\Phi(t).$$

Hence

$$-2\Lambda(t)\Phi(t) \leq \Phi'(t) \leq 2\Lambda(t)\Phi(t). \tag{4.3.17}$$

If $(\widehat{t}, \phi(\widehat{t}))$ is a point of the trajectory that belongs to C, then upon integrating (4.3.17) we get

$$\Phi(t) \leq \Phi(\widehat{t}) \exp\left(2\left|\int_{\widehat{t}}^{t} \Lambda(s) ds\right|\right) \leq \Phi(\widehat{t}) \exp\left(2\int_{\mathcal{I}} \Lambda(s) ds\right)$$

for all points of the trajectory. Since C is compact, there exists a constant D such that if (t, x) is in C then $|x| \leq D$. Hence

$$\Phi(t) \leq (D^2 + 1) \exp\left(2\int_{\mathcal{I}} \Lambda(s) ds\right).$$

Since the right-hand side of this inequality is a constant independent of the trajectory ϕ, it follows that all trajectories lie in some compact set $\mathcal{R}_0$.

If the initial points (t_0, x_0) all lie in a compact set, we need only utilize the rightmost inequality (4.3.16) to obtain a bound on $\Phi(t)$ that is independent of ϕ. $\square$

Corollary 4.3.15. *Under the hypotheses of Lemma 4.3.14 all admissible relaxed trajectories lie in a compact set.*

Example 4.3.16. Let x be one-dimensional with $dx/dt = u(t)$. Let $\Omega(t) = \{z : |z| \le A\}$ where $A > 1$. Let $\mathcal{B}$ consist of a single point $(t_0, x_0, t_1, x_2) = (0, 0, 1, 0)$ and let

$$J(\phi, u) = \int_0^1 [\phi^2(t) + (1 - u^2(t))^2] dt.$$

Then $J(\phi, u) > 0$ for all ϕ, u with $u(t) \in \Omega(t)$ and $\phi(0) = \phi(1) = 0$. To show that the infimum of $J(\phi, u)$ among all such (ϕ, u) equals 0, we define (ϕ_k, u_k) for $k = 1, 2, \ldots$ as follows. For $i = 0, 1, \ldots, 2k - 1$ let

$$u_k(t) = \begin{cases} 1 & \text{if } t \in [i/2k, (i+1)/2k],\ i \text{ even}, \\ -1 & \text{if } t \in [i/2k, (i+1)/2k],\ i \text{ odd}, \end{cases}$$
$$\phi_k(t) = \int_0^t u_k(s) ds.$$

Then $\phi_k(0) = \phi_k(1)$ and $J(\phi_k, u_k) = \frac{1}{3}(2k)^{-2}$, which tends to 0 as $k \to \infty$. Although there is no admissible pair (ϕ, u) with $J(\phi, u) = 0$, the following pair (ψ, μ) is minimizing with $J(\psi, \mu) = 0$. Let

$$\mu_t = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$$
$$\psi(t) = \int_0^t \int_{-A}^{A} z\, d\mu_t ds = \int_0^t \left[\frac{1}{2}(1) + \frac{1}{2}(-1)\right] ds = 0.$$

In this example, the local controllability condition in Theorem 4.4.6 is satisfied.

The corollary follows from the observation in Remark 3.5.8 that a relaxed problem can be considered as an ordinary problem with state equation (3.5.1) and controls $(u_1, \ldots, u_{n+2}, p^1, \ldots, p^{n+2})$ satisfying the constraints in Eq. (3.5.2). If (4.3.16) holds for the ordinary problem, then (4.3.16) will hold for the relaxed problem viewed as an ordinary problem.

Corollary 4.3.17. *Either let Assumption 4.3.1 hold or let f be continuous on $\mathcal{I} \times \mathcal{X} \times \mathcal{U}$ and let the hypotheses of Lemma 4.3.14 hold. Let there exist a control $\overline{u}$ defined on an interval $I \subset \mathcal{I}$. Then a corresponding ordinary or relaxed admissible trajectory defined on I can be extended to a trajectory defined on all of $\mathcal{I}$.*

Proof. As noted in the proof of Corollary 4.3.15, it suffices to consider ordinary admissible trajectories. Let $I = [0, a]$ and let (ϕ, u) be an admissible trajectory-control pair defined on a maximal open interval $I_{\max} = (\alpha, \beta)$. If we set $\widehat{u}(t) = u(t)$ for t in $I_{\max}$ and $\widehat{u}(t) = \overline{u}(t)$ for all other t in I, then we may consider $(\phi, \widehat{u})$ to be an admissible pair. Suppose that $\alpha > 0$ and let $\{t_n\}$ be a sequence of points in $I_{\max}$ such that $t_n \to \alpha$. Since the graph of ϕ lies in a compact set, there is a subsequence of $\{t_n\}$ and a point x_0 such that $\phi(t_n) \to x_0$.

It follows from standard existence and uniqueness theorems for differential equations that if f satisfies Assumption 4.3.1, then

$$x' = f(t, x, \widehat{u}(t)) \quad x(t_0) = x_0$$

has a unique solution on an open interval containing t_0. If f is assumed to be continuous, then a solution exists but need not be unique. This contradicts the maximality of (α, β), so we must have $\alpha = 0$. Similarly we have that $\beta = a$ and the desired extension of (ϕ, u). □

4.4 Existence of Ordinary Optimal Controls

Lemma 4.4.1. *If (ϕ^*, u^*) is an ordinary admissible pair that is a solution of the relaxed problem, then (ϕ^*, u^*) is a solution of the ordinary problem. Moreover, the minima of the relaxed problem and the ordinary problem are equal.*

Proof.

$$\begin{aligned} J(\varphi^*, u^*) &= \inf\{J(\psi, \mu)\colon (\psi, \mu) \text{ relaxed admissible}\} \\ &\leq \inf\{J(\phi, \mu)\colon (\phi, u) \text{ ordinary admissible}\} \\ &\leq J(\phi^*, u^*). \end{aligned}$$

We introduce a convexity condition guaranteeing the existence of an optimal relaxed control that is an optimal ordinary control. Let $\widehat{f} = (f^0, f^1, \ldots, f^n)$ and let

$$\begin{aligned} Q(t,x) &= \{\widehat{y} = (y_0, y)\colon \widehat{y} = \widehat{f}(t,x,z),\ z \in \Omega(t)\} \\ Q^+(t,x) &= \{\widehat{y} = (y_0, y)\colon y^0 \geq f^0(t,x,z),\ y = f(t,x,z),\ z \in \Omega(t)\}. \end{aligned}$$

□

Theorem 4.4.2. *Let the hypothesis of Theorem 4.3.5 hold and let the sets $Q^+(t,x)$ be convex. Then there exists an ordinary admissible pair that is optimal for both the ordinary and relaxed problem.*

Proof. We first prove the theorem under the assumption that $\widehat{f}$ is continuous. By Theorem 4.3.5 there exists a relaxed optimal pair (ψ, μ) on an interval $[t_0, t_1]$. Define

$$\psi^0(t) = \int_{t_0}^{t} f^0(s, \psi(s), \mu_s)ds \quad t_0 \leq t \leq t_1$$

and set $\widehat{\psi} = (\psi^0, \psi)$. Then

$$\widehat{\psi}' = \widehat{f}(t, \psi(t), \mu_t) \qquad \text{a.e.}$$

Hence by Theorem 3.2.11 there exist measurable functions $u_1, \ldots, u_{n+2}$ defined on $[t_0, t_1]$ such that each $u_i(t) \in \Omega(t)$ a.e. and real valued functions $p^1, \ldots, p^{n+2}$ defined on $[t_0, t_1]$ such that

$$p^i(t) \geq 0 \qquad \sum_{i=1}^{n+2} p^i(t) = 1$$

with the property that

$$\widehat{\psi}'(t) = \sum_{i=1}^{n+2} p^i(t) \widehat{f}(t, \psi(t), u_i(t)) \quad \text{a.e. on } [t_0, t_1].$$

Thus, $\widehat{\psi}'(t) \in \operatorname{co} Q(t, \psi(t))$. Since $Q(t, \psi(t)) \subseteq Q^+(t, \psi(t))$ and since $Q^+(t, \psi(t))$ is convex, we have

$$\widehat{\psi}'(t) \in \operatorname{co} Q(t, \psi(t)) \subseteq \operatorname{co} Q^+(t, \psi(t)) = Q^+(t, \psi(t)).$$

Therefore, for a.e. t in $[t_0, t_1]$ there exists a $z(t) \in \Omega(t)$ such that

$$\begin{aligned} \psi^{0\prime}(t) &\geq f^0(t, \psi(t), z(t)) \\ \psi'(t) &= f(t, \psi(t), z(t)). \end{aligned} \tag{4.4.1}$$

We shall show, using Filippov's Lemma, that (4.4.1) holds with z replaced by a measurable function v. Assume for the moment that (4.4.1) holds with $z(t)$ replaced by $v(t)$, where v is measurable and $v(t) \in \Omega(t)$. The second equation in (4.4.1) shows that ψ is the trajectory corresponding to the control v. The first equation shows that the function $t \to f^0(t, \psi(t), v(t))$ is integrable. Since $v(t) \in \Omega(t)$ and $e(\psi) \in \mathcal{B}$, it follows that (ψ, v) is an ordinary admissible pair for the relaxed problem and is optimal for the relaxed problem. The theorem in the case of continuous $\widehat{f}$ now follows from Lemma 4.4.1.

We use the theorem established under the assumption that $\widehat{f}$ is continuous to prove the theorem under the more general assumption that $\widehat{f}$ is measurable on I and continuous on $\mathcal{X} \times \mathcal{U}$ by the argument used in the analogous situation in the proof of Lemma 3.2.10.

To complete the proof we must establish the existence of the measurable function v. With reference to Filippov's Lemma, let $T = \{t \colon \widehat{\psi}'(t) \in Q^+(t, \psi(t)\}$. Let $Z = \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. Let $\mathcal{R}_0$ denote the compact set containing the graphs of all relaxed trajectories. Let $\Delta = \{(t, x, z) \colon (t, x) \in \mathcal{R}_0, z \in \Omega(t)\}$ and let

$$D = \{(t, x, z, \eta) \colon (t, x, z) \in \Delta,\ \eta \geq f^0(t, x, z)\}.$$

The set T is Lebesgue measurable and thus is a measure space. The set Z is clearly Hausdorff. Since Ω is u.s.c.i., it follows from Lemma 3.3.11 that Δ is compact. From this and the continuity of f^0 we get that D is closed. If D is bounded, then D is compact. Otherwise D is the countable union of compact sets D_i, where each D_i is the intersection of D with the closed ball centered at the origin with radius i.

Let Γ denote the mapping from T to Z defined by $\Gamma(t) = (t, \psi(t), \psi'(t), \psi^{0'}(t))$. Since each of the functions $\widehat{\psi}$ and $\widehat{\psi}'$ is measurable, so is Γ. Let φ denote the mapping from D to Z defined by

$$\varphi(t, x, z, \eta) = (t, x, f(t, x, z), \eta).$$

Since f is continuous, so is φ. From (4.4.1) we get that $\Gamma([t_0, t_1]) \subseteq \varphi(D)$. Hence all the hypotheses of Filippov's Lemma hold, and so there exists a measurable map m from T to D

$$m \colon t \to (\tau(t), x(t), v(t), \eta(t))$$

such that

$$\begin{aligned}\varphi(m(t)) &= (\tau(t), x(t), f(\tau(t), x(t), v(t)), \eta(t)) \\ &= \Gamma(t) = (t, \psi(t), \psi'(t),\ \psi^{0'}(t)).\end{aligned}$$

Hence

$$\psi'(t) = f(t, \psi(t), v(t)), \quad \psi^{0'}(t) \geq f^0(t, \psi(t), v(t)),$$

where v is measurable.

In the proof of Theorem 4.4.2 we used the hypotheses of Theorem 4.3.5 to obtain the existence of an optimal relaxed control. The hypothesis that Ω is u.s.c.i. was used in Theorem 4.3.5 and in the proof of Theorem 4.4.2 to show that the set D is closed. To show that D is closed, however, we can impose a less restrictive assumption on the constraint map Ω, namely that it is *upper semicontinuous*. The definition of upper semicontinuity of a set valued mapping is given in Definition 5.2.1. It follows from Lemma 5.2.4 that if Ω is upper semicontinuous, then the set Δ is closed. From this and the continuity of $\widehat{f}$ it follows that D is closed. $\square$

If we are considering a problem with non-compact constraints and take relaxed controls and trajectories to be as given in Section 3.5, the proof of Theorem 4.4.2 and the observations of the preceding paragraph yield the following corollary.

Corollary 4.4.3. *Let $\widehat{f}$ be continuous and let the mapping Ω be upper semicontinuous. Let the relaxed problem have a solution (ψ, μ). If the sets $Q^+(t, x)$ are convex, then there exists an ordinary control that is optimal for both the ordinary and the relaxed problem.*

The situation in Lemma 4.4.1 and Theorem 4.4.2 does not always hold. Both the relaxed and ordinary problems can have solutions with the minimum of the relaxed problem strictly less than the minimum of the ordinary problem, as the next example shows.

Example 4.4.4. Let the state equations be

$$\begin{aligned} dx^1/dt &= (x^2)^2 - (u(t))^2 \\ dx^2/dt &= u(t) \\ dx^3/dt &= (x^2)^2. \end{aligned} \tag{4.4.2}$$

Let the constraints be given by $\Omega(t) = \{z \colon |z| \leq 1\}$, and let

$$\mathcal{B} = \{(t_0, x_0, t_1, x_1) \colon t_0 = 0,\ x_0 = 0,\ t_1 = 1,\ x_1^3 = 0\}. \tag{4.4.3}$$

Let $f^0 = 0$ and let $g(t_0, x_0, t_1, x_1) = x_1^1$. Thus, the problem is to minimize $\phi^1(1)$ over all admissible pairs (ψ, u).

From the third equation in (4.4.2) we see that to satisfy the end conditions $\phi^3(0) = 0$ and $\phi^3(1) = 0$ we must have $\phi^2(t) \equiv 0$. From the second equation in (4.4.2) we then get $u(t) \equiv 0$. Hence, from the first equation and the initial condition $x_0 = 0$ we get that $\phi^1(t) \equiv 0$. Thus, $\phi \equiv 0$ and $u \equiv 0$ is the only admissible pair, and so is optimal. Moreover, $J(\phi, u)) = 0$.

The ordinary problem has a solution even though the sets $Q^+(t, x) = (\eta^0, \eta^1, \eta^2, \eta^3)$, where

$$\eta^0 \geq 0,\ \eta^1 = (x^2)^2 - z^2,\ \eta^2 = z,\ \eta^3 = (x^2)^2, \qquad |z| \leq 1$$

are not convex. This follows from the fact that the sets

$$P(t, x) \equiv \{(\eta^1, \eta^2) \colon \eta^1 = (x^2)^2 - z^2,\ \eta^2 = z, \qquad |z| \leq 1\}$$

are those points on the parabola $\eta^1 = (x^2)^2 - (\eta^2)^2$ in the (η^1, η^2) plane with $|\eta^2| \leq 1$.

The relaxed problem corresponding to (4.4.2) has state equations

$$\begin{aligned} dx^1/dt &= (x^2)^2 - \sum_{i=1}^{4} p^i(t)(u_i(t))^2 \\ dx^2/dt &= \sum_{i=1}^{4} p^i(t) u_i(t) \\ dx^3/dt &= (x^2)^2. \end{aligned} \tag{4.4.4}$$

From the first equation in (4.4.4) we see that $\inf J(\psi, u) = \inf \psi^1(1) \geq -1$. If we take

$$u_1(t) = \quad 1 \qquad u_2(t) = \ -1 \qquad u_3(t) = 0 \qquad u_4(t) = 0$$

$$p^1(t) = 1/2 \qquad p^2(t) = 1/2 \qquad p^3(t) = 0 \qquad p^4(t) = 0,$$

then (4.4.4) becomes

$$dx^1/dt = (x^2)^2 - 1 \qquad dx^2/dt = 0 \qquad dx^3/dt = (x^2)^2. \tag{4.4.5}$$

The solution of the system (4.4.5) satisfying the end conditions (4.4.3) is

$$\psi^1(t) = -t \qquad \psi^2(t) = 0 \qquad \psi^3(t) = 0. \tag{4.4.6}$$

Since $\psi^1(1) = -1$, we have a solution of the relaxed problem. Note that the solution of the relaxed problem is not a solution of the ordinary problem and note that the minimum of the relaxed problem is strictly less than the minimum of the ordinary problem.

The Chattering Lemma asserts that there exists a sequence of ordinary trajectories $\{\phi_n\}$ that converge uniformly to ψ. The trajectories $\{\phi_n\}$ need not be admissible, as they may fail to satisfy the end conditions. We now exhibit such a sequence.

For each positive integer n, partition $[0, 1]$ into 2^n equal subintervals and alternately take $u_n(t) = 1$ and $u_n(t) = -1$ on these subintervals. Then for any solution of (4.4.2) with $\phi^2_n(0) = 0$, we have $\phi^2_n(t) \le 2^{-n}$ and $0 \le \phi^3_n(t) \le 4^{-n}$ on $[0, 1]$. Thus, $\phi^2_n(t) \to 0$ and $\phi^3_n(t) \to 0$, uniformly on $[0, 1]$. From the first equation of (4.4.2) and the initial condition $\phi^1_n(0) = 0$ we have

$$\phi^1_n(t) = -t + \int_0^t [\phi^2_n(s)]^2 ds.$$

From this and the estimate $0 \le \phi^2_n(t) \le 2^{-n}$, we get that $\phi^1_n(t) \to -t$, uniformly on $[0, 1]$. Thus, the sequence $\{\phi_n\}$ defined above is the desired sequence. Note that the sequence is not admissible because $\phi^3_n(1) > 0$.

We conclude this section with a result that shows that the situation illustrated by Example 4.4.4 does not occur if a "local controllability" condition is assumed. For simplicity, we assume that a fixed initial point (t_0, x_0) and a terminal set $\mathcal{T}_1$ are given. Thus, $\mathcal{B} = \{(t_0, x_0, t_1, x_1) : (t_1, x_1) \in \mathcal{T}_1\}$.

Definition 4.4.5. The system is locally controllable at $\mathcal{T}_1$ if, for each $(t_1, x_1) \in \mathcal{T}_1$ there exists a neighborhood N of (t_1, x_1) such that the following property holds. If $(\tau, \xi) \in N$ with $\tau < t_1$, then there exists a control $\widetilde{u}(t)$ on $[\tau, t_1]$ and a corresponding solution $\widetilde{\phi}(t)$ to (2.3.1) with $\widetilde{u}(t) \in \Omega(t)$ and $\widetilde{\phi}(\tau) = \xi$, $\widetilde{\phi}(t_1) = x_1$.

Theorem 4.4.6. *If the system is locally controllable at $\mathcal{T}_1$, then the infimum of $J(\phi, u)$ among admissible ordinary controls equals the minimum of $J(\psi, \mu)$ among admissible relaxed controls.*

Proof. We sketch a proof, leaving certain details to the reader. Consider the

equivalent form of the control problem in Section 2.4, with augmented state $\widehat{\phi}(t) = (\phi^0(t), \phi(t))$ where

$$\phi^0(t) = \int_{t_0}^{t} f^0(s, u(s), \phi(s))ds.$$

Similarly, in the relaxed control problem (Definition 3.2.5) the augmented state is $\widehat{\psi}(t) = (\psi^0(t), \psi(t))$ where

$$\psi^0(t) = \int_{t_0}^{t} \int_{\Omega(t)} f^0(s, \psi(s), d\mu_s ds.$$

The criterion to be minimized is

$$J(\widehat{\phi}, u) = g(t_1, \phi(t_1)) + \phi^0(t_1)$$

and in the relaxed problem

$$J(\widehat{\psi}, \mu) = g(t_1, \psi(t_1)) + \psi^0(t_1).$$

Let $(\widehat{\psi}^*, \mu^*)$ be an admissible relaxed pair, which minimizes $J(\widehat{\psi}, \mu)$ as in Theorem 4.3.5. Let $x_1 = \psi^*(t_1)$ and note that $(t_1, x_1) \in \mathcal{T}_1$. Since $(\widehat{\psi}^*, \mu^*)$ is minimizing,

$$J(\widehat{\psi}^*, \mu^*) \leq J(\widehat{\psi}, \mu)$$

for all admissible pairs $(\widehat{\psi}, \mu)$ and in particular for all ordinary pairs $(\widehat{\phi}, u)$. To prove the theorem, we must show that, for any $a > 0$, there is an admissible pair $(\widehat{\phi}, u)$ such that

$$J(\widehat{\phi}, u) < J(\widehat{\psi}^*, \mu^*) + a.$$

We apply Theorem 3.6.8 to the augmented formulation of the problem. For each $\epsilon > 0$ there exists an ordinary control pair $(\widehat{\phi}_\epsilon, u_\epsilon)$ with $\phi_\epsilon(t_0) = x_0$, $\phi^0_\epsilon(t_0) = 0$, and

$$|\widehat{\phi}_\epsilon(t) - \widehat{\psi}^*(t)| < \epsilon \quad \text{for } t_0 \leq t \leq t_1.$$

Let $\tau = t_1 - \delta$ with $\delta > 0$, $\xi = \phi_\epsilon(t_1 - \delta)$, and $x_1 = \psi^*(t_1)$ as in Definition 4.4.5. Let

$$\begin{aligned} &u(t) = u_\epsilon(t), \quad \phi(t) = \phi_\epsilon(t), \quad \text{if } t_0 \leq t < t_1 - \delta, \quad \text{and} \\ &u(t) = \widetilde{u}(t), \quad \phi(t) = \widetilde{\phi}(t), \quad \text{if } t_1 - \delta < t \leq t_1. \end{aligned}$$

Note that $\widehat{\phi}_\epsilon(t) = \widehat{\phi}(t)$ if $t_0 \leq t \leq t_1 - \delta$. Then

$$\begin{aligned} J(\widehat{\phi}, u) - J(\widehat{\psi}^*, \mu^*) &= \phi^0(t_1) - \psi^{*0}(t_1) \\ &\leq |\phi^0(t_1) - \phi^0_\epsilon(t - \delta)| + |\phi^0_\epsilon(t_1 - \delta) - \psi^{*0}(t_1 - \delta)| \\ &\quad + |\psi^{*0}(t_1 - \delta) - \psi^{*0}(t_1)|. \end{aligned}$$

Since $\Omega(t)$ is a subset of a fixed compact set Z, the right side is less than a if ϵ and δ are chosen small enough. $\square$

4.5 Classes of Ordinary Problems Having Solutions

We now point out important classes of problems where Theorem 4.4.2 is applicable. We suppose that the hypotheses concerning $\mathcal{R}_0, \Omega, g$, and $\mathcal{B}$ hold.

The first class of problems is the linear problems. The state equations are

$$\frac{dx}{dt} = A(t)x + B(t)u(t) + h(t), \tag{4.5.1}$$

where A is an $n \times n$ matrix continuous on some interval $\mathcal{I} = [T_0, T_1]$, B is an $n \times m$ matrix continuous on $\mathcal{I}$, and h is an n-dimensional column vector continuous on $\mathcal{I}$. The cost functional J is given by

$$J(\phi, u) = g(e(\phi)) + \int_{t_0}^{t_1} \{\langle a_0(t), \phi(t)\rangle + \langle b_0(t), u(t)\rangle + h_0(t)\}dt,$$

where a_0, b_0, and h_0 are continuous functions on $\mathcal{I}$. Here the set $\mathcal{R}$ is the slab $T_0 \leq t \leq T_1$, $-\infty < x^i < \infty$ in (t, x) space and $\mathcal{U}$ is all of $\mathbb{R}^m$. The function $\widehat{f} = (f^0, f)$ is clearly continuous. The function f also satisfies (4.3.16), so that if all trajectories are required to pass through a fixed compact set, Lemma 4.3.14 guarantees that all trajectories lie in a fixed compact set $\mathcal{R}_0$. This would be the case if we assume $\mathcal{B}$ to be compact, or that the set of initial points (t_0, x_0) in $\mathcal{B}$ is compact. If we further assume that each of the sets $\Omega(t)$ is convex, then it is readily checked that the sets $Q^+(t, x)$ are convex, and hence an ordinary optimal pair exists. We shall see in Section 4.7 that an ordinary optimal pair exists, *even if the sets* $\Omega(t)$ *are not convex.*

An important problem in the class of linear problems is the "*time optimal problem with linear plant.*" In this problem, the state equations are of the form (4.5.1) and it is required to bring the system from a given initial position x_0 at a given initial time t_0 to a given terminal position x_1 in such a way as to minimize the time to carry this out. The regulator problem of Section 1.5, Chapter 1 is an example of such a problem. If t_1 denotes the time at which the trajectory reaches x_1, then we wish to minimize $t_1 - t_0$, and the cost functional becomes $J(\phi, u) = t_1 - t_0$. Then, we can consider J as being obtained either by setting $g(t_0, x_0, t_1, x_1) = t_1 - t_0$ and $f^0 \equiv 0$ or by setting $g \equiv 0$ and $f^0 \equiv 1$.

Another class of problems to which Theorem 4.4.2 can be applied is the so-called class of problems with "*linear plant and convex integral cost criterion.*" In these problems, the state equations are given by (4.5.1) and the cost functional is given by

$$J(\phi, u) = g(t_0, \phi(t_0), t_1, \phi(t_1)) + \int_{t_0}^{t_1} f^0(t, \phi(t), u(t))dt, \tag{4.5.2}$$

f^0 is continuous or satisfies Assumption 4.3.1 and is a convex function of z for each (t, x) in $\mathcal{R}$. If the constraint sets $\Omega(t)$ are convex, then the sets $Q^+(t, x)$

will be convex. To see this, let $\widehat{y} = (y_1^0, y_1)$ and $\widehat{y}_2 = (y_2^0, y_2)$ be two points in $Q^+(t,x)$. Then there exist points z_1 and z_2 in $\Omega(t)$, such that

$$y_1^0 \geq f^0(t,x,z_1) \quad y_1 = A(t)x + B(t)z_1 + h(t) \tag{4.5.3}$$

$$y_2^0 \geq f^0(t,x,z_2) \quad y_2 = A(t)x + B(t)z_2 + h(t). \tag{4.5.4}$$

Let α and β be two real numbers such that $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta = 1$. If we multiply the relations in (4.5.3) by α, the relations in (4.5.4) by β, and add, we get

$$\begin{aligned} &\alpha y_1^0 + \beta y_2^0 \geq \alpha f^0(t,x,z_1) + \beta f^0(t,x,z_2) \\ &\alpha y_1 + \beta y_2 = A(t)x + B(t)(\alpha z_1 + \beta z_2) + h(t). \end{aligned}$$

Since $\Omega(t)$ is convex, there exists a point z_3 in $\Omega(t)$ such that $z_3 = \alpha z_1 + \beta z_2$. From the convexity of f^0 in z, we get

$$\alpha f^0(t,x,z_1) + \beta f^0(t,x,z_2) \geq f^0(t,x,\alpha z_1 + \beta z_2) = f^0(t,x,z_3).$$

Hence

$$\alpha y_1^0 + \beta y_2^0 \geq f^0(t,x,z_3)$$
$$\alpha y_1 + \beta y_2 = A(t)x + B(t)z_3 + h(t),$$

and so $Q^+(t,x)$ is convex. Thus, by Theorem 4.4.2 the linear plant and convex integral cost criterion problem has a solution in the class of ordinary controls.

An important problem in the class of linear problems with convex integral cost criterion is the minimum fuel problem for linear systems. In this problem a linear system is to be brought from a given initial state x_0 to any state x_1 in a specified set of terminal states in such a way as to minimize the fuel consumed during the transfer. The terminal time can be either fixed or free. The control u is required to satisfy constraints $|u^i(t)| \leq 1$, $i = 1, \ldots, m$. The rate of fuel flow at time t, which we denote by $\beta(t)$, is assumed to be proportional to the magnitude of the control vector as follows:

$$\beta(t) = \sum_{i=1}^{m} c^i |u^i(t)|, \qquad c^i > 0, \text{ constant}.$$

Thus, the fuel consumed in transferring the system from x_0 to x_1 is

$$J(\phi, u) = \int_{t_0}^{t_1} \left(\sum_{i=1}^{m} c^i |u^i(t)| \right) dt.$$

The functional J is to be minimized. Here

$$f^0(t,x,z) = \sum_{i=1}^{n} c^i |z^i|$$

and f^0 is convex in z. The constraint sets $\Omega(t)$ are hypercubes and thus are convex. Theorem 4.4.2 gives the existence of an ordinary optimal control.

Another important problem in the class of linear problems with convex integral cost criterion is the "*quadratic criterion*" problem, which arises in the following way. An absolutely continuous function ζ is specified on a fixed interval $[t_0, t_1]$. This is usually a desired trajectory for the system. It is required to choose an admissible control u so that the mean square error over $[t_0, t_1]$ between the trajectory ϕ and the given trajectory ζ be minimized and that this be accomplished with minimum energy consumption. If one takes the integral $\int_{t_0}^{t_1} |u|^2 dt$ to be a measure of the energy consumption, one is led to consider the cost functional

$$J(\phi, u) = |\phi(t_1) - \xi(t_1)|^2 + \int_{t_0}^{t_1} |\phi(t) - \xi(t)|^2 dt + \int_{t_0}^{t_1} |u(t)|^2 dt.$$

If we set $\bar{\phi}(t) = \phi(t) - \xi(t)$, then since ϕ is a solution of (4.5.1) $\bar{\phi}$ will also be a solution of a linear system of the form (4.5.1). Hence we can suppose that the functional J has the form

$$J(\phi, u) = |\phi(t_1)|^2 + \int_{t_0}^{t_1} |\phi(t)|^2 dt + \int_{t_0}^{t_1} |u(t)|^2 dt.$$

If one assigns nonnegative weights to the coordinates of the trajectory and to the components of u, the functional becomes

$$\begin{aligned} J(\phi, u) = \langle \phi(t_1), R\phi(t_1) \rangle &+ \int_{t_0}^{t_1} \langle \phi(t), X(t)\phi(t) \rangle dt \\ &+ \int_{t_0}^{t_1} \langle u(t), Q(t)u(t) \rangle dt, \end{aligned}$$

where X and Q are continuous diagonal matrices with nonnegative diagonal entries and R is a constant diagonal matrix with nonnegative diagonal entries. If the constraint sets $\Omega(t)$ are convex, we again obtain the existence of an ordinary optimal control.

More generally, we can take X and Q to be continuous positive semi-definite symmetric matrices on $[t_0, t_1]$. Later, when we consider noncompact constraint sets, the matrix Q will be required to be positive definite. The generality in assuming that Q is not diagonal is somewhat spurious, as the following discussion shows. There exists a real orthogonal matrix P such that $D = P^*QP$ where D is diagonal and P^* is the transpose of P. Under the change of variable $v = P^*u$ the quadratic form $\langle u, Qu \rangle$ becomes $\langle v, Dv \rangle$ with D diagonal. The state equations (4.5.1) become

$$\frac{dx}{dt} = A(t)x + C(t)v(t) + h(t),$$

where $C(t) = B(t)P(t)$. If X is a constant matrix, then there is a change of

variable $y = Sx$, where S is orthogonal and constant, such that the quadratic form $\langle x, Xx \rangle$ is replaced by $\langle y, Yy \rangle$, with Y diagonal, and the state equations are transformed into equations that are linear in y and u.

The linear problems and the linear problems with convex integral cost criteria are special cases of the following problem, in which the existence of an optimal ordinary control follows from Theorem 4.4.2.

Corollary 4.5.1. *Let h be a continuous function from $\mathcal{R}$ to $\mathbb{R}^n$, let B be an $n \times m$ continuous matrix on $\mathcal{R}$, and let f^0 be a real valued continuous function on $\mathcal{R} \times \mathcal{U}$ such that for each (t, x) in $\mathcal{R}$, f^0 is convex on $\mathcal{U}$. Let all trajectories intersect a fixed compact set, and let there exist an integrable function L on $\mathcal{I}$ such that*

$$\langle x, h(t,x) + B(t,x)z \rangle \leq L(t)(|x|^2 + 1)$$

for all (t, x, z) in $\mathcal{R} \times \mathcal{U}$. Let the state equations be

$$\frac{dx}{dt} = h(t,x) + B(t,x)z$$

and let $J(\phi, u)$ be given by (4.5.2). *Let each set $\Omega(t)$ be compact and convex and let Ω be u.s.c.i. on $\mathcal{I}$. Then there exists an ordinary pair (ϕ, u) that is optimal for both the ordinary and relaxed problems.*

4.6 Inertial Controllers

An ordinary control is a measurable function and hence there are no restrictions on the rate at which it can vary. That is, the control is assumed to have no inertia. While this may be a reasonable model for electrical systems, it may not be reasonable for mechanical or economic systems. We therefore consider controls that do have inertia. Continuous functions u with piecewise continuous derivatives u' such that $|u'| \leq K$, where K is a constant, appear to be reasonable models of inertial controllers. At points of discontinuity of u' we interpret the inequality $|u'(t)| \leq K$ to hold for both the right- and left-hand limits $u'(t+0)$ and $u'(t-0)$. We now have a bound on the rate at which a control can be changed. It turns out that the class of functions u just described is too restrictive to enable us to prove an existence theorem for problems with inertial controllers. For this purpose it is necessary to take our model of an inertial controller to be an absolutely continuous function u such that $|u'(t)| \leq K$ a.e., where K is a constant independent of u.

We now state a minimization problem for inertial controllers.

Problem 4.6.1. Minimize the functional

$$J(\phi, u) = g(e(\phi)) + \int_{t_0}^{t_1} f^0(t, \phi(t), u(t))dt$$

subject to

$$
\begin{aligned}
&d\phi/dt = f(t, \phi(t), u(t)) \\
&u(t) \in \Omega(t) \\
&(t_0, \phi(t_0), t_1, \phi(t_1)) \in \mathcal{B} \\
&u \text{ is absolutely continuous on } [t_0, t_1] \\
&|u'(t)| \leq K \text{ a.e. on } [t_0, t_1],
\end{aligned}
$$

where K is a pre-assigned constant and g is a functional on the end conditions.

Theorem 4.6.1. *Let the class of admissible pairs for Problem 4.6.1 be nonempty and let the following hold.*

(i) There exists a compact set $\mathcal{R}_0 \subset \mathcal{R}$ such that for all admissible trajectories ϕ we have $(t, \phi(t)) \in \mathcal{R}_0$ for all t in $[t_0, t_1]$.

(ii) The set $\mathcal{B}$ is closed.

(iii) The mapping Ω is u.s.c.i. on $\mathcal{R}_0$.

(iv) For each t in $[t_0, t_1]$ the set $\Omega(t)$ is compact.

(v) The function $\widehat{f} = (f^0, f)$ is continuous and g is lower semicontinuous on $\mathcal{B}$.

Then Problem 4.6.1 has a solution that is also a solution of the relaxed version of Problem 4.6.1.

Proof. We reformulate this problem by taking the control variable z to be a state variable and the derivative of the control to be the control. Thus, we define a new system by

$$
\begin{aligned}
\frac{dx}{dt} &= f(t, x, z) \\
\frac{dz}{dt} &= v(t).
\end{aligned}
\tag{4.6.1}
$$

Let

$$
\begin{aligned}
\widetilde{\mathcal{R}} &= \{(t, x, z) : (t, x) \in \mathcal{R}, z \in \Omega(t)\}, \\
\widetilde{\mathcal{B}} &= \{(t_0, x_0, z_0, t_1, x_1, z_1) \colon (t_0, x_0, t_1, x_1) \in \mathcal{B}, z_i \in \Omega(t_i), i = 0, 1\}, \\
\widetilde{\Omega}(t) &= \{w \in \mathbb{R}^m \colon |w| \leq K\}.
\end{aligned}
$$

The functional to be minimized is

$$
\widetilde{J}(\phi, \zeta, v) = g(e(\phi)) + \int_{t_0}^{t_1} f^0(t, \phi(t), \zeta(t))dt,
$$

where (ϕ, ζ) is a solution of (4.6.1), $(t, \phi(t), \zeta(t)) \in \widetilde{\mathcal{R}}$, $v(t) \in \widetilde{\Omega}(t)$ a.e., and $(t_0, \phi(t_0), \zeta(t_0), t_1, \phi(t_1), \zeta(t_1)) \in \widetilde{\mathcal{B}}$.

By hypothesis, all points $(t, \phi(t))$ of an admissible trajectory lie in a compact set $\mathcal{R}_0 \subseteq \mathcal{R}$. Since $\widetilde{\Omega}(t) = \{w\colon |w| \leq K\}$ for all t, all points $(t, \zeta(t))$ lie in a compact set of (t, z) space. Hence all points $(t, \phi(t), \zeta(t))$ of trajectories of the reformulated problem lie in a compact set in (t, x, z)-space.

The set $\widetilde{\mathcal{B}}$ is closed. To see this let the sequence of points $\{(t_{0n}, x_{0n}, z_{0n}, t_{1n}, x_{1n}, z_{1n})\}$ tend to a point $(t_0, x_0, z_0, t_1, x_1, z_1)$. Then since $\mathcal{B}$ is closed, $(t_0, x_0, t_1, x_1) \in \mathcal{B}$. Let $\varepsilon > 0$ be given. Since the mapping Ω is u.s.c.i, there exists a positive integer n_0 such that for $n > n_0$, $z_{0n} \in [\Omega(t_0)]_\varepsilon$. Hence $z_0 \in [\Omega(t_0)]_\varepsilon$. Since $\Omega(t_0)$ is compact and ε is arbitrary, we get that $z_0 \in \Omega(t_0)$. Similarly, $z_1 \in \Omega(t_1)$, and so $\widetilde{\mathcal{B}}$ is closed.

In the reformulated problem the control variable does not appear in $\widehat{f} = (f^0, f)$. It appears linearly in the second equation in (4.6.1) and satisfies the constraint $\{w\colon |w| \leq K\}$. Therefore, the sets $Q^+(t, x, z)$ of the reformulated problem are convex. All the other hypotheses of Theorem 4.4.2 are clearly fulfilled. It follows from Remark 4.3.10 and the fact that in the present situation an optimal trajectory is also a relaxed optimal trajectory that $\zeta(t) \in \Omega(t)$ for all t. Hence the reformulated problem has a solution, and therefore so does Problem 4.6.1. □

4.7 Systems Linear in the State Variable

A system that is linear in the state variable has state equations

$$\frac{dx}{dt} = A(t)x + h(t, u(t)). \tag{4.7.1}$$

A control problem is linear in the state variable if the state equations are given by (4.7.1) and the payoff is given by

$$J(\phi, u) = g(e(\phi)) + \int_{t_0}^{t_1} \{\langle a_0(t), \phi(t)\rangle + h^0(t, u(t))\}\, dt. \tag{4.7.2}$$

In this section we shall assume the following.

Assumption 4.7.1. (i) The $n \times n$ matrix A and the n-vector a_0 are continuous on some compact interval $\mathcal{I} = [T_0, T_1]$.

(ii) The n-vector h and the scalar h^0 are continuous on $\mathcal{I} \times \mathcal{U}$, where $\mathcal{U}$ is an interval in $\mathbb{R}^m$.

(iii) The terminal set $\mathcal{B}$ is closed and g is continuous on $\mathcal{B}$.

(iv) For each t in $\mathcal{I}$ the set $\Omega(t)$ is a compact subset of $\mathcal{U}$ and the mapping Ω is u.s.c.i.

Lemma 4.7.2. *Let the linear in the state system (4.7.1) satisfy (i), (ii), and (iv) of Assumption 4.7.1. Then there exists a control u defined on I such that $u(t) \in \Omega(t)$ for all t.*

Proof. For each t in I let

$$d(t) = \min\{|z|\, z \in \Omega(t)\}.$$

Since the absolute value is a continuous function and $\Omega(t)$ is compact, we may write min and the minimum is achieved at some point $z_0(t)$ in $\Omega(t)$. We assert that the function d is lower semicontinuous on I. To show this we shall show that for each real α the set

$$E_\alpha = \{t : d(t) \leq \alpha\}$$

is closed. Let $\{t_n\}$ be a sequence of points in E_α tending to a point t_0. Let $z_0(t_n)$ be a point in $\Omega(t_n)$ such that $d(t_n) = |z_0(t_n)|$. Since the mapping Ω is u.s.c.i. on the compact interval I and each $\Omega(t)$ is compact, it follows from Lemma 3.3.11 that the sequence $\{z_0(t_n)\}$ lies in a compact set. Hence there exists a subsequence $\{z_0(t_n)\}$ and a point z_0 in $\mathbb{R}^m$ such that $z_0(t_n) \to z_0$.

We now show that $z_0 \in \Omega(t)$. Let $\varepsilon > 0$ be given. Since Ω is u.s.c.i and $t_n \to t_0$, it follows that there is a positive integer n_0 such that for $n > n_0$, $z_0(t_n) \in [\Omega(t_0)]_\varepsilon$. Hence $z_0 \in [\Omega(t_0)]_\varepsilon$. Since ε is arbitrary and $\Omega(t_0)$ is compact, $z_0 \in \Omega(t_0)$. Since $|z_0(t_n)| = d(t_n) \leq \alpha$, we have $|z_0| \leq \alpha$. Hence $d(t_0) \leq \alpha$, and so E_α is closed.

In summary, we have shown that for each t in I there exists a $z_0(t) \in \Omega(t)$ such that $d(t) = |z_0(t)|$ for all t in I and that the function d is lower semicontinuous and hence measurable on I. Since the absolute value is a continuous function, it follows from Lemma 3.2.10 that there is a measurable function u such that $u(t) \in \Omega(t)$ for t in I and $d(t) = |u(t)|$. From standard theorems on the existence of solutions of linear differential equations, we get that

$$\frac{dx}{dt} = A(t)x + h(t, u(t))$$

has a solution defined on all of I. Hence u is a control. □

To facilitate the study of systems linear in the state we introduce the notion of attainable set.

Definition 4.7.3. Given a control system (not necessarily linear in the state)

$$\frac{d\phi}{dt} = f(t, \phi(t), u(t))$$

and initial condition (t_0, x_0), then the *attainable set at time $t > t_0$, written*

$\mathcal{K}(t; t_0, x_0)$, is the set of points x such that there exists an admissible pair (ϕ, u) with $\phi(t) = x$. The *relaxed attainable set, at time* t, written $\mathcal{K}_R(t; t_0, x_0)$, is the set of points x such that there exists an admissible relaxed pair (ψ, μ) with $\psi(t) = x$.

Since an ordinary trajectory is also a relaxed trajectory, we have that for $t > t_0$,

$$\mathcal{K}(t; t_0, x_0) \subseteq \mathcal{K}_R(t; t_0, x_0).$$

The principal result of this section is that for systems linear in the state $\mathcal{K}(t; t_0, x_0)$ is not empty and $\mathcal{K}(t; t_0, x_0) = \mathcal{K}_R(t; t_0, x_0)$; from this several important consequences will follow. We begin with a result that will be used in our arguments.

Lemma 4.7.4. *Let E be a measurable subset of the line with finite measure. Let y be a function defined on E with values in $\mathbb{R}^k$ and such that $y \in L_1(E)$. Let w be a real valued measurable function defined on E such that $0 \leq w \leq 1$. Then there exists a measurable subset $F \subset E$ such that*

$$\int_E y(t)w(t)dt = \int_F y(t)dt.$$

In the proof of Lemma 4.7.4 and elsewhere in this section we shall need the Krein-Milman theorem, which we state as Lemma 4.7.5. We refer the reader to Dunford-Schwartz [31, Theorem V 8.4, p. 440] for a proof of this theorem. We shall also need, here and elsewhere, a theorem of Mazur, which we state as Lemma 4.7.6. Certain basic facts about the weak-* topology of a Banach space will also be used. For these topics we refer the reader to Dunford-Schwartz [31, Chapter V, pp. 420–439]. A short, readable treatment of some of the topics used can also be found in Hermes and LaSalle [42, pp. 1–22].

Lemma 4.7.5 (Krein-Milman)**.** *Let $\mathcal{C}$ be a compact convex set in a locally convex topological vector space. Then $\mathcal{C}$ is the closed convex hull of its extreme points. If C is a compact convex set in $\mathbb{R}^n$, then C is the convex hull of its extreme points.*

Lemma 4.7.6 (Mazur)**.** *Every strongly closed convex subset of a Banach space is weakly closed.*

Proof of Lemma 4.7.4*.* Define a mapping T as follows. For each real valued function ρ in $L_\infty(E)$, let $T\rho = \int_E y(t)\rho(t)dt$. The mapping T so defined is a continuous mapping from $L_\infty(E)$ with the weak-* topology to $\mathbb{R}^k$ with the euclidean topology. Let $a = \int_E y(t)w(t)dt$. Then $T^{-1}(a)$ is a convex, weak-* closed set of $L_\infty(E)$. Let Σ denote the intersection of $T^{-1}(a)$ and the unit ball in $L_\infty(E)$. Since $w \in \Sigma$, the set Σ is not empty. The weak-* topology is a Hausdorff topology. Therefore, since the unit ball in $L_\infty(E)$ is weak-* compact and $T^{-1}(a)$ is weak-* closed, the set Σ is weak-* compact. It is also convex. Therefore, by the Krein-Milman theorem, Σ has extreme points. We show that

the extreme points of Σ are characteristic functions χ_F of measurable subsets F of E. This will prove the lemma, for then

$$a = \int_E y\chi_F dt = \int_F y dt.$$

The proof proceeds by induction on k, the dimension of the range of y. We shall give the general induction step. The proof for the initial step, $k = 1$, is essentially the same as the proof for the general step and will be left to the reader.

Assume that the lemma is true for $k - 1$. We suppose that θ is an extreme point of Σ and that θ is not a characteristic function of some set F. Then there exists an $\epsilon > 0$ and a measurable set $E_1 \subset E$ with $\text{meas}(E_1) > 0$ such that $\epsilon \leq \theta(t) \leq 1 - \epsilon$ for a.e. t in E_1. Let E_2 and E_3 be two subsets of E_1 such that E_2 and E_3 have positive measure and $E_3 = E_1 - E_2$. From the inductive hypothesis applied to E_2 and E_3 we obtain the existence of measurable sets F_2 and F_3 such that $F_2 \subset E_2$, $F_3 \subset E_3$ and

$$\frac{1}{2}\int_{E_j} y^i(t)dt = \int_{F_j} y^i(t)dt \qquad i = 1, \ldots, k-1, \quad j = 2, 3.$$

Let $h_2 = 2\chi_{F_2} - \chi_{E_2}$ and let $h_3 = 2\chi_{F_3} - \chi_{E_3}$. Then h_2 and h_3 are not identically zero on E_1, do not exceed one in absolute value, and for $i = 1, \ldots, k-1$ satisfy

$$\int_{E_2} y^i h_2 dt = \int_{E_3} y^i h_3 dt = 0.$$

Let $h(t) = \alpha h_2(t) + \beta h_3(t)$ where α and β are chosen so that $|\alpha| < \varepsilon/2$, $|\beta| < \varepsilon/2$, $\alpha^2 + \beta^2 > 0$, and

$$\int_{E_1} y^k h dt = \alpha \int_{E_2} y^k h_2 dt + \beta \int_{E_3} y^k h_3 dt = 0.$$

This can always be done. The function h so defined also satisfies $\int_{E_1} y^i h dt = 0$ for $i = 1, \ldots, k-1$ and $|h(t)| < \epsilon$. Hence $0 < \theta \pm h < 1$ and

$$\int_{E_1} y(\theta \pm h)dt = \int_{E_1} y\theta dt = a.$$

Hence $\theta + h$ and $\theta - h$ are in Σ. But then θ cannot be an extreme point of Σ because it is the midpoint of the segment with end points $\theta + h$ and $\theta - h$. Therefore, θ must be a characteristic function of some set F. □

Theorem 4.7.7. *Let $\mathcal{I}$ be a compact interval $[T_0, T_1]$, let A be an $n \times n$ matrix continuous on $\mathcal{I}$, and let h be a continuous mapping from $\mathcal{I} \times \mathcal{U}$ to $\mathbb{R}^n$, where $\mathcal{U}$ is an interval in $\mathbb{R}^k$. Let Ω be a mapping from $\mathcal{I}$ to compact subsets $\Omega(t)$ of $\mathcal{U}$ that is u.s.c.i. on $\mathcal{I}$. Then for $t_0 \in \mathcal{I}$ and $x_0 \in \mathbb{R}^n$ the sets $\mathcal{K}_R(t; t_0, x_0)$ are nonempty, compact and convex, and $\mathcal{K}_R(t; t_0, x_0) = \mathcal{K}(t; t_0, x_0)$.*

Proof. By Lemma 4.7.2 the sets $\mathcal{K}(t;t_0,x_0)$, and hence $\mathcal{K}_R(t;t_0,x_0)$ are nonempty. By the variation of parameters formula, a relaxed trajectory ψ corresponding to a relaxed control μ is given by

$$\psi(t) = \Psi(t)[x_0 + \int_{t_0}^{t} \Psi^{-1}(s)h(s,\mu_s)ds], \tag{4.7.3}$$

where Ψ is the fundamental matrix solution of the homogeneous system $x' = Ax$ satisfying $\Psi(t_0) = I$, the $n \times n$ identity matrix, and

$$h(s,\mu_s) = \int_{\Omega(s)} h(s,z)d\mu_s.$$

The convexity of $\mathcal{K}_R(t,t_0,x_0)$ is an immediate consequence of (4.7.3). The compactness of $\mathcal{K}_R(t,t_0,x_0)$ follows from (4.7.3) and Theorem 3.3.12.

From Theorem 3.2.11 and Remark 3.5.7 we have that the relaxed system corresponding to (4.7.1) can also be written as

$$\frac{dx}{dt} = A(t)x + \sum_{i=1}^{n+1} p^i(t)h(t,u_i(t)), \tag{4.7.4}$$

where for almost all t in $\mathcal{I}$

$$\sum_{i=1}^{n+1} p^i(t) = 1 \qquad 0 \le p^i(t) \le 1, \quad i = 1,\ldots,n+1 \tag{4.7.5}$$

and $u_i(t) \in \Omega(t)$. Thus, the solution of (4.7.4) that satisfies $\psi(t_0) = x_0$ is defined for all t in $\mathcal{I}$ as is the solution of (4.7.1) that satisfies $\phi(t_0) = x_0$.

Let Θ denote the set of measurable functions θ that satisfy

$$\theta(s) \in co\ h(s,\Omega(s)) \qquad \text{a.e.}, \tag{4.7.6}$$

where co A denotes the convex hull of A. We shall show that there exists a constant A such that

$$|\theta(t)| \le A \quad \text{a.e} \tag{4.7.7}$$

for all θ in Θ and that $\theta \in \Theta$ if and only if

$$\theta(s) = \sum_{i=1}^{n+1} p^i(s)h(s,u_i(s)) \qquad \text{a.e.}, \tag{4.7.8}$$

where $p^1,\ldots,p^{n+1}$ are real valued measurable functions on $[t_0,t]$ satisfying the relations (4.7.5) and where $u_1,\ldots,u_{n+1}$ are measurable functions satisfying $u_i(s) \in \Omega(s)$.

Any function θ satisfying (4.7.8) is measurable and satisfies (4.7.6). Conversely, by Caratheodory's Theorem (Sec. 3.2) any measurable function θ

that satisfies (4.7.6) can be written in the form (4.7.8) without any assertion about the measurability of the functions p^i and u_i, $i = 1, \ldots, n+1$. From Lemma 3.2.10, however, we conclude that the p^i and u_i may be chosen to be measurable.

Since Ω is u.s.c.i. on $\mathcal{I}$, it follows from Lemma 3.3.11 that the set $\{(s,z)\colon s \in \mathcal{I},\ z \in \Omega(s)\}$ is compact. The continuity of h implies that h is bounded on this set. Therefore, (4.7.7) holds. Thus, any measurable function of the form (4.7.6) is integrable. Therefore, (4.7.6) and (4.7.7) give equivalent characterizations of Θ.

From (4.7.8) we see that Θ is contained in a closed ball B of finite radius in $L_2[t_0, t_1]$. We also note that Θ is a convex set in $L_2[t_0, t]$. Since Θ is convex, if we can show that Θ is strongly closed, then by Mazur's Theorem (Lemma 4.7.6), Θ will be weakly closed. The closed ball B is weakly compact, and since $\Theta \subseteq B$, we can conclude that Θ is weakly compact. We now show that Θ is indeed strongly closed.

In the next to the last paragraph we showed that the set $(s, \Omega(s)) = \{(s,z)\colon t_0 \leq s \leq t_1,\ z \in \Omega(s)\}$ is compact. Since h is continuous, the set $h(s, \Omega(s))$ is also compact. Therefore, so are the sets *co* $h(s, \Omega(s))$. Let $\{\theta_k\}$ be a sequence of functions in Θ converging to a function θ_0 in $L_2[t_0, t_1]$. There exists a subsequence, relabeled as $\{\theta_k\}$, such that $\theta_k(s) \to \theta_0(s)$, except possibly on a set E_0 of measure zero. Now $\theta_k(s) \in$ *co* $h(s, \Omega(s))$ except possibly on a set E_k of measure zero. Let $E = \bigcup_{k=0}^{\infty} E_k$. Then for $s \notin E$, $\theta_k(s) \in$ *co* $h(s, \Omega(s))$ for $k = 1, 2, 3, \ldots$ and $\theta_k(s) \to \theta_0(s)$. Since for all $s \notin E$, the set co $h(s, \Omega(s))$ is closed, $\theta_0(s) \in$ *co* $h(s, \Omega(s))$. Thus, $\theta_0 \in \Theta$ and so Θ is strongly closed.

From (4.7.4), (4.7.6), and (4.7.8) it follows that a relaxed trajectory can also be defined as an absolutely continuous function such that

$$\psi'(s) = A(s)\psi(s) + \theta(s)$$

for some function θ in Θ. From this and from the variation of parameters formula we have that

$$\psi(t) = \Psi(t)[x_0 + \int_{t_0}^{t} \Psi^{-1}(s)\theta(s)ds]$$

for some $\theta \in \Theta$. Hence if x is in $\mathcal{K}_R(t; t_0, x_0)$, then

$$x = \Psi(t)[x_0 + \int_{t_0}^{t} \Psi^{-1}(s)\theta(s)ds \tag{4.7.9}$$

for some $\theta \in \Theta$.

Since a trajectory of the system (4.7.1) is also a relaxed trajectory, we have that $\mathcal{K}(t, t_0, x_0) \subseteq \mathcal{K}_R(t, t_0, x_0)$. Therefore, to complete the proof it suffices to show that $\mathcal{K}_R(t, t_0, x_0) \subseteq \mathcal{K}(t, t_0, x_0)$.

Let x be an element of $\mathcal{K}_R(t, t_0, x_0)$. Then x is given by (4.7.9). Let

$$a = \Psi^{-1}(t)x - x_0 = \int_{t_0}^{t} \Psi^{-1}(s)\theta(s)ds.$$

Define a linear mapping T from $L_2[t_0, t]$ to $\mathbb{R}^n$ by the formula

$$T\rho = \int_{t_0}^{t} \Psi^{-1}(s)\rho(s)ds.$$

The mapping T is a continuous map from $L_2[t_0, t]$ to $\mathbb{R}^n$.

The point a is in the set $T(\Theta)$. The set $T^{-1}(a)$ is non-empty, is closed, and is convex in $L_2[t_0, t]$. Let Σ denote the intersection of $T^{-1}(a)$ and Θ. Then Σ is weakly closed and convex. Since Θ is bounded, so is Σ. Since Σ is also strongly closed and convex, by Mazur's Theorem, Σ is also weakly closed. Since Θ is weakly compact, Σ is weakly compact. By the Krein-Milman Theorem (Lemma 4.7.5), Σ has an extreme point θ_0. Since $\theta_0 \in \Theta$ it follows that θ_0 has a representation given by (4.7.8). We now assert that on no measurable subset E of $[t_0, t]$ with positive measure can we have $0 < \epsilon \leq p^i(s) \leq 1 - \epsilon$ for some i in the set $i, \ldots, n+1$ and some $\epsilon > 0$. This assertion implies that θ_0 can be written

$$\theta_0(s) = h(s, u(s)), \tag{4.7.10}$$

where u is a measurable function on $[t_0, t]$ satisfying $u(s) \in \Omega(s)$. Once we establish (4.7.10), the theorem will be proved, for then we will have that

$$a = \Psi^{-1}(t)x - x_0 = \int_{t_0}^{t} \Psi^{-1}(s)h(s, u(s))ds,$$

which says that $x \in \mathcal{K}(t, t_0, x_0)$.

Let us suppose that θ_0 has the form (4.7.8) and that there exists an $\epsilon > 0$ and a measurable set $E \subseteq [t_0, t]$ such that $\text{meas}(E) > 0$ and $\epsilon \leq p^i(t) \leq 1 - \epsilon$ on E for some index i. Then for at least one other index i we must also have $\epsilon \leq p^i(t) \leq 1 - \epsilon$ on E. For definiteness let us suppose that $i = 1$ and $i = 2$ are the indices. Since a belongs to $T(\Theta)$, we get from (4.7.8) that

$$\begin{aligned} a &= \int_{t_0}^{t} \left\{ \Psi^{-1}(s) \sum_{i=1}^{n+1} p^i(s)h(s, u_i(s)) \right\} ds \\ &= \int_{t_0}^{t} \left\{ \sum_{i=1}^{n+1} p^i(s)(\Psi^{-1}(s)h(s, u_i(s)) \right\} ds. \end{aligned}$$

We next apply Lemma 4.7.4 with $w \equiv 1/2$ and y the vector function in $L_1(E)$ with range in $\mathbb{R}^{2n}$ defined by

$$y(s) = \begin{pmatrix} \Psi^{-1}(s)h(s, u_1(s)) \\ \Psi^{-1}(s)h(s, u_2(s)) \end{pmatrix}.$$

We obtain the existence of a set $F \subseteq E$ such that

$$\frac{1}{2} \int_E \Psi^{-1}(s)h(s, u_i(s))ds = \int_F \Psi^{-1}(s)h(s, u_i(s))ds \qquad i = 1, 2.$$

Let χ_F denote the characteristic function of the set F and let χ_E denote the characteristic function of the set E. Let the function γ be defined as follows:

$$\gamma(s) = 2\chi_F(s) - \chi_E(s).$$

The function γ is equal to one in absolute value on the set E. Also, for $i = 1, 2$,

$$\int_{t_0}^{t} \Psi^{-1}(s)h(s, u_i(s))\gamma(s)ds = 2\int_F \Psi^{-1}(s)h(s, u_i)ds - \int_E \Psi^{-1}(s)h(s, u_i(s))ds = 0.$$

Let

$$\begin{array}{llll} \pi_1^1 = p^1 + \epsilon\gamma & \pi_1^2 = p^2 - \epsilon\gamma & \pi_1^i = p^i, & i = 3, \ldots, n+1 \\ \pi_2^1 = p^1 - \epsilon\gamma & \pi_2^2 = p^2 + \epsilon\gamma & \pi_2^i = p^i, & i = 3, \ldots, n+1. \end{array}$$

Then $0 \le \pi_1^i \le 1$ and $0 \le \pi_2^i \le 1$ for $i = 1, \ldots, n+1$. Also, $\Sigma\pi_1^i = 1$ and $\Sigma\pi_2^i = 1$.

Let

$$\theta_1(s) = \sum_{i=1}^{n+1} \pi_1^i(s)h(s, u_i(s))$$

$$\theta_2(s) = \sum_{i=1}^{n+1} \pi_2^i(s)h(s, u_i(s)).$$

Then θ_1 and θ_2 are in Θ and

$$a = \int_{t_0}^{t} \Psi^{-1}(s)\theta_1(s)ds = \int_{t_0}^{t} \Psi^{-1}(s)\theta_2(s)ds.$$

Hence θ_1 and θ_2 are in Σ. But, $\theta_0 = (\theta_1 + \theta_2)/2$, which contradicts the fact that θ_0 is an extreme point, and the theorem is proved. □

An important consequence of Theorem 4.7.7 is that *optimal control problems that are linear in the state have ordinary solutions in the absence of convexity assumptions on the sets* $Q^+(t, x)$.

Theorem 4.7.8. *Let the optimal control problem have state equations (4.7.1) and payoff (4.7.2). Let Assumption 4.7.1 hold and let all trajectories of the system (4.7.1) have some point in a fixed compact set K. Then there exists an ordinary control that is optimal for both the ordinary and relaxed problem.*

Proof. By introducing an additional coordinate x^0 and state equation $dx^0/dt = \langle a_0(t), x\rangle + h^0(t, u(t))$, we may assume that the state equations are linear in the state and the payoff is a terminal payoff. Since the state equations are linear in the state and h is bounded on the set $\{(s, z);\ s \in \mathcal{I},\ z \in \Omega(s)\}$ the growth condition (4.3.16) holds. The assumption that all trajectories intersect a fixed compact set imply (Lemma 4.3.14) that all trajectories lie in a compact set. It is readily checked that all other assumptions of Theorem 4.3.5 hold. Hence a relaxed optimal pair (ψ^*, μ^*) exists on an interval $[t_0, t_1]$. □

Let $m = \inf\{J(\psi,\mu)\colon (\psi,\mu) \text{ admissible}\}$ and let $m_0 = \inf\{J(\phi,u)\colon (\phi,u)$ admissible$\}$. Then, $m \leq m_0$ and

$$m = J(\psi^*,\mu^*) = g(e(\psi^*)).$$

The relation $\mathcal{K}(t_1,t_0,x_0) = \mathcal{K}_R(t_1,t_0,x_0)$ implies that there exists an ordinary pair (ϕ^*,u^*) with $e(\phi^*) = e(\psi^*)$. Hence

$$m = g(e(\psi^*)) = g(e(\phi^*)) \geq m_0 \geq m.$$

Thus $g(e(\phi^*)) = m$ and the pair (ϕ^*,u^*) is an ordinary optimal pair, which is optimal for the relaxed problem.

Another corollary of Theorem 4.7.7 is the so-called "bang-bang principle," which is contained in Theorem 4.7.9. The reason for the terminology and the significance of the principle in applications will be discussed after the proof of Theorem 4.7.9 is given.

If $\mathcal{C}$ is a compact convex set in $\mathbb{R}^m$, then we shall denote the set of extreme points of $\mathcal{C}$ by $\mathcal{C}_e$. By the Krein-Milman Theorem, $\mathcal{C}_e$ is non-void and $\mathcal{C} = co(\mathcal{C}_e)$.

Theorem 4.7.9. *Let $\mathcal{I} = [t_0,t_1]$ be a compact interval in $\mathbb{R}^1$, let A be an $n\times n$ continuous matrix on $\mathcal{I}$, and let B be an $n\times m$ continuous matrix on $\mathcal{I}$. Let $\mathcal{C}$ be a compact convex set in $\mathbb{R}^m$. Let $\mathcal{K}(t_1,t_0,x_0)$ denote the attainable set at t_1 for the system*

$$\frac{dx}{dt} = A(t)x + B(t)u(t) \tag{4.7.11}$$

with initial point (t_0,x_0) and with the control constraint $u(t) \in \mathcal{C}$. Let $\mathcal{K}_e(t_1,t_0,x_0)$ denote the attainable set for the system (4.7.11) *with initial point (t_0,x_0) and with control constraint $u(t) \in \mathcal{C}_e$. Then $\mathcal{K}_e(t_1,t_0,x_0)$ is non-empty and $\mathcal{K}(t_1,t_0,x_0) = \mathcal{K}_e(t_1,t_0,x_0)$.*

Proof. Since the function defined by $u(t) = z_0$, where z_0 is any point of $\mathcal{C}_e$ is admissible for the system (4.7.11) with initial point (t_0,x_0) and control constraint $u(t) \in \mathcal{C}_e$, it follows that $\mathcal{K}_e(t_1,t_0,x_0)$ is non-empty. Since $\mathcal{C}$ is compact and convex, $\mathcal{C} = co(\mathcal{C}_e)$. By Carathéodory's theorem every point in $co(\mathcal{C}_e)$ can be written as a convex combination of at most $(n+1)$ points in $\mathcal{C}_e$. Therefore, any control u such that $u(t) \in \mathcal{C}$ can be written as

$$u(t) = \sum_{i=1}^{n+1} p^i(t)u_i(t),$$

where $0 \leq p^i(t) \leq 1$, $\Sigma p^i(t) = 1$, and $u_i(t) \in \mathcal{C}_e$. By Lemma 3.2.10 the functions p^i and u_i can be chosen to be measurable. Hence the set $\mathcal{K}(t_1,t_0,x_0)$ is contained in the relaxed attainable set $\mathcal{K}_{eR}(t_1,t_0,x_0)$ corresponding to $\mathcal{K}_e(t_1,t_0,x_0)$.

Conversely, every relaxed control for the system (4.7.11) with control constraint $u(t) \in \mathcal{C}_e$ is a control for the system (4.7.11) with control constraint $u(t) \in \mathcal{C}$. Hence

$$\mathcal{K}(t_1, t_0, x_0) = \mathcal{K}_{eR}(t_1, t_0, x_0).$$

It is readily checked that the system (4.7.11) with initial point (t_0, x_0) and control constraint $u(t) \in \mathcal{C}_e$ satisfies the hypotheses of Theorem 4.7.7. Hence $\mathcal{K}_{eR}(t_1, t_0, x_0) = \mathcal{K}_e(t_1, t_0, x_0)$ and the present theorem is established. □

In many applications the constraint set $\mathcal{C}$ is a compact convex polyhedron, or even a cube, in $\mathbb{R}^m$. The set of extreme points $\mathcal{C}_e$ is the set of vertices of the polyhedron, and is therefore closed. Theorem 4.7.9 in this situation states that if a control u with values $u(t) \in \mathcal{C}$ will transfer the system from a point x_0 at time t_0 to a point x_1 at time t_1, then there exists a control u_e with values $u_e(t)$ in $\mathcal{C}_e$ that will do the same thing. Thus, in designing a control system the designer need only allow for a finite number of control positions corresponding to the vertices of $\mathcal{C}$. The term "bang-bang" to describe controls with values on the vertices of polyhedron derives from the case where $\mathcal{C}$ is a one-dimensional interval. In this case, controls u_e with $u_e(t) \in \mathcal{C}_e$ are controls that take on the values $+1$ and -1. Such controls represent the extreme positions of the control device and are therefore often referred to in the engineering vernacular as "bang-bang" controls. In the control literature, the terminology has been carried over to theorems such as Theorem 4.7.9.

Chapter 5

Existence Theorems; Non-Compact Constraints

5.1 Introduction

In this chapter we shall prove existence theorems for ordinary and relaxed versions of Problem 2.3.2, which we restate for the reader's convenience.

Minimize

$$J(\varphi, u) = g(e(\varphi)) + \int_{t_0}^{t_1} f^0(t, \varphi(t), u(t))\, dt \tag{5.1.1}$$

subject to

$$\frac{d\varphi}{dt} = f(t, \varphi(t), u(t)) \tag{5.1.2}$$

and

$$(t_0, \varphi(t_0), t_1, \varphi(t_1)) \in \mathcal{B} \quad u(t) \in \Omega(t, \varphi(t)). \tag{5.1.3}$$

The constraint sets $\Omega(t, x)$ depend on t and x and are not assumed to be convex.

In Chapter 4, when studying this problem, the constraint sets $\Omega(t)$ were assumed to depend on t alone and to be compact. The weak compactness of relaxed controls in this case enabled us to pattern the proof of the existence of an optimal relaxed pair after the proof of the theorem that a real valued continuous function on a compact set attains a minimum. If the constraints are not compact, then a set of relaxed controls need not be compact, as was shown in Remark 3.3.7. To overcome this deficiency when the constraints are not compact, we impose conditions on the data of the problem that guarantee the following. Given a relaxed minimizing sequence $\{(\psi_n, \mu_n)\}$, there exists a subsequence, which we relabel as $\{(\psi_n, \mu_n)\}$, and an admissible pair (ψ, μ) such that $\psi_n \to \psi$ uniformly and

$$\lim_{n\to\infty} \int_{t_0}^{t_1} f^0(t, \psi_n(t), \mu_{nt}) dt \geq \int_{t_0}^{t_1} f^0(t, \psi(t), \mu_t) dt.$$

If we assume that g is a lower semi-continuous mapping from $\mathcal{B}$ to the reals, then we get that (ψ, μ) is optimal by the usual argument. Namely, let

$$m = \inf\{J(\psi, \mu) : (\psi, \mu) \text{ admissible}\}.$$

Then

$$
\begin{aligned}
m &= \lim_{n\to\infty} J(\psi_n,\mu_n) = \liminf_{n\to\infty} J(\psi_n,\mu_n) \qquad (5.1.4)\\
&\geq \liminf_{n\to\infty} g(e(\psi_n)) + \liminf_{n\to\infty}\int_{t_0}^{t_1} f^0(t,\psi_n(t),\mu_{nt})dt\\
&\geq g(e(\psi)) + \int_{t_0}^{t_1} f^0(t,\psi(t),\mu(t))dt = J(\psi,\mu) \geq m.
\end{aligned}
$$

Hence $J(\psi,\mu) = m$, and so (ψ,μ) is optimal.

5.2 Properties of Set Valued Maps

In this section we shall consider regularity properties of set valued maps Λ from a subset X of a euclidean space $\mathbb{R}^p$ to subsets of a euclidean space $\mathbb{R}^q$. The existence theorems of this chapter will involve the regularity of the constraint mapping Ω. The reader may omit this section and return to it as various regularity conditions arise in the sequel.

One of the regularity conditions is *upper semi-continuity with respect to inclusion (u.s.c.i.)*, which was introduced in Definition 3.3.8. In Lemma 3.3.11 we gave a necessary and sufficient condition that a mapping Λ defined on a compact subset of $\mathbb{R}^p$ and whose values are compact sets be u.s.c.i.

We recall notation introduced in Chapter 3. Let ξ_0 be a point in X. Then $N_\delta(\xi_0)$ will denote the δ-neighborhood of ξ_0 relative to X. That is,

$$N_\delta(\xi_0) = \{\xi : |\xi - \xi_0| < \delta,\ \xi \in X\}.$$

By $\Lambda(N_\delta(\xi_0))$, the image of $N_\delta(\xi_0)$ under Λ, we mean

$$\Lambda(N_\delta(\xi_0)) = \bigcup [\Lambda(\xi) : \xi \in N_\delta(\xi_0)].$$

Definition 5.2.1. A mapping Λ is said to be *upper semi-continuous* at a point ξ_0 in X if

$$\bigcap_{\delta>0} \text{cl } \Lambda(N_\delta(\xi_0)) \subseteq \Lambda(\xi_0), \qquad (5.2.1)$$

where *cl* denotes closure. A mapping Λ is upper semi-continuous on a set X if it is upper semi-continuous at every point of X.

Since the inclusion opposite to that in (5.2.1) always holds, we may replace the inclusion in (5.2.1) by equality and obtain an equivalent definition. From this it follows that *if Λ is upper semi-continuous at ξ_0, then $\Lambda(\xi_0)$ must be closed.*

In Example 3.3.10 for both maps Λ_1 and Λ_2 we have $\Lambda_i(N_\delta(0)) = \mathbb{R}^1$, $i = 1, 2$. Hence $\bigcup_{\delta>0}$ cl $N_\delta(\Lambda_i(0)) = \mathbb{R}^1$, $i = 1.2$. Since $\Lambda_1(0) = \mathbb{R}^1$ and $\Lambda_2(0) = 0$, we see that Λ_1 is upper semi-continuous at 0, but Λ_2 is not.

Lemma 5.2.2. *Let Λ be u.s.c.i. at ξ_0 and let $\Lambda(\xi_0)$ be closed. Then Λ is upper semi-continuous at ξ_0.*

Proof. Let $\varepsilon > 0$ be given. Then there exists a $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$

$$\Lambda(\xi_0) \subseteq \Lambda(N_\delta(\xi_0)) \subseteq [\Lambda(\xi_0)]_\varepsilon.$$

Hence, since $[\Lambda(\xi_0)]_\varepsilon$ is closed,

$$\Lambda(\xi_0) \subseteq \text{ cl } \Lambda(N_\delta(\xi_0)) \subseteq \text{ cl } [\Lambda(\xi_0)]_\varepsilon = [\Lambda(\xi_0)]_\varepsilon.$$

Therefore, since $\varepsilon > 0$ is arbitrary and $\Lambda(\xi_0)$ is closed

$$\Lambda(\xi_0) \subseteq \bigcap_{\delta>0} \text{ cl } \Lambda(N_\delta(\xi_0)) \subseteq \Lambda(\xi_0).$$

Thus, equality holds throughout and the upper semi-continuity of Λ at ξ_0 is proved. $\square$

We next give an example of a mapping Λ and a point ξ_0 such that $\Lambda(\xi_0)$ is closed, Λ is upper semi-continuous at ξ_0, but Λ is *not* u.s.c.i. at ξ_0.

Example 5.2.3. Let $t \in \mathbb{R}$ and let

$$\Lambda(t) = \begin{cases} [0,1] \cup \{1/t\} & \text{if } t > 0 \\ [0,1] & \text{if } t = 0. \end{cases}$$

Then for all $t \geq 0$, the set $\Lambda(t)$ is closed and $\Lambda(N_\delta(0)) = [0,1] \cup [1/\delta, \infty)$. Hence Λ is not u.s.c.i. at $t = 0$. On the other hand, Λ is upper semi-continuous at $t = 0$ since $\bigcap_{\delta>0}$ cl $\Lambda(N_\delta(0)) = [0,1] = \Lambda(0)$.

The next lemma is the "closed graph theorem" for upper semi-continuous mappings, and is used in the proofs of existence theorems.

Lemma 5.2.4. *Let X be closed. A necessary and sufficient condition that a mapping Λ be upper semi-continuous on X is that the set $G_\Lambda = \{(\xi, \lambda) : \xi \in X, \lambda \in \Lambda(\xi)\}$ be closed.*

Proof. We first suppose that Λ is upper semi-continuous on X, so that (5.2.1) holds at every point of X. Let $\{(\xi_n, \lambda_n)\}$ be a sequence of points in G_Λ converging to a point (ξ_0, λ_0). Since X is closed, $\xi_0 \in X$. Moreover, for each $\delta > 0$ there exists a positive integer $n(\delta)$ such that for $n > n(\delta)$, $\xi_n \in N_\delta(\xi_0)$. Since $\lambda_n \in \Lambda(\xi_n)$, we have that for $n > n(\delta)$, $\lambda_n \in \Lambda(N_\delta(\xi_0))$. Hence $\lambda_0 \in$ cl $\Lambda(N_\delta(\xi_0))$, for each $\delta > 0$. Thus, $\lambda_0 \in \bigcap_{\delta>0}$ cl $\Lambda(N_\delta(\xi_0))$, and from (2.1) we get that $\lambda_0 \in \Lambda(\xi_0)$. Thus, $(\xi_0, \lambda_0) \in G_\Lambda$, and G_Λ is closed.

Conversely, let G_Λ be closed and let $\lambda_0 \in$ cl $\Lambda(N_\delta(\xi_0))$ for each $\delta > 0$. Then there exists a sequence of points $\{\xi_n\}$ in X, a sequence of positive numbers $\{\delta_n\}$, and a sequence of points $\{\lambda_n\}$ such that the following hold: (i) $\delta_n \to 0$; (ii) $\xi_n \in N_{\delta_n}(\xi_0)$; (iii) $\lambda_n \in \Lambda(\xi_n)$; and (iv) $\lambda_n \to \lambda_0$. Thus, $(\xi_n, \lambda_n) \to (\xi_0, \lambda_0)$. Since X and G_Λ are closed, $\lambda_0 \in \Lambda(\xi_0)$, and therefore (5.2.1) holds. $\square$

Definition 5.2.5. A mapping Λ is said to possess the *Cesari property* at a point ξ_0 if

$$\bigcap_{\delta>0} \text{cl co } \Lambda(N_\delta(\xi_0)) \subseteq \Lambda(\xi_0). \tag{5.2.2}$$

A mapping Λ possesses the Cesari property on a set X if it possesses the property at every point of X.

Since the inclusion opposite to that in (5.2.2) always holds, we may replace the inclusion in (5.2.2) by an equality and obtain an equivalent definition. From this it follows that if Λ *possesses the Cesari property at a point* ξ, *then* $\Lambda(\xi_0)$ *must be closed and convex.*

Remark 5.2.6. If Λ possesses the Cesari property at ξ_0, then it is upper semi-continuous at ξ_0. To see this, note that

$$\Lambda(\xi_0) \subseteq \bigcap_{\delta>0} \text{cl } \Lambda(N_\delta(\xi_0)) \subseteq \bigcap_{\delta>0} \text{cl co } \Lambda(N_\delta(\xi_0)) = \Lambda(\xi_0).$$

The mapping Λ of Example 5.2.3 is upper semi-continuous at 0, but does not have the Cesari property at 0, even though $\Lambda(0)$ is closed and convex. At points ξ near 0 and different from 0, the sets $\Lambda(\xi)$ are not convex.

Lemma 5.2.7. *Let* Λ *be u.s.c.i. at* ξ_0 *and let* $\Lambda(\xi_0)$ *be closed and convex. Then* Λ *has the Cesari property at* ξ_0.

Proof. From the definition of u.s.c.i. we have that for each $\varepsilon > 0$ there exists a δ_0 such that for all $0 < \delta < \delta_0$

$$\Lambda(\xi_0) \subseteq \Lambda(N_\delta(\xi_0)) \subseteq [\Lambda(\xi_0)]_\varepsilon.$$

This chain holds if we take convex hulls of all the sets. Since $\Lambda(\xi_0)$ and $[\Lambda(\xi_0]_\varepsilon$ are closed and convex, we have

$$\Lambda(\xi_0) \subseteq \bigcap_{\delta>0} \text{cl co } \Lambda(N_\delta(\xi_0)) \subseteq [\Lambda(\xi_0)]_\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $\Lambda(\xi_0)$ is closed, we get that $\Lambda(\xi_0) = \bigcap_{\delta>0}$ cl co $\Lambda(N_\delta(\xi_0))$.

The Cesari property will be needed for mappings from (t, x)-space to various euclidean spaces. We shall take the point ξ to be $\xi = (t, x)$. We define an *x-delta neighborhood of* (t_0, x_0), denoted by $N_{\delta x}(t_0, x_0)$ as follows,

$$N_{\delta x}(t_0, x_0) = \{(t_0, x) \text{ in } \mathcal{R} : |x - x_0| < \delta\}.$$

□

Definition 5.2.8. A mapping Λ has the *weak Cesari property* at (t_0, x_0) if

$$\bigcap_{\delta>0} \text{cl co } \Lambda(N_{\delta x}(t_0, x_0)) \subseteq \Lambda(t_0, x_0).$$

We now give an example of a mapping that has the weak Cesari property, but not the Cesari property. Clearly, any map that has the Cesari property has the weak Cesari property.

Example 5.2.9. Let

$$\Lambda(t,x) = \left\{(y^0, y) : y^0 \geq t^2 z^2,\ y = z,\ |z| \leq \frac{1}{t}\right\} \text{ if } t \neq 0$$
$$\Lambda(0,x) = \{(y^0, y) : y^0 \geq 0,\ y = 0\} \text{ if } t = 0.$$

All of the sets $\Lambda(t,x)$ are independent of x. If $t \neq 0$, then $\Lambda(t,x)$ is the set of points on or above the segment of the parabola $y^0 = t^2 y^2$, $|y| \leq 1/t$ in the (y^0, y) plane. Since $\Lambda(N_\delta(0,0)) = \bigcup_{0 \leq t < \delta} \Lambda(t,x)$, we see that $\Lambda(N_\delta(0,0))$ is the open upper half plane plus the origin. Hence $\bigcap_{\delta > 0}$ cl co $\Lambda(N_\delta(0,0)) =$ closed upper half plane. But $\Lambda(0,0) = \{(y^0, y) : y^0 \geq 0,\ y = 0\}$, so the Cesari property fails. On the other hand, $N_{\delta x}(0,0) = \{(0,x) : |x| < \delta\}$. Since Λ is independent of x,

$$\Lambda(N_{\delta x}(0,0)) = \{(y^0, y) : y^0 \geq 0,\ y = 0\} = \Lambda(0,0).$$

Hence

$$\bigcap_{\delta > 0} \text{cl co } (N_{\delta x}(0,0)) = \Lambda(0,0),$$

and the weak Cesari property holds.

5.3 Facts from Analysis

Definition 5.3.1. A set F of functions f in $L_1[a,b]$, where $[a,b] = \{t : a \leq t \leq b\}$, is said to have equi-absolutely continuous integrals if given an $\varepsilon > 0$ there is a $\delta > 0$ such that for any Lebesgue measurable set $E \subset [a,b]$ with meas $(E) < \delta$ and any f in F,

$$|\int_E f dt| < \varepsilon.$$

Since $[a,b]$ is a finite interval and we are dealing with Lebesgue measure, it follows that if the functions f in F have equi-absolutely continuous integrals, then there is a constant $K > 0$ such that for each f in F

$$\int_a^b |f| dt < K. \tag{5.3.1}$$

That is, the set F is bounded in $L_1[a,b]$.

Definition 5.3.2. A set F of absolutely continuous functions f defined on $[a,b]$ is said to be equi-absolutely continuous if given an $\varepsilon > 0$ there is a $\delta > 0$ such that for any finite collection of non-overlapping intervals (α_i, β_i) contained in $[a,b]$, with $\sum_i |\beta_i - \alpha_i| < \delta$, the inequality $\sum_i |f(\beta_i) - f(\alpha_i)| < \varepsilon$ holds for all f in F.

We leave it to the reader to verify that a set of absolutely continuous functions is equi-absolutely continuous if and only if the derivatives f' have equi-absolutely continuous integrals.

Lemma 5.3.3. *Let $\{f_n\}$ be a sequence of equi-absolutely continuous functions defined on an interval $[a,b]$ and converging to a function f. Then f is absolutely continuous.*

Proof. Let (α_i, β_i), $i = 1, \ldots, k$ be a finite collection of non-overlapping intervals all contained in $[a,b]$. Then

$$\sum_{i=1}^{k} |f(\beta_i) - f(\alpha_i)| \le \sum_{i=1}^{k} |f(\beta_i) - f_n(\beta_i)| + \sum_{i=1}^{k} |f_n(\beta_i) - f_n(\alpha_i)| + \sum_{i=1}^{k} |f(\alpha_i) - f_n(\alpha_i)|,$$

from which the lemma follows. $\square$

Remark 5.3.4. The inequality in the proof of Lemma 5.3.3 also shows that if $\{f_n\}$ is a sequence of absolutely continuous functions converging uniformly on $[a,b]$ to a function f, then f is absolutely continuous. The next example shows that if the convergence is uniform, then the limit can be absolutely continuous even if the $\{f_n\}$ are not equi-absolutely continuous.

For each positive integer n let $f_n(x) = \sin n^2 x/n$ for $0 \le x \le 1$. Each function is absolutely continuous and the sequence $\{f_n\}$ converges uniformly to zero. To show that the functions f_n are not equi-absolutely continuous we must exhibit an $\varepsilon > 0$ such that for each $\delta > 0$ there exists a positive integer $n = n(\delta)$ with the following property. There exists a finite collection of pairwise disjoint intervals $[\alpha_i, \beta_i]$ with $\Sigma|\beta_i - \alpha_i| < \delta$ and with $\Sigma|f_n(\beta_i) - f_n(\alpha_i)| > \varepsilon$. To this end, let $\varepsilon = 1/2$ and let $\delta > 0$ be arbitrary, but less than one. Let $n = n(\delta)$ be the smallest positive integer such that $\pi/2n < \delta$. Let

$$[0, \pi/2n^2], [2\pi/2n^2, 3\pi/2n^2], \ldots, [2k\pi/2n^2, (2k+1)\pi/2n^2], \ldots, [2(n-1)\pi/2n^2, (2n-1)\pi/2n^2]$$

be a collection of n pairwise disjoint intervals in $[0,1]$. The sum of the lengths of these intervals is $\pi/2n$, and

$$\sum_{k=0}^{n-1} |f_n((2k+1)\pi/2n^2) - f_n(2k\pi/2n^2))| = n \cdot \frac{1}{n} = 1 > 1/2.$$

Thus, the functions $\{f_n\}$ are not equi-absolutely continuous.

For us, the importance of the notion of equi-absolute continuity stems from the following theorem.

Theorem 5.3.5. *Let $[a, b]$ be a finite interval and let $\{f_n\}$ be a sequence of functions in $L_1[a, b]$. The sequence of functions $\{f_n\}$ converges weakly to a function f in $L_1[a, b]$ if and only if the following conditions are satisfied: (i) The functions f_n have equi-absolutely continuous integrals and (ii) for every t in $[a, b]$*

$$\lim_{n\to\infty} \int_a^t f_n(s)ds = \int_a^t f(s)ds.$$

We shall sketch a proof of the theorem, referring the reader to standard texts for some of the arguments and leaving other parts to the reader.

We first consider the necessity of conditions (i) and (ii). Weak convergence of f_n to f means that for every bounded measurable function g defined on $[a, b]$

$$\int_a^b g f_n dt \to \int_a^b g f dt. \tag{5.3.2}$$

Hence by taking g to be the characteristic function of $[a, t]$ we obtain (ii). By taking g to be the characteristic function of a measurable set E we get that (5.3.2) holds when the integrals are taken over any measurable set E and $g = 1$.

From this fact, from the absolute continuity of the integral and from

$$|\int_E f_n dt| \leq |\int_E (f_n - f)dt| + |\int_E f dt|$$

it follows that for each $\varepsilon > 0$ there exists a $\delta > 0$ and a positive integer N such that for $n > N$ and $|E| < \delta$, $|\int_E f_n dt| < \varepsilon$. Since the integrals of the finite set of functions $f_1, \ldots, f_N$ are equi-absolutely continuous, (i) follows.

Now suppose that (i) and (ii) hold. Condition (i) implies that (5.3.1) holds with f replaced by f_n; that is, the sequence $\{f_n\}$ is bounded in $L_1[a, b]$. Condition (ii) implies that condition (ii) holds when the interval of integration is taken to be $[t', t'']$, where $[t', t'']$ is any interval contained in $[a, b]$. From this statement and (i) it follows that (ii) holds when the integrals are taken over any measurable set E in $[a, b]$. It then follows that (5.3.2) holds for any step function g. If g is an arbitrary measurable function, then g is the almost everywhere limit of a sequence of step functions $\{\sigma_k\}$. By Egorov's theorem, for every $\delta > 0$ there is a set E of measure $< \delta$ such that on the complement of E relative to $[a, b]$, $\sigma_k \to g$ uniformly. From the last observation, from (5.3.2) with g replaced by a step function, the uniform L_1 bound for the functions f_n, and the equi-absolute continuity of the $\{f_n\}$ there follows the validity of (5.3.2) for arbitrary bounded measurable g.

Lemma 5.3.6. *Let $\{f_n\}$ be a sequence of functions in $L_1[a, b]$ that converges*

weakly to a function f in $L_1[a,b]$. Then for each positive integer j there exists a positive integer n_j, a set of integers $1,\ldots,k$, where k depends n_j, and a set of real numbers $\alpha_{1j},\ldots,\alpha_{kj}$ satisfying

$$\alpha_{ij} \geq 0, \quad i=1,\ldots,k \quad \sum_{i=1}^{k} \alpha_{ij} = 1 \tag{5.3.3}$$

such that $n_{j+1} > n_j + k$ and such that the sequence

$$\psi_j = \sum_{i=1}^{k} \alpha_{ij} f_{n_j+i} \tag{5.3.4}$$

converges strongly in $L_1[a,b]$ to f.

Proof. The set cl co $\{f_n\}$, where closure is taken in $L_1[a,b]$, is a strongly closed convex set in the Banach space $L_1[a,b]$. Therefore, by Mazur's theorem (Lemma 4.7.6) the set cl co $\{f_n\}$ is weakly closed. Hence f is in this set and can be approximated to any degree of accuracy in $L_1[a,b]$ by a convex combination of functions in $\{f_n\}$.

We now define the sequences n_j and ψ_j inductively. Let $j=1$. Then there exists a convex combination of the functions in $\{f_n\}$ that approximates f in the $L_1[a,b]$ norm to within $1/2$. By choosing some of the coefficients in the convex combination to be zero, we may suppose that we are taking a convex combination of consecutive functions $f_{n_1+1},\ldots,f_{n_1+k}$, where k depends on n_1. That is, there exist real numbers $\alpha_{i1}, i=1,\ldots,k$ such that (5.3.3) holds and the function ψ_1 defined by (5.3.4) satisfies

$$\int_a^b |\psi_1 - f| dt < 1/2.$$

Now suppose that $n_1,\ldots,n_j$ and $\psi_1,\ldots,\psi_j$ have been defined and that for $i=1,\ldots,j$, $\int_a^b |\psi_i - f| dt < 2^{-i}$. Let n_{j+1} be any integer greater than n_j+k. Then cl co $\{f_{n_j+1}, f_{n_j+2},\ldots\}$ is a weakly closed set in $L_1[a,b]$, and so contains f. We can then apply the argument used for $j=1$ to this set to obtain a function ψ_{j+1} defined by (5.3.4) such that

$$\int_a^b |\psi_{j+1} - f| dt < 1/2^{j+1}.$$

□

Lemma 5.3.7. *Let $\{s_n\}$ be a sequence of points in $\mathbb{R}^k$ converging to s. Let $\{n_j\}$ be a subsequence of the positive integers and let*

$$\sigma_j = \sum_{i=1}^{k} \alpha_{ij} s_{n_j+i},$$

where the α_{ij} and k are as in (5.3.3). Then $\sigma_j \to s$.

We leave the proof as an exercise for the reader.

We conclude this section with the following result.

Lemma 5.3.8. *Let I be a compact real interval and let $h : (t, \xi) \to h(t, \xi)$ be a continuous mapping from $I \times \mathbb{R}^r$ to $\mathbb{R}^1$. Let $\{v_k\}$ and $\{w_k\}$ be sequences in $L^p[I, \mathbb{R}^r]$, $1 \le p \le \infty$ such that $\|v_k\|_p \le M$ and $\|w_k\|_p \le M$ for some $M > 0$ and such that $(v_k - w_k) \to 0$ in measure on I. Then*

$$h(t, v_k(t)) - h(t, w_k(t)) \to 0$$

in measure on I.

Proof. We must show that for arbitrary $\eta > 0$ and $\varepsilon > 0$ there exists a positive integer K such that if $k > K$, then

$$\text{meas } \{t : |h(t, v_k(t)) - h(t, w_k(t))| \ge \eta\} < \varepsilon.$$

Let

$$A = M(3/\varepsilon)^{1/p}, \tag{5.3.5}$$

where we interpret $1/\infty$ as zero. Let G_A denote the set of points ξ in $\mathbb{R}^r$ such that $|\xi| \le A$. Since h is uniformly continuous on $I \times G_A$, there exists a $\delta > 0$ such that if ξ and ξ' belong to G_A and $|\xi - \xi'| < \delta$, then

$$|h(t, \xi) - h(t, \xi')| < \eta \tag{5.3.6}$$

for all t in I.

Let I_{kv} denote the set of points in I at which $|v_k(t)| > A$ and let I_{kw} denote the set of points in I at which $|w_k(t)| > A$. Let $I_k = I_{kv} \cup I_{kw}$ and let

$$G_k = \{t : |v_k(t) - w_k(t)| \ge \delta\}.$$

Then for $t \notin I_k \cup G_k$,

$$|h(t, v_k(t)) - h(t, w_k(t)| < \eta.$$

Therefore, to establish the lemma we must show that for k sufficiently large, meas $(I_k \cup G_k) < \varepsilon$.

For $1 \le p < \infty$ we have

$$M \ge \left(\int_I |v_k(t)|^p dt \right)^{1/p} \ge \left(\int_{I_{kv}} A^p dt \right)^{1/p} = A(\text{meas } I_{kv})^{1/p}.$$

From this and (5.3.5) we get that meas $(I_{kv}) < \varepsilon/3$. Similarly, meas $(I_{kw}) < \varepsilon/3$.

Since $(v_k - w_k) \to 0$ in measure, for sufficiently large k, meas$(G_k) < \varepsilon/3$. Thus, meas$(I_k \cup G_k) < \varepsilon$ for k sufficiently large. For $p = \infty$, we have from (5.3.5) that $A = M$, so meas $I_k = 0$. Since $(v_k - w_k) \to 0$ in measure, there exists a positive integer K such that for $k > K$, meas $G_k < \varepsilon$. Hence meas$(I_k \cup G_k) < \varepsilon$, and the lemma is proved. ∎

5.4 Existence via the Cesari Property

In this section we shall prove existence theorems for ordinary and relaxed problems without assuming the constraint sets to be compact. Instead, we shall assume that certain set valued mappings possess the weak Cesari property. The assumptions about the data of the problem will be less restrictive than in the case of problems with compact constraints. The functions $\widehat{f} = (f^0, f^1, \ldots, f^n)$ will not be required to be Lipschitz in the state variable. The constraint mappings Ω will be allowed to depend on (t, x), rather than on t alone, and will be assumed to be upper semicontinuous rather than u.s.c.i.

The definition of the relaxed problem for the case of non-compact constraints given in Section 3.5 is consistent with the definition for compact constraints given in Section 3.2 in the sense that the sets of relaxed trajectories will be the same under both definitions.

To facilitate our discussion of problems with constraints that are not assumed to be compact, we introduce some notation. Let p and $\overline{v}$ be measurable functions of the form

$$p = (p^1, \ldots, p^{n+2}) \quad \overline{v} = (u_1, \ldots, u_{n+2}),$$

where the p^i are real valued measurable functions and the u_i are vector valued functions with range in $\mathcal{U}$. Let

$$\begin{aligned} v &= (p, \overline{v}) = (p^1, \ldots, p^{n+2}, u_1, \ldots, u_{n+2}) \qquad (5.4.1)\\ \pi &= (\pi^1, \ldots, \pi^{n+2}) \quad \zeta = (z_1, \ldots, z_{n+2}) \quad \pi^i \in \mathbb{R}, z_i \in \mathbb{R}^m \\ \overline{z} &= (\pi, \zeta) \end{aligned}$$

$$\Pi_{n+2} = \left\{ \pi = (\pi^1, \ldots, \pi^{n+2}) : \pi^i \geq 0, \ \sum_{i=1}^{n+2} \pi^i = 1 \right\}$$

$$f_r^i(t, x, \overline{z}) \equiv f_r^i(t, x, \pi, \zeta) \equiv \sum_{j=1}^{n+2} \pi^j f^i(t, x, z_j) \quad z_j \in \mathbb{R}^m \quad i = 0, 1, \ldots, n+2,$$

and let $\widehat{f}_r = (f_r^0, f_r) = (f_r^0, f_r^1, \ldots, f_r^n)$.

In terms of the notation just introduced, the relaxed problem corresponding to Eqs. (5.1.1) to (5.1.3) can be written as:

Minimize

$$J(\psi, v) = g(e(\psi)) + \int_{t_0}^{t_1} f_r^0(t, \psi(t), v(t)) dt \qquad (5.4.2)$$

subject to the differential equation

$$\frac{d\psi}{dt} = f_r(t, \psi(t), v(t)), \qquad (5.4.3)$$

end condition

$$(t_0, \psi(t_0), t_1, \psi(t_1)) \in \mathcal{B} \tag{5.4.4}$$

and control constraints on $v(t) = (p(t), \overline{v}(t))$

$$p(t) \in \Pi_{n+2} \quad u_i(t) \in \Omega(t, \psi(t)) \quad i = 1, \ldots, n+2. \tag{5.4.5}$$

If we set

$$\widetilde{\Omega}(t,x) = \Omega(t,x) \underbrace{\times \ldots \times}_{n+2 \text{ times}} \Omega(t,x)$$

then we may write (5.4.5) as

$$v(t) \in \widetilde{\Omega}(t, \psi(t)). \tag{5.4.6}$$

We emphasize the observation made in Remark 3.5.8 that *the relaxed problem just formulated can be viewed as an ordinary problem with controls* $v = (p, u_1, \ldots, u_{n+2})$.

We assume that the $\widehat{f}, g, \mathcal{B}$, and Ω satisfy the following:

Assumption 5.4.1. (i) The function $\widehat{f} = (f^0, f) = (f^0, f^1, \ldots, f^n)$ is defined on a set $\mathcal{G} = \mathcal{I} \times \mathcal{X} \times \mathcal{U}$, where $\mathcal{I}$ is a real compact interval, $\mathcal{X}$ is a closed interval in $\mathbb{R}^n$ and $\mathcal{U}$ is an open interval in $\mathbb{R}^m$.

(ii) The function f is continuous on $\mathcal{G}$.

(iii) The function f^0 is lower semi-continuous on $\mathcal{G}$ and there exists an integrable function β on $\mathcal{I}$ such that $f^0(t,x,z) \geq \beta(t)$ for all (t,x,z) in $\mathcal{G}$.

(iv) The terminal set $\mathcal{B}$ is a closed set of points (t_0, x_0, t_1, x_1) in $\mathbb{R}^{n+2}$ with $t_0 < t_1$ and with (t_0, x_0) and (t_1, x_1) in $\mathcal{R} = \mathcal{I} \times \mathcal{X}$.

(v) The function g is lower semi-continuous on $\mathcal{B}$.

(vi) Ω is a mapping from $\mathcal{R} = \mathcal{I} \times \mathcal{X}$ to subsets $\Omega(t,x)$ of $\mathcal{U}$ that is upper semi-continuous on $\mathcal{R}$.

We next recall sets introduced in Section 4.4 that will again play a crucial role in our existence theorems.

Definition 5.4.2. For each (t,x) in $\mathcal{R}$ let

$$Q^+(t,x) = \{(y^0, y) \colon y^0 \geq f^0(t,x,z) \quad y = f(t,x,z),\ z \in \Omega(t,x)\}$$

and let

$$\begin{aligned} &Q_r^+(t,x) = \{(y^0, y) \colon y^0 \geq f_r^0(t,x,\pi,\overline{z}) \quad y = f_r(t,x,\pi,\overline{z}), \\ &\pi \in \Pi_{n+2},\ \zeta = (z_1, \ldots, z_{n+2}) \quad z_i \in \Omega(t,x),\ i = 1, \ldots, n+2\}. \end{aligned}$$

Lemma 5.4.3. *If the mapping Q_r^+ has the weak Cesari property (Cesari property) at (t,x) and if $Q^+(t,x')$ is convex for all x' in a neighborhood of x, then Q^+ has the weak Cesari property (Cesari property) at (t,x).*

Proof. From the definition of $Q_r^+(t,x)$ we have that $Q_r^+(t,x) = co\ Q^+(t,x)$. Thus, if $Q^+(t,x)$ is convex, then $Q_r^+(t,x) = Q^+(t,x)$. Hence for δ_x sufficiently small,

$$Q^+(N_{\delta x}(t,x) = \bigcup_{|x'-x|<\delta x} Q^+(t,x') = \bigcup_{|x'-x|<\delta x} Q_r^+(t,x') = Q_r^+(N_{\delta x}(t,x)).$$

Thus, cl $co\ Q^+(N_{\delta x}(t,x)) =$ cl co $Q_r^+(N_{\delta x}(t,x))$. Therefore, since Q_r^+ has the weak Cesari property at (t,x)

$$\bigcap_{\delta x>0} \text{cl co } Q^+(N_{\delta x}(t,x)) = \bigcap_{\delta x>0} \text{cl co } Q_r^+(N_{\delta x}(t,x)) = Q_r^+(t,x) = Q^+(t,x).$$

The equality of the leftmost and rightmost sets shows that Q^+ has the weak Cesari property. $\square$

To prove the statements about the Cesari property one considers points (t',x') such that $|(t',x') - (t,x)| < \delta$.

Theorem 5.4.4. *Let Assumption 5.4.1 hold. Let the set of admissible relaxed pairs (ψ,μ) be non-empty. Let the relaxed problem have a minimizing sequence $\{(\psi_n,\mu_n)\}$ whose trajectories lie in a compact set $\mathcal{R}_0 \subseteq \mathcal{R}$ and are equi-absolutely continuous. Let the mapping Q_r^+ possess the weak Cesari property at all points of $\mathcal{R}_0$. Then the relaxed problem has an optimal solution. If $Q^+(t,x)$ is convex at all points (t,x) in $\mathcal{R}$, then there exists an ordinary admissible pair that is optimal for both the ordinary and relaxed problems.*

To invoke Theorem 5.4.4 in a specific problem, we must find a minimizing sequence all of whose trajectories lie in a compact set and are equi-absolutely continuous, and that the mapping Q^+ or Q_r^+ has the weak Cesari property. In a specific problem this may or not be easy to do. Lemma 4.3.14 gives a sufficient condition for *all trajectories* of the ordinary or relaxed system to lie in a compact set. While this condition is useful in problems with compact constraints, it is not as applicable to problems in which the constraint sets are not assumed to be compact. For such problems a sufficient condition for trajectories (ordinary or relaxed) in a minimizing sequence to lie in a compact set will be given in Lemma 5.4.14. Before presenting the proof of Theorem 5.4.4 we shall give a condition in terms of the data of the problem that is sufficient for the mappings Q^+ and Q_r^+ to have the weak Cesari property. Part of this condition is sufficient for the trajectories in a minimizing sequence to be equi-absolutely continuous.

Definition 5.4.5. Let G be a real valued nonnegative function defined on $\mathcal{G} = \mathcal{R}\times\mathcal{U}$. Let F be a function defined on $\mathcal{G}$ with range in $\mathbb{R}^n$. Then F is said

to be of *slower growth than* G *uniformly on* $\mathcal{G}$ if for each $\varepsilon > 0$ there exists a positive number ν such that if $|z| > \nu$, then

$$|F(t,x,z)| < \varepsilon G(t,x,z).$$

Lemma 5.4.6. *Let* $\widehat{f} = (f^0, f)$ *be as in Assumption 5.4.1 with* $\beta \equiv 0$. *Let* f *be of slower growth than* f^0, *uniformly on* $\mathcal{G}$, *and let the function identically equal to one be of slower growth than* f^0, *uniformly on* $\mathcal{G}$. *Let the constraint mapping* Ω *be upper semi-continuous on* $\mathcal{R}$. *Then:*

(i) At each (t,x) *the mapping* Q_r^+ *has the weak Cesari property on* $\mathcal{R}$.

(ii) If at each (t,x) *in* $\mathcal{R}$ *the set* $Q^+(t,x)$ *is convex, then the mapping* Q^+ *has the weak Cesari property on* $\mathcal{R}$.

Remark 5.4.7. The functions f_r^0, f_r need not satisfy the growth condition of the lemma, even though the functions f^0 and f do. To see this, let $f^0 = z^2$, $f = z$. Then $f_r^0 = \pi^1(z_1)^2 + \pi^2(z_2)^2$ and $f_r = \pi^1 z_1 + \pi^2 z_2$. For the relaxed problem take the sequence of controls $\{(\pi_k, \overline{z}_k) = \{(0,1,k,0)\}$, $k = 1,2,\ldots$. Then $|(\pi_k, \overline{z}_k)| \to \infty$ and $f_r^0(\pi_k, \overline{z}_k) = 0$, $f_r(\pi_k, \overline{z}_k) = 0$, so the function identically one cannot be of slower growth than f_r^0.

Proof of Lemma. By Lemma 5.4.3, if (i) holds, then so does (ii). Hence we need only prove (i). We first note that as a consequence of the definition and of Lemma 3.2.10, that the upper semi-continuity of the mapping Ω implies that for each (t,x) in $\mathcal{R}$, the sets $\Omega(t,x)$ and

$$\mathcal{D} = \{(t,x,z) : (t,x) \in \mathcal{R},\ z \in \Omega(t,x)\} \tag{5.4.7}$$

are closed. To prove conclusion (i) we must show that if

$$\widehat{y} = (y^0, y) \in \bigcap_{\delta x > 0} \text{cl co } Q_r^+(N_{\delta x}(t,x)) \tag{5.4.8}$$

then $\widehat{y} \in Q_r^+(t,x)$.

Since for any collection $\{S_\alpha\}$ of sets we have $\bigcup_\alpha \text{ co } S_\alpha \subseteq \text{ co } \bigcup_\alpha S_\alpha$ and since $Q_r^+(t,x) = \text{ co } Q^+(t,x)$, the following holds:

$$\begin{aligned} Q_r^+(N_{\delta x}(t,x)) &\equiv \bigcup_{|x'-x|<\delta x} Q_r^+(t,x') = \bigcup_{|x'-x|<\delta x} \text{co } Q^+(t,x') \\ &\subseteq \text{ co } \bigcup_{|x'-x|<\delta x} Q^+(t,x') = \text{ co } Q^+(N_{\delta x}(t,x)). \end{aligned}$$

Thus, cl co $Q_r^+(N_{\delta x}(t,x)) \subseteq$ cl co $Q^+(N_{\delta x}(t,x))$, and so

$$\bigcap_{\delta x>0} \text{cl co } Q_r^+(N_{\delta x}(t,x)) \subseteq \bigcap_{\delta x>0} \text{cl co } Q^+(N_{\delta x}(t,x)).$$

Hence, if $\widehat{y}$ is as in (5.4.8), then

$$\widehat{y} = (y^0, y) \in \bigcap_{\delta x > 0} \text{cl co } Q^+(N_{\delta x}(t, x)). \tag{5.4.9}$$

Let $\widehat{y} = (y^0, y)$ satisfy (5.4.9). Then $y^0 \geq 0$. Let $\{\delta_k\}$ be a decreasing sequence of real numbers such that $\delta_k \to 0$. Then for each positive integer k there exists a point $\widehat{y}_k = (y^0_k, y_k)$ with

$$\widehat{y}_k \in \text{ co } Q^+(N_{\delta_k x}(t, x)) \tag{5.4.10}$$

and $|\widehat{y}_k - \widehat{y}| < 1/k$. In other words, the sequence $\{\widehat{y}_k\}$ satisfies (5.4.10) and

$$\widehat{y}_k = (y^0_k, y_k) \to (y^0, y) = \widehat{y}. \tag{5.4.11}$$

From (5.4.10) and the Carathéodory theorem it follows that for each integer k there exist real numbers $\alpha_{k,1}, \ldots, \alpha_{k,n+2}$ with

$$\alpha_{ki} \geq 0 \quad \text{and} \quad \sum_{i=1}^{n+2} \alpha_{ki} = 1, \tag{5.4.12}$$

points $(t, x_{k_1}), \ldots, (t, x_{k,n+2})$ in $\mathcal{R}$, and points $\widehat{y}_{k,1}, \ldots, \widehat{y}_{k,n+2}$ such that

$$|x_{ki} - x| < \delta_k \quad \text{and} \quad \widehat{y}_{ki} \in Q^+(t, x_{ki}) \quad i = 1, \ldots, n+2 \tag{5.4.13}$$

and

$$\widehat{y}_k = \sum_{i=1}^{n+2} \alpha_{ki} \widehat{y}_{ki}. \tag{5.4.14}$$

From the second relation in (5.4.13) it follows that there exist points $z_{k,1}, \ldots,$ $z_{k,n+2}$ with $z_{ki} \in \Omega(t, x_{ki})$ $i = 1, \ldots, n+2$ such that

$$y^0_{ki} \geq f^0(t, x_{ki}, z_{ki}) \quad y_{ki} = f(t, x_{ki}, z_{ki}) \quad i = 1, \ldots, n+2. \tag{5.4.15}$$

It follows from (5.4.12) that for each i, the sequence $\{\alpha_{ki}\}$ is bounded. Hence there exists a subsequence of $\{k\}$, which we again label as $\{k\}$ and non-negative numbers $\alpha_1, \ldots, \alpha_{n+2}$ such that

$$\alpha_{ki} \to \alpha_i \quad \alpha_i \geq 0 \quad \sum_{i=1}^{n+2} \alpha_i = 1. \tag{5.4.16}$$

Since

$$y^0_k = \sum_{i=1}^{n+2} \alpha_{ki} y^0_{ki} \text{ and } y^0_k \to y^0,$$

the sequence $\{y^0_k\}$ is bounded. Since $\alpha_{ki} \geq 0$ and $y^0_{ki} \geq 0$, it follows that each

sequence $\{\alpha_{ki} y^0_{ki}\}$, $i = 1, \ldots, n+2$ is bounded. Hence for each i there exists a subsequence $\{\alpha_{ki} y^0_{ki}\}$ and a nonnegative number η_i such that

$$\alpha_{ki} y^0_{ki} \to \eta_i \quad \eta_i \geq 0.$$

From the last relation in (5.4.16) it follows that the set of indices for which $\alpha_i > 0$ is non-void. Let $i = 1, 2, \ldots, s$ denote this set. Then

$$y^0_{ki} \to \frac{\eta_i}{\alpha_i} \quad i = 1, \ldots, s. \tag{5.4.17}$$

We assert that this implies that the sequence $\{(z_{k1}, \ldots, z_{ks})\}$ is bounded. If this assertion were false, there would exist a subsequence and an index i, such that $|z_{ki}| \to \infty$. Since one is of slower growth than f^0, this would imply that for each $\varepsilon > 0$ and all sufficiently large k

$$\frac{1}{\varepsilon} \leq f^0(t, x_{ki}, z_{ki}) \leq y^0_{ki}.$$

Thus, y^0_{ki} would be unbounded, contradicting (5.4.17). Hence $\{(z_{k,1}, \ldots, z_{k,s})\}$ is bounded.

Since $\{(z_{k,1}, \ldots, z_{k,s})\}$ is bounded there exists a subsequence $\{k\}$ and points $z_1, \ldots, z_s$ such that $z_{ki} \to z_i, i = 1, \ldots, s$. Also $x_{ki} \to x$, so for each $i = 1, \ldots, s$, the sequence of points (t, x_{ki}, z_{ki}) in $\mathcal{D}$ converges to a point (t, x, z_i). Since $\mathcal{D}$ is closed, $(t, x, z_i) \in \mathcal{D}$, and so $z_i \in \Omega(t, x)$ for $i = 1, \ldots, s$. Since f is continuous, for $i = 1, \ldots, s$

$$f(t, x_{ki}, z_{ki}) \to f(t, x, z_i) \quad z_i \in \Omega(t, x). \tag{5.4.18}$$

We now consider indices $i > s$. If for an index $i > s$ the sequence $\{z_{ki}\}$ is unbounded, we may select a subsequence such that $|z_{ki}| \to \infty$. Then, since $|f|$ is of slower growth than f^0, for each $\varepsilon > 0$ and k sufficiently large

$$\alpha_{ki} |f(t, x_{ki}, z_{ki})| < \varepsilon \alpha_{ki} f^0(t, x_{ki}, z_{ki}) \leq \varepsilon \alpha_{ki} y^0_{ki}.$$

Since $\alpha_{ki} y^0_{ki} \to \eta_i$ and $\varepsilon > 0$ is arbitrary,

$$\alpha_{ki} f(t, x_{ki}, z_{ki}) \to 0. \tag{5.4.19}$$

If for an index $i > s$ the sequence $\{z_{ki}\}$ is bounded, then since $x_{ki} \to x$, the sequence of points $\{(t, x_{ki}, z_{ki})\}$ is bounded. Since f is continuous the set $\{f(t, x_{ki}, z_{ki})\}$ is bounded. Since $\alpha_{ki} \to 0$, the relation (5.4.19) holds in this case also.

It now follows from Eqs. (5.4.14) to (5.4.16), (5.4.18), (5.4.19), and $\alpha_i = 0$ for $i > s$ that

$$y_k = \sum_{i=1}^{n+2} \alpha_{ki} y_{ki} \to \sum_{i=1}^{s} \alpha_i f(t, x, z_i),$$

where $z_i \in \Omega(t,x)$, $\alpha_i > 0$ and $\sum \alpha_i = 1$. But $y_k \to y$, so

$$y = \sum_{i=1}^{s} \alpha_i f(t,x,z_i). \tag{5.4.20}$$

It follows from Eqs. (5.4.11), (5.4.14), (5.4.16), and (5.4.5) and the lower semicontinuity of f^0 that

$$\begin{aligned} y^0 = \sum_{i=1}^{n+2} \eta_i \geq \sum_{i=1}^{s} \eta_i &= \sum_{i=1}^{s} \lim_{k\to\infty} (\alpha_{ki} y^0_{ki}) \\ &\geq \sum_{i=1}^{s} \liminf_{k\to\infty} (\alpha_{ki} f^0(t, x_{ki}, z_{ki})) \\ &\geq \sum_{i=1}^{s} \alpha_i f^0(t,x,z_i), \end{aligned} \tag{5.4.21}$$

where $z_i \in \Omega(t,x)$ and $\alpha_i \geq 0$ for $i = 1, \ldots, s$ and $\sum \alpha_i = 1$. From (5.4.20) and (5.4.21) we get that $\widehat{y} = (y^0, y)$ is in co $Q^+(t,x)$. But $Q^+_r(t,x) = \text{co } Q^+(t,x)$, so $\widehat{y} \in Q^+_r(t,x)$. Thus, Q^+_r has the weak Cesari property at (t,x). $\square$

Remark 5.4.8. If we consider neighborhoods $N_\delta(t,x)$ and in (5.4.12) consider sequences of points $(t_{k1}, x_{k1}), \ldots, (t_{k,n+2}, x_{k,n+2})$ such that $|(t_{ki}, x_{ki}) - (t,x)| < \delta_k$, then the preceding argument gives the stronger result that the mapping Q^+ satisfies the Cesari property.

Lemma 5.4.9. *Let Assumption 5.4.1 hold with $\beta \equiv 0$ in (iii). Let f be of slower growth than f^0, uniformly on $\mathcal{G} = \mathcal{R} \times \mathcal{U}$. Let there exist a minimizing sequence of either the ordinary or the relaxed problem whose trajectories lie in a compact subset of $\mathcal{R}$. Then these trajectories are equi-absolutely continuous.*

Proof. Since the trajectories of the minimizing sequence lie in a compact set and $\mathcal{B}$ is closed, the set of endpoints is compact. Since g is lower semicontinuous on $\mathcal{B}$, the function g is bounded below on the set of endpoints. Hence the set of integrals in (5.1.1) or (5.4.2) evaluated along the pairs in a minimizing sequence is bounded above. We *denote this bound by A*.

We first consider the ordinary problem with a minimizing sequence $\{(\varphi_n, u_n)\}$ of admissible pairs. We shall show that the integrals $\int \varphi'_n dt$ are equi-absolutely continuous. Let $\varepsilon > 0$ be given and let $\eta = \varepsilon/2A$. Then there exists a positive number ν such that if $|u_n(t)| > \nu$, then

$$|f(t, \varphi_n(t), u_n(t))| < \eta f^0(t, \varphi_n(t), u_n(t)).$$

Also, since all points $(t, \phi_n(t))$ are in a compact set, there exists a positive constant K_η such that if $|u_n(t)| \leq \nu$, then

$$|f(t, \varphi_n(t), u_n(t))| \leq K_\eta.$$

Thus, for each n and measurable set $E \subseteq \mathcal{I}$,

$$\begin{aligned}\int_E |\varphi_n'(t)|dt &\leq K_\eta \text{ meas}(E) + \eta \int_E f^0(t, \varphi_n(t), u_n(t))dt \\ &\leq K_\eta \text{ meas}(E) + \eta A.\end{aligned} \tag{5.4.22}$$

If we take $\text{meas}(E) < \varepsilon/2K_\eta$ and recall that $\eta = \varepsilon/2A$, we obtain the asserted equi-absolute continuity.

For the relaxed problem with ε as before and $\eta = \varepsilon/2(n+2)A$ we have

$$\begin{aligned}\int_E |\psi_n'(t)|dt &\leq \int_E \sum_{i=1}^{n+2} p^i(t)|f(t, \psi_n(t), u_{ni}(t))|dt \\ &\leq \int_E \sum_{i=1}^{n+2} p^i(t)(K_\eta + \eta f^0(t, \psi_n(t), u_{ni}(t)))dt \\ &\leq K_\eta \text{ meas}(E) + \eta \int_E \left(\sum_{i=1}^{n+2} p^i(t) f^0(t, \psi_n(t), u_{ni}(t))\right) dt \\ &\leq K_\eta \text{ meas}(E) + \eta(n+2)A.\end{aligned}$$

From the preceding we obtain the equi-absolute continuity, as from (5.4.22).

If the control problem is of Mayer type, that is, $f^0 \equiv 0$, then, in general, the condition "f is of slower growth than f^0" will not hold. □

The proof of Lemma 5.4.9, however, shows that for the Mayer problem, the following is true. If there exists a nonnegative function F^0 defined on $\mathcal{G}$ such that: (i) f is of slower growth than F^0, uniformly on $\mathcal{G}$ and (ii) there exists a constant $A > 0$ such that for all (ϕ_k, u_k) in u minimizing sequence

$$\int_{t_0}^{t_1} F^0(t, \psi_k(t), u_k(t)dt \leq A,$$

then the trajectories $\{\phi_k\}$ are equi-absolutely continuous. A similar statement holds for relaxed trajectories $\{\psi_k\}$ in a minimizing sequence.

The next lemma has a corollary that gives a sufficient condition that does not involve f^0 for trajectories in a minimizing sequence to be absolutely continuous.

Lemma 5.4.10. *Let Φ be a positive, continuous non-decreasing function defined on $[0, \infty)$ such that $\Phi(\xi) \to +\infty$ as $\xi \to +\infty$. Let $\{\theta_k\}$ be a sequence of real valued functions such that θ_k is defined and integrable on $[t_{0k}, t_{1k}]$ and such that $\theta_k(t) = 0$ for $t \notin [t_{0,k}, t_{1k}]$. Let there exist a constant C such that*

$$\int_{t_{0k}}^{t_{1k}} |\theta_k(t)|\Phi(|\theta_k(t)|)dt \leq C. \tag{5.4.23}$$

Then the functions $\{\theta_k\}$ have equi-absolutely continuous integrals.

Proof. Since Φ is continuous and each θ_k is measurable, it follows that the functions $\{\Phi(|\theta_k|\}$ are measurable.

Let $\eta > 0$ be given. Then there exists a positive number K_η such that if $\xi > K_\eta$, then $\Phi(\xi) > 1/\eta$. For each k let

$$E_{1k} = \{t : |\theta_k(t)| \le K_\eta\}$$
$$E_{2k} = \{t : |\theta_k(t)| > K_\eta\}.$$

If $t \in E_{2k}$, then $\Phi(|\theta_k(t)|) > 1/\eta$. Therefore,

$$|\theta_k(t)| = \frac{|\theta_k(t)|\Phi(|\theta_k(t)|)}{\Phi(|\theta_k(t)|)} \le K_\eta + \eta|\theta_k(t)|\Phi(|\theta_k(t)|).$$

Hence, for any set $E \subseteq \mathcal{I}$

$$\begin{aligned}\int_E |\theta_k(t)|dt &\le K_\eta \text{ meas } E + \eta \int_E |\theta_k(t)|\Phi(|\theta_k(t)|)dt \\ &\le K_\eta \text{ meas } E + \eta C.\end{aligned}$$

From this and the argument following (5.4.22) we get the asserted equi-absolute continuity. □

Corollary 5.4.11. *Let $\{\varphi_k\}$ be the trajectories in a minimizing sequence such that each component $\{d\varphi_k^i/dt\}$, $i = 1, \ldots, n$ of the sequence of derivatives satisfies (5.4.23). Then the sequence $\{\varphi_k\}$ is equi-absolutely continuous. Similar statements hold for relaxed minimizing sequences $\{\psi_k\}$.*

Corollary 5.4.12. *Let there exist a constant C and a real number $1 < p < \infty$ such that the components of the trajectories of a minimizing sequence satisfy*

$$\int_{t_{0k}}^{t_{1k}} |d\phi_k^i/dt|^p dt \le C.$$

Then the trajectories $\{\phi_k\}$ of the minimizing sequence are equi-absolutely continuous. A similar statement holds for the trajectories $\{\psi_k\}$ of a relaxed minimizing sequence.

Proof. Write $|d\phi_k^i/dt|^p = |d\phi_k^i/dt||d\phi_k^i/dt|^{p-1}$. □

Remark 5.4.13. The problem of minimizing (5.1.1) is the same as the problem of minimizing (5.1.1) with integrand $f^0 - \beta$. Similarly the problem of minimizing (5.4.2) is the same as the problem of minimizing (5.4.2) with integrand $f^0 - \beta$. Hence in Assumption 5.4.1 there is no loss of generality in assuming $\beta \equiv 0$; that is, $f^0 \ge 0$.

The variational problem of finding the curve that minimizes the distance between the points $(0, 0)$ and $(1, 0)$ in the (t, x) plane has the following form as a control problem. Minimize $\int_0^1 (1 + u^2)^{1/2} dt$ subject to $dx/dt = u$, $x(0) = x(1) = 0$ and $u(t)$ in $\mathbb{R}^1$. Lemma 4.3.14 is not applicable because the dynamics do not satisfy condition (4.3.16). The next lemma is applicable to this problem.

Lemma 5.4.14. *Let Assumption 5.4.1 hold with the added condition that $\mathcal{B}$ is compact. Let there exist a constant $K > 0$ such that for $i = 1, \dots, n$*

$$|f^i(t,x,z)| \le Kf^0(t,x,z)$$

for all (t,x,z) in $\mathcal{G}$. Let the notation be as in Section 3.5. Let $\{(\psi_n, v_n)\}$ be a relaxed minimizing sequence with (ψ_n, v_n) defined on $[t_{0n}, t_{1n}]$. There exists a compact set $\mathcal{R}_0 \subseteq \mathcal{R}$ such that all trajectories $\{\psi_n\}$ are contained in $\mathcal{R}_0$.

Proof. It follows from (5.4.1) and $|f^i| \le Kf^0$ that $|f^i_r(t,x,\overline{z})| \le Kf^0_r(t,x,\overline{z})$, and so

$$|f_r(t,x,\overline{z})| \le K_1 f^0_r(t,x,\overline{z}) \tag{5.4.24}$$

for some constant $K_1 > 0$. Since $\mathcal{B}$ is compact, there exists a closed interval $[a,b]$ such that if $(t_0, x_0, t_1, x_1) \in \mathcal{B}$, then $[t_0, t_1] \subseteq [a,b]$. Since g is lower semicontinuous on $\mathcal{B}$ and $f^0_r \ge 0$, it follows that there exists a constant $A > 0$ such that for all (ψ_n, v_n) of the minimizing sequence

$$\int_{t_{0n}}^{t_{1n}} f^0_r(t, \psi_n(t), v_n(t))dt \le A. \tag{5.4.25}$$

□

For any admissible trajectory ψ, let $\Psi(t) = |\psi(t)|^2 + 1$. Then $\Psi'(t) = 2\langle \psi(t), f_r(t, \psi(t), v)\rangle$. From the Cauchy-Schwarz inequality we get that

$$-2|\psi(t)||f_r(t,\psi(t),v(t))| \le \Psi'(t) \le 2|\psi(t)||f_r(t,\psi(t),v(t))|.$$

From this and (5.4.24) we get that

$$-2(|\psi(t)|^2+1)f^0_r(t,\psi(t),v(t) \le \Psi'(t) \le 2(|\psi|^2+1)f^0_r(t,\psi(t),v(t)).$$

Hence

$$-2f^0_r(t,\psi(t),v(t)) \le \frac{\Psi'(t)}{\Psi(t)} \le 2f^0_r(t,\psi(t),v(t)).$$

Now let (ψ, v) be an element of the minimizing sequence. If we then integrate the preceding relation and use (5.4.25) and $f^0_r \ge 0$, we get that for $t_{0n} \le t \le t_{1n}$,

$$-2A \le \log \frac{\Psi(t)}{\Psi(t_{0n})} \le 2A.$$

Thus,

$$e^{-2A} \le (|\psi_n(t)|^2 + 1)(|\psi(t_{0n})|^2 + 1)^{-1} \le e^{2A}.$$

Since all points $(t_{0n}, \psi_n(t_{0n}))$ lie in a compact set, it follows that all trajectories ψ_n of the minimizing sequence lie in a compact set $\mathcal{R}_0 \subseteq \mathcal{R}$.

Remark 5.4.15. If $g \equiv 0$, then (5.4.25) follows because $\{(\psi_n, v_n)\}$ is a minimizing sequence. To show that all trajectories ψ_n lie in a compact set, the assumption that $\mathcal{B}$ is compact is not needed; we need only assume that all initial points (t_0, x_0) lie in a compact set.

The next theorem is an immediate consequence of Theorem 5.4.4, Lemma 5.4.6, and Lemma 5.4.9. In a specific example, the hypotheses of Theorem 5.4.16 are usually easier to verify than those of Theorem 5.4.4.

Theorem 5.4.16. *Let Assumption 5.4.1 hold. Let the set of admissible relaxed pairs* (ψ, μ) *be non-empty and let all the trajectories in a minimizing sequence lie in a compact set* $\mathcal{R}_0 \subseteq \mathcal{R}$. *Let* f *and the function identically equal to one be of slower growth than* f^0, *uniformly on* $\mathcal{R}_0 \times \mathfrak{U}$. *Then the relaxed problem has a solution. If the sets* $Q^+(t,x)$ *are convex, then there exists an ordinary pair that is optimal for both the ordinary and relaxed problems.*

Proof of Theorem 5.4.4. The last conclusion of the theorem follows from Corollary 4.4.3, once the existence of an optimal relaxed solution is shown. We proceed to do this. □

In Remark 3.5.8 and in the discussion preceding Assumption 5.4.1, it was pointed out that the relaxed problem Eqs. (5.4.1) to (5.4.5) can be viewed as an ordinary problem. The notation henceforth will be as in Eqs. (5.4.1) to (5.4.5). The function $\widehat{f}_r$ satisfies Assumption 5.4.1. Since Π_{n+2} is constant and since Ω is upper semi-continuous, so is $\widetilde{\Omega}$. All other hypotheses of Assumption 5.4.1 clearly hold for the relaxed problem treated as an ordinary problem.

In the proof we shall select subsequences of various sequences and shall relabel the subsequence with the labeling of the original sequence. By Remark 5.4.13 we may assume that $f^0 \geq 0$.

Let $\{(\psi_k, v_k)\}$ be a minimizing sequence. The trajectory ψ_k is defined on an interval $[t_{0k}, t_{1k}]$. We extend ψ_k to a function $\widetilde{\psi}_k$ defined on all of $\mathcal{I}$ by setting $\widetilde{\psi}_k(t) = \psi_k(t_{0k})$ if $t \leq t_{0_k}$, and $\widetilde{\psi}_k(t) = \psi_k(t_{1k})$ if $t \geq t_{1k}$. The sequence of end points $\{e(\psi_k)\}$ lies in the compact set $\mathcal{R}_0 \cap \mathcal{B}$. Hence there exists a subsequence $\{\psi_k\}$ of the minimizing sequence and a point (t_0, x_0, t_1, x_1) in $\mathcal{B}$ such that $e(\psi_k) \to (t_0, x_0, t_1, x_1)$, or

$$t_{0k} \to t_0 \quad \psi_k(t_{0k}) \to x_0 \quad t_{1k} \to t_1 \quad \psi_k(t_{1k}) \to x_1. \tag{5.4.26}$$

The functions $\{\psi_k\}$ are equi-absolutely continuous by hypothesis. The functions $\{\widetilde{\psi}_k\}$ are constant outside of the intervals $[t_{0k}, t_{1k}]$, and so are equi-absolutely continuous on $\mathcal{I}$. Since they all lie in $\mathcal{R}_0$, they are uniformly bounded. Hence by Ascoli's theorem, there exists a subsequence $\{\widetilde{\psi}_k\}$ and a continuous function $\widetilde{\psi}$ on $\mathcal{I}$ such that

$$\widetilde{\psi}_k \to \widetilde{\psi} \text{ uniformly on } \mathcal{I}. \tag{5.4.27}$$

Since $\widetilde{\psi}(t_{ik}) = \psi(t_{ik}),\ i = 0, 1$, it follows from (5.4.26) and (5.4.27) that

$$\widetilde{\psi}(t_{0k}) \to \widetilde{\psi}(t_0) = x_0 \text{ and } \widetilde{\psi}(t_{1k}) \to \widetilde{\psi}(t_1) = x_1.$$

If $t < t_0$, then for k sufficiently large, $t < t_{0k}$ and $\widetilde{\psi}_k(t) = \widetilde{\psi}(t_{0k})$. Hence $\widetilde{\psi}(t) = \widetilde{\psi}(t_0)$ for $t < t_0$. Similarly, $\widetilde{\psi}(t) = \widetilde{\psi}(t_1)$ for $t > t_1$.

Since the functions $\{\widetilde{\psi}_k\}$ are equi-absolutely continuous and converge uniformly to $\widetilde{\psi}$ on $\mathcal{I}$, the function $\widetilde{\psi}$ is absolutely continuous. Hence it is differentiable almost everywhere and

$$\widetilde{\psi}(t) = \widetilde{\psi}(t_0) + \int_{t_0}^{t} \widetilde{\psi}'(s)ds.$$

Let $\mathcal{I} = [a, b]$. Since $\widetilde{\psi}$ is constant on $[a, t_0]$ and on $[t_1, b]$, we have $\widetilde{\psi}'(t) = 0$ on (a, t_0) and (t_1, b). Hence we can write the preceding relation as

$$\widetilde{\psi}(t) = \widetilde{\psi}(a) + \int_{a}^{t} \widetilde{\psi}' ds$$

for all t in $[a, b]$. We also have

$$\widetilde{\psi}_k(t) = \widetilde{\psi}_k(a) + \int_{a}^{t} \widetilde{\psi}'_k ds.$$

It then follows from (5.4.27) that for all t in $\mathcal{I}$,

$$\int_{a}^{t} \widetilde{\psi}'_k ds \to \int_{a}^{t} \widetilde{\psi}'_k ds.$$

Since the functions $\{\widetilde{\psi}_k\}$ are equi-absolutely continuous, so are the integrals of the functions $\{\widetilde{\psi}'_k\}$. Hence by Theorem 5.3.5

$$\widetilde{\psi}'_k \to \widetilde{\psi}' \text{ weakly in } L_1[a, b].$$

We also note that $\widetilde{\psi}$ is in $\mathcal{R}_0$ and $(t_0, \widetilde{\psi}(t_0), t_1, \widetilde{\psi}(t_1))$ is in $\mathcal{B}$.

We summarize the preceding results as Step 1 of the proof.

Step 1. There exists a subsequence $\{\widetilde{\psi}_k\}$ of the extended functions of the minimizing sequence and an absolutely continuous function $\widetilde{\psi}$ such that (5.4.27) holds and $\widetilde{\psi}'_k \to \widetilde{\psi}'$ weakly in $L_1[a, b]$. The function $\widetilde{\psi}$ lies in $\mathcal{R}_0$ and is the extension of a function defined on an interval $[t_0, t_1] \subseteq [a, b]$. Moreover, $(t_0, \widetilde{\psi}(t_0), t_1, \widetilde{\psi}(t_1))$ is in $\mathcal{B}$.

Step 2. Let

$$I(\psi, v) \equiv \int_{t_0}^{t_1} f_r^0(s, \psi(s), v(s))ds.$$

In the first paragraph of the proof of Lemma 5.4.9 we showed that the sequence $\{I(\psi_k, v_k)\}$ is bounded above. Since $f^0 \geq 0$ and all intervals $[t_{0k}, t_{1k}]$ are contained in the fixed compact interval $\mathcal{I}$, the sequence $\{I(\psi_k, v_k)\}$ is bounded below. Hence there exists a subsequence $\{(\psi_k, v_k)\}$ and a real number γ such that $I(\psi_k, v_k) \to \gamma$. The subsequence $\{(\psi_k, v_k)\}$ is a subsequence of the subsequence in Step 1.

Henceforth we let ψ denote the restriction of $\widetilde{\psi}$ to the interval $[t_0, t_1]$. That is,

$$\psi(t) \equiv \widetilde{\psi}(t) \quad t \in [t_0, t_1].$$

Step 3. There exists a real valued function λ that is integrable on $[t_0, t_1]$ such that $(\lambda(t), \psi'(t)) \in Q_r^+(t, \psi(t))$ a.e. on $[t_0, t_1]$ and such that

$$\int_{t_0}^{t_1} \lambda(s)ds \leq \gamma, \tag{5.4.28}$$

where γ is as in Step 2.

Since $\widetilde{\psi}'_k \to \widetilde{\psi}'$ weakly in $L_1[a, b]$, Lemma 5.3.6 gives the following statement. For each integer j there exists an integer n_j, a set of integers $1, \ldots, k$ where k depends on j and a set of real numbers $\alpha_{1j}, \ldots, \alpha_{kj}$ satisfying

$$\alpha_{ij} \geq 0,\ \ i = 1, \ldots, k \text{ and } \sum_{i=1}^{k} \alpha_{ij} = 1$$

such that $n_{j+1} > n_j + k$ and such that the sequence

$$\omega_j = \sum_{i=1}^{k} \alpha_{ij} \widetilde{\psi}'_{n_j+i}$$

converges to $\widetilde{\psi}'$ in $L_1[a, b]$. Recall that for every positive integer q, if $t \notin [t_{0q}, t_{1q}]$ then $\widetilde{\psi}'_q(t) = 0$ and that v_q and ψ_q are defined on $[t_{0q}, t_{1q}]$. If for $t \notin [t_{0q}, t_{1q}]$ we define $f_r(t, \psi_q(t), v_q(t))$ to be zero and recall that on $[t_{0q}, t_{1q}]$, $\widetilde{\psi}(t) = \psi(t)$, we can write ω_j as follows:

$$\omega_j(t) = \sum_{i=1}^{k} \alpha_{ij} f_r(t, \psi_{n_j+i}, v_{n_{j+i}}). \tag{5.4.29}$$

Since $\omega_j \to \widetilde{\psi}'$ in $L_1[a, b]$, there is a subsequence $\{\omega_j\}$ such that

$$\omega_j(t) \to \widetilde{\psi}'(t) \quad \text{a.e. in } [a, b]. \tag{5.4.30}$$

Corresponding to the sequence (5.4.30) we define a sequence $\{\lambda_j\}$ as follows:

$$\lambda_j(t) = \sum_{i=1}^{k} \alpha_{ij} f_r^0(t, \psi_{n_j+i}(t),\ v_{n_j+i}(t)), \tag{5.4.31}$$

where if $t \notin [t_{0q}, t_{1q}]$ we set $f^0(t, \psi_q(t), v_q(t)) = 0$ and where for each j the numbers α_{ij}, the indices $n_j + i$, and the functions ψ_{n_j+i} and v_{n_j+i} are as in (5.4.29). Note that if $t \notin [t_0, t_1]$, then there exists a positive integer $j_0 = j_0(t)$ such that if $j > j_0$, then $\lambda_j(t) = 0$.

Let

$$\lambda(t) = \liminf \lambda_j(t). \tag{5.4.32}$$

Since $f^0 \geq 0$ it follows that $\lambda(t) \geq 0$. If $t \notin [t_0, t_1]$, then $\lambda(t) = 0$. Therefore, if we set $f_{rq}^0(t) \equiv f_r^0(t, \psi_q(t), v_q(t))$ and use Fatou's Lemma we get

$$\int_{t_0}^{t_1} \lambda dt = \int_a^b \lambda dt \leq \liminf_{j\to\infty} \left[\sum_{i=1}^{k} \alpha_{ij} \int_a^b f_{r,n_j+i}^0 dt \right]$$

$$= \liminf_{j\to\infty} \left[\sum_{i=1}^{k} \alpha_{ij} \int_{t_{0,n_j+i}}^{t_{1,n_j+i}} f^0_{r,n_j+i} dt \right]$$
$$= \liminf_{j\to\infty} \left[\sum_{i=1}^{k} \alpha_{ij} I(\psi_{n_j+i}, v_{n_j+i}) \right].$$

From Step 2 we have $I(\varphi_{n_j+i}, u_{n_j+i}) \to \gamma$ as $j \to \infty$. It then follows from Lemma 5.3.7 that

$$\lim_{j\to\infty} \sum_{i=1}^{k} \alpha_{ij} I(\psi_{n_j+i}, v_{n_j+i}) = \gamma,$$

which establishes (5.4.28). Since $\lambda \geq 0$, it follows that λ is in $L_1[t_0, t_1]$, and so is finite a.e.

We now show that $(\lambda(t), \psi'(t)) \in Q^+_r(t, \psi(t))$ a.e. on $[t_0, t_1]$. Let T_1 denote the set of points in $[t_0, t_1]$ at which $\lambda(t)$ is finite and $\omega_j(t) \to \psi'(t)$. The set T_1 has the full measure $|t_1 - t_0|$. For each positive integer k define a set E_k as follows:

$$E_k = \{t : t \in [t_{0k}, t_{1k}],\ v_k(t) \notin \widetilde{\Omega}(t, \varphi_k(t))\}.$$

Then meas $E_k = 0$. Let E denote the union of the sets E_k. Then meas $E = 0$. Let T_2 denote the set of points in $[t_0, t_1]$ that do not belong to E. Let $T' = T_1 \cap T_2$. Then meas $T' = |t_1 - t_0|$.

Let t be a fixed point in T', $t \neq t_i, i = 0, 1$. There exists a subsequence $\{\lambda_j(t)\}$, which in general depends on t such that $\lambda_j(t) \to \lambda(t)$. For the corresponding sequence $\omega_j(t)$ we have from (5.4.30) and the fact that t is interior to (t_0, t_1) that $\omega_j(t) \to \psi'(t)$. Since t is interior to (t_0, t_1) and $t_{ik} \to t_i, i = 0, 1$ it follows that there exists a positive integer j_0 such that if $j > j_0$, then $t \in (t_{0,n_j+i}, t_{1,n_j+i})$. For each $\delta > 0$ there exists a positive integer k_0 such that if $k > k_0$, then $|\psi_k(t) - \psi(t)| < \delta$. Thus, for $k > k_0$

$$(t, \psi_k(t)) \in N_{\delta x}(t, \psi(t)).$$

Therefore, for j sufficiently large

$$\widehat{f}_r(t, \psi_{n_j+i}(t), v_{n_j+i}(t)) \in Q^+_r(N_{\delta x}(t, \psi(t))),$$

where $\widehat{f}_r = (f^0_r, f_r)$. Therefore, by (5.4.29) and (5.4.31)

$$(\lambda_j(t), \omega_j(t)) \in \text{ co } Q^+_r(N_{\delta x}(t, \psi(t))).$$

Since $\lambda_j(t) \to \lambda(t)$ and $\omega_j(t) \to \psi'(t)$ we have that

$$(\lambda(t), \psi'(t)) \in \text{ cl co } Q^+_r(N_{\delta x}(t, \psi(t))).$$

Since δ is arbitrary, $(\lambda(t), \psi'(t))$ is in cl co $Q^+_r(N_{\delta x}(t, \psi(t))$ for each $\delta > 0$, and hence

$$(\lambda(t), \psi'(t)) \in \bigcap_{\delta x > 0} \text{ cl co} Q^+_r(N_{\delta x}(t, \varphi(t))).$$

Since the mapping Q_r^+ has the weak Cesari property, we get that $(\lambda(t), \psi'(t)) \in Q_r^+(t, \varphi(t))$.

Step 4. There exists a measurable function u defined on $[t_0, t_1]$ such that for almost all t in $[t_0, t_1]$: (i) $\psi'(t) = f(t, \psi(t), v(t))$; (ii) $v(t) \in \widetilde{\Omega}(t, \varphi(t))$; (iii) $\lambda(t) \geq f_r^0(t, \psi(t), v(t))$.

The existence of a function v satisfying the conclusion of Step 4 is a restatement of $(\lambda(t), \psi'(t)) \in Q^+(t, \psi(t))$. The problem is to show that there is a measurable function u with this property. This will be done using Filippov's Lemma, Theorem 3.4.1.

With reference to Theorem 3.4.1, let $T = \{t : (\lambda(t), \psi'(t)) \in Q_r^+(t, \varphi(t))\}$. Step 3 shows that T is not empty. Let $Z = \mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^1$. Let $\mathcal{D} = \{(t, x, \overline{z}) : (t, x) \in \mathcal{R},\ \overline{z} \in \widetilde{\Omega}(t, x)\}$ and let $D = \{(t, x, \overline{z}, \eta) : (t, x, \overline{z}) \in \mathcal{D},\ \eta \geq f_r^0(t, x, \overline{z})\}$. From the upper semi-continuity of the mapping $\widetilde{\Omega}$ and Lemma 5.2.4 it follows that $\mathcal{D}$ is closed. Since f_r^0 is lower semi-continuous, it follows that the set D is also closed. Moreover, D can be written as the union of an at most countable number of compact sets D_i, where D_i is the intersection of D with the compact closed ball of radius i centered at the origin.

Let Γ denote the mapping from T to Z defined by $t \to (t, \psi(t), \psi'(t), \lambda(t))$. Each of the functions ψ, ψ', λ is measurable, so Γ is a measurable map. Let Φ denote the map from D to Z defined by $(t, x, \overline{z}, \eta) \to (t, x, f_r(t, x, \overline{z}), \eta)$. Since f_r is continuous, so is Φ. By the definition of T, $\Gamma(t) \subseteq \Phi(D)$. Hence all the hypotheses of Filippov's Lemma are fulfilled. Thus, there exists a measurable mapping m from T to D,

$$m : t \to (\tau(t), x(t), v(t), \eta(t))$$

such that for $t \in T$

$$\begin{gathered}\Phi(m(t)) = (\tau(t), x(t), f_r(\tau(t), x(t), v(t)), \eta(t)) = \\ \Gamma(t) = (t, \psi(t), \psi'(t), \lambda(t)).\end{gathered}$$

From this, the conclusion of Step 4 follows.

Step 5. Completion of Proof.

The function ψ in Step 4 is the restriction to $[t_0, t_1]$ of the function $\widetilde{\psi}$ obtained in Step 1. Hence $e(\psi) \in \mathcal{B}$ and ψ lies in the compact set $\mathcal{R}_0 \subseteq \mathcal{R}$. Thus, to show that (ψ, v) is admissible it remains to show that the mapping $t \to f_r^0(t, \psi(t), v(t))$ is integrable. Since ψ and v are measurable and f_r^0 is lower semi-continuous $t \to f^0(t, \psi(t), v(t))$ is measurable. Since λ is integrable and $f_r^0 \geq 0$, it follows from (iii) of Step 4 that $t \to f_r^0(t, \psi(t), v(t))$ is integrable.

Finally, we show that (ψ, v) is optimal. From Step 3 we get that

$$\int_{t_0}^{t_1} f_r^0(t, \psi(t), v(t))dt \leq \int_{t_0}^{t_1} \lambda(t)dt \leq \gamma.$$

In Step 2 we obtained a subsequence $\{(\psi_k, v_k)\}$ of the minimizing sequence

such that

$$\gamma = \lim_{k\to\infty} \int_{t_{0k}}^{t_{1k}} f_r^0(t, \psi_k(t), v_k(t))dt,$$

and so

$$\int_{t_0}^{t_1} f_r^0(t, \psi(t), v(t))dt \leq \lim_{k\to\infty} \int_{t_{0k}}^{t_{1k}} f_r^0(t, \psi_k(t), v_k(t))dt. \qquad (5.4.33)$$

Let $m = \inf\{J|(\psi, v)\colon (\psi, v) \text{ admissible}\}$. Then from the uniform convergence of $\{\psi_k\}$ to ψ, the lower semi-continuity of g and (5.4.33) we get:

$$\begin{aligned} m &= \lim_{k\to\infty} J(\psi_k, v_k) = \lim_{k\to\infty} [g(e(\psi_k)) + \int_{t_{0k}}^{t_{1k}} f_r^0(t, \psi_k, v_k)dt] \\ &\geq \liminf_{k\to\infty} g(e(\psi_k)) + \int_{t_0}^{t_1} f_r^0(t, \psi(t), v(t))dt \\ &\geq g(e(\psi)) + \int_{t_0}^{t_1} f_r^0(t, \psi(t), v(t))dt \geq m. \end{aligned}$$

Hence (ψ, v) is optimal.

Remark 5.4.17. In Theorem 5.3.5 the optimal trajectory was obtained as the uniform limit of trajectories in a minimizing sequence and the optimal relaxed control was obtained as a weak limit of the corresponding relaxed controls. In contrast in Theorem 5.4.4, the optimal trajectory was obtained as the uniform limit of trajectories in a minimizing sequence and it was then shown that there exists a control that will yield this trajectory. No relationship is established between the controls in the minimizing sequence and the control that gives the optimal trajectory.

Next, we shall obtain the classical Nagumo-Tonelli existence theorem for the simple problem in the calculus of variations as an almost immediate consequence of Theorem 5.4.16.

Theorem 5.4.18. *Let f^0 be lower semicontinuous in $\mathcal{G} = \mathcal{R} \times \mathbb{R}^n$ and let $f^0(t, x, z) \geq 0$ for all (t, x, z) in $\mathcal{G}$. For each (t, x) in $\mathcal{R}$, let f^0 be a convex function of z. Let $\mathcal{B}$ be closed and let g be lower semi-continuous on $\mathcal{B}$. Let the graphs of all trajectories lie in a compact subset $\mathcal{R}_0$ of $\mathcal{R}$. Let there exist a nonnegative function Φ defined on $[0, \infty)$ such that $\Phi(\xi)/\xi \to \infty$ as $\xi \to \infty$ and such that for all (t, x, z) in $\mathcal{G}$, $f^0(t, x, z) \geq \Phi(|z|)$. Then there exists an absolutely continuous function ϕ^* that satisfies $e(\varphi^*) \in \mathcal{B}$ and that minimizes*

$$J(\phi) = g(t_0, \phi(t_0), t_1, \phi(t_1) + \int_{t_0}^{t_1} f^0(t, \phi(t), \phi'(t))dt.$$

Moreover ϕ^* is optimal for the relaxed version of this problem.

Proof. Write the variational problem as a control problem by writing the integrand as $f^0(t,x,z)$, adding the state equation $x' = z$ and taking $\Omega(t,x) = \mathbb{R}^n$ for all (t,x). It is readily checked that Assumption 5.4.1 holds for the control problem. Since the trajectories of the control problem are the same as those for the variational problem, the trajectories for the control problem lie in a compact set.

In the control formulation,

$$\begin{aligned} Q^+(t,x) &= \{(y^0,y) : y^0 \geq f^0(t,x,z), y = z\} \\ &= \{y^0 : y^0 \geq f^0(t,x,y) \quad y \in \mathbb{R}^n\}. \end{aligned}$$

Since for each (t,x), $f^0(t,x,y)$ is convex in y, it is readily checked that the sets $Q^+(t,x)$ are convex.

Let $\varepsilon > 0$ be given. Then since $\Phi(\xi)/\xi \to \infty$ as $\xi \to \infty$, and $f^0(t,x,z) \geq \Phi(|z|)$, we have that for $|z|$ sufficiently large

$$\frac{f^0(t,x,z)}{|z|} \geq \frac{\Phi(|z|)}{|z|} \geq \frac{1}{\varepsilon}.$$

Thus, $f(t,x,z) = z$ is of slower growth than f^0. Similarly, the function identically equal to one is of slower growth than f^0. It now follows from Theorem 5.4.16 that there exists an absolutely continuous function ϕ^* that minimizes $J(\varphi)$. □

Exercise 5.4.19. Show that if $\mathcal{B}$ is compact, then all trajectories in a minimizing sequence (ordinary or relaxed) will lie in a compact subset $\mathcal{R}'$ of $\mathcal{R}$. Thus, we can replace the second and third sentences in the statement of Theorem 5.4.18 by the statement, "Let $\mathcal{B}$ be compact."

Exercise 5.4.20. In this exercise we obtain an existence theorem for the linear plant-quadratic criterion problem in which the state equations are

$$\frac{dx}{dt} = A(t)x + B(t)z + d(t)$$

and the function f^0 is given by

$$f^0(t,x,z) = \langle x, X(t)x\rangle + \langle z, R(t)z\rangle.$$

The matrices A, B, X, and R are continuous on an interval $[a,b]$, as is the vector d. For each t in $[a,b]$, the matrix $X(t)$ is symmetric, positive semidefinite and the matrix $R(t)$ is symmetric positive definite. The set B is an n dimensional closed manifold consisting of points (t_0,x_0,t_1,x_1) with (t_0,x_0) fixed and (t_1,x_1) in a specified n-dimensional manifold $\mathfrak{J}_1$. The controls u have values in the open set $\mathcal{U}$

1. Show that

$$\int_a^b \langle u(t), R(t)u(t)\rangle dt < +\infty$$

if and only if $u \in L_2[a,b]$. (Hint: Recall that

$$\rho_1(t)|u(t)|^2 \leq \langle u(t), R(t)u(t)\rangle \leq \rho_n(t)|u(t)|^2$$

where $\rho_1(t)$ is the smallest eigenvalue of $R(t)$ and $\rho_n(t)$ is the largest eigenvalue of $R(t)$. Show that ρ_1 is continuous and $\rho_1(t) > 0$ for all t in $[a,b]$.

2. Use the result in (1) and the variation of parameters formula to show that all trajectories (relaxed or ordinary) of a minimizing sequence lie in a compact set $\mathcal{R}_0 \subseteq \mathcal{R}$.

3. Use Theorem 5.4.16 to show that the linear-plant quadratic criterion problem has a relaxed solution that is an ordinary solution (φ, u).

4. Obtain the same conclusion as in (3) using Theorems 5.5.3 and 5.5.7 of the next section.

5.5 Existence Without the Cesari Property

In this section we shall state and prove two existence theorems for relaxed and ordinary problems in which it is not assumed that the weak Cesari property holds. In both of these theorems it will be assumed that the constraint mapping Ω depends on t alone. In one of the theorems we assume that the function $\widehat{f} = (f^0, f)$ satisfies a generalized Lipschitz condition. In the other we assume that the controls in a minimizing sequence all lie in a closed ball of some L_p space, $1 \leq p \leq \infty$.

We assume that $\widehat{f}, g, \mathcal{B}$, and Ω, the data of both the ordinary and relaxed problems satisfy the following:

Assumption 5.5.1. (i) The function $\widehat{f} = (f^0, f) = (f^0, f^1, \ldots, f^n)$ is defined on a set $\mathcal{G} = \mathcal{I} \times \mathcal{X} \times \mathcal{U}$, where $\mathcal{I}$ is a real compact interval, $\mathcal{X}$ is a closed interval in $\mathbb{R}^n$, and $\mathcal{U}$ is an open interval in $\mathcal{R}^m$.

(ii) The function $\widehat{f}$ is continuous on $\mathcal{G}$.

(iii) There exists an integrable function β on $\mathcal{I}$ such that $f^0(t,x,z) \geq \beta$ for all (t,x,z) in $\mathcal{G}$.

(iv) The set $\mathcal{B}$ is closed.

(v) The function g is lower semi-continuous on $\mathcal{B}$.

(vi) Ω is a mapping from $\mathcal{I}$ to subsets $\Omega(t)$ of $\mathcal{U}$ that is upper semi-continuous on $\mathcal{I}$.

Remark 5.5.2. Assumption 5.5.1 differs from Assumption 5.4.1 in that f^0 is continuous rather than semi-continuous and Ω depends on t alone rather than on (t, x).

Theorem 5.5.3. *Let Assumption 5.5.1 hold. Let there exist a non-decreasing function ρ defined on $[0, \infty)$ such that $\rho(\delta) \to 0$ as $\delta \to 0$ and a nonnegative function L defined on $\mathcal{I} \times \mathcal{U}$ such that*

$$|\widehat{f}(t, x, z) - \widehat{f}(t, x', z)| \leq L(t, z)\rho(|x - x'|) \tag{5.5.1}$$

for all (t, x, z) and (t, x', z) in $\mathcal{G}$.

(i) Let the sets $Q_r^+(t, x)$ be closed and let there exist a minimizing sequence $\{\psi_k\}$ for the relaxed problem such that all the trajectories $\{\psi_k\}$ lie in a compact set and are equi-absolutely continuous. Let there exist a constant $A > 0$ such that for all the functions $\{u_{ki}\}$, $i = 1, \ldots, n+2$, $k = 1, 2, \ldots$ appearing in the sequence v_k of relaxed controls

$$\int_{t_0}^{t_1} L(t, u_{ki}(t))dt \leq A. \tag{5.5.2}$$

Then the relaxed problem has a solution.

(ii) If the sets $Q^+(t, x)$ are convex, then there exists an ordinary control that is optimal for the relaxed and ordinary problem.

Remark 5.5.4. If $\widehat{f}$ is Lipschitz continuous in x, uniformly for (t, z) in $\mathcal{I} \times \mathcal{U}$, then (5.5.1) holds with $L(t) = K$, the Lipschitz constant, and $\rho(\delta) = \delta$. If $\widehat{f}$ is uniformly continuous on

$$\mathcal{D} = \{(t, x, z) : (t, x) \in \mathcal{R},\ z \in \Omega(t)\},$$

which occurs if $\mathcal{D}$ is compact, then (5.5.1) holds with $L \equiv 1$ and ρ the modulus of continuity.

Remark 5.5.5. A sufficient condition that the sets $Q^+(t, x)$ be closed is that the function identically equal to one be of slower growth than f^0. This condition and the condition $|f^j| \leq K f^0$ for some constant $K > 0$, $j = 1, \ldots, n$ are sufficient for the sets $Q_r^+(t, x)$ to be closed.

To see this let $\widehat{y}_k = (y_k^0, y_k)$ be a sequence of points in $Q^+(t, x)$ converging to a point (y^0, y). Then there exists a sequence of points $\{z_k\}$ in $\Omega(t)$ such that

$$y_k^0 \geq f^0(t, x, z_k) \quad y_k = f(t, x, z_k). \tag{5.5.3}$$

We assert that the sequence $\{z_k\}$ is bounded. For if not, there would exist a subsequence such that $|z_k| \to \infty$. Let $\varepsilon > 0$ be arbitrary. Then for all k sufficiently large

$$1 \leq \varepsilon f^0(t, x, z_k) \leq \varepsilon y_k^0.$$

Since ε is arbitrary, the preceding would imply that the sequence $\{y_k^0\}$ is unbounded. This contradicts $y_k^0 \to y^0$. Hence $\{z_k\}$ is bounded and there exists a subsequence and a point z in $\mathbb{R}^m$ such that $z_k \to z$. Since the mapping Ω is upper semi-continuous, the sets $\Omega(t)$ are closed. Since the points $\{z_k\}$ are in $\Omega(t)$, so is the point z. If we let $k \to \infty$ in (5.5.3), it follows from the continuity of $\widehat{f}$ that $\widehat{y} = (y^0, y)$ is in $Q^+(t,x)$, and so $Q^+(t,x)$ is closed.

We now show that the sets $Q_r^+(t,x)$ are closed. Let $\widehat{y}_k = (y_k^0, y_k)$ be a sequence of points in $Q_r^+(t,x)$ converging to a point $\widehat{y} = (y^0, y)$. Then there exists a sequence of points $\{\pi_k, \zeta_k\}$, where each π_k is as in (5.4.1) and each ζ_k is as in (5.4.1) with $z_{ki} \in \Omega(t)$ such that

$$y_k^0 = \sum_{i=1}^{n+2} \pi_k^i f^0(t,x,z_{ki}) \qquad y_k = \sum_{i=1}^{n+2} \pi_k^i f(t,x,z_{ki}).$$

Since $\pi_k^i \geq 0$, $i = 1, \ldots, n+2$ and $f^0 \geq 0$, it follows that

$$\pi_k^i f^0(t,x,z_{ki}) \leq y_k^0 \tag{5.5.4}$$

for $i = 1, \ldots, n+2$. In what follows we shall be taking a series of subsequences with the implicit assumption that each subsequence is a subsequence of the previous subsequence.

Since Π_{n+2} is compact there exists a subsequence of $\{k\}$ such that $\{\pi_k\}$ converges to a point π in Π_{n+2}. Let $\pi^1, \ldots, \pi^s$ be the positive components of π. Then $\sum_{i=1}^s \pi^i = 1$. We assert that each of the sequences $\{z_{ki}\}$, $i = 1, \ldots, s$ is bounded. For if a sequence $\{z_{ki}\}$ were unbounded there would exist a subsequence such that $|z_{ki}| \to \infty$. Hence for each $\varepsilon > 0$, we would have $\varepsilon f^0(t,x,z_{ki}) > 1$ for all sufficiently large k. From this and from (5.5.4) we would get

$$\varepsilon y_k^0 \geq \varepsilon \pi_k^i f^0(t,x,z_{ki}) > \pi_k^i.$$

Since $\varepsilon > 0$ is arbitrary and $\pi_k^i \to \pi^i > 0$, this would imply that y_k^0 is unbounded, thus contradicting $y_k^0 \to y^0$. Since each sequence $\{z_{ki}\}$, $i = 1, \ldots, s$ is bounded, there exists a subsequence of $\{k\}$ and a point $(z_1, \ldots, z_s)$ such that $z_{ki} \to z$. Since the points $\{z_{ki}\}$ are in $\Omega(t)$ and $\Omega(t)$ is closed, each $z_i \in \Omega(t)$. From the continuity of $\widehat{f}$ it follows that $\widehat{f}(t,x,z_{ki}) \to \widehat{f}(t,x,z_i)$ for $i = 1, \ldots, s$.

We now consider those components π^i, if any, with $\pi^i = 0$. If the sequence $\{z_{ki}\}$ is bounded, then there exists a subsequence $\{z_{ki}\}$ and a point $z_i \in \Omega(t)$ such that $z_{ki} \to z_i$. Hence $\widehat{f}(t,x,z_{ki}) \to \widehat{f}(t,x,z_i)$ and so $\pi_k^i \widehat{f}(t,x,z_{ki}) \to 0$. If the sequence $\{z_{ki}\}$ is unbounded, then there exists a subsequence $\{z_{ki}\}$ such that $|z_{ki}| \to \infty$. Since one is of slower growth than f^0, there is a subsequence $\{z_{ki}\}$ of the preceding subsequence such that $1 < (\pi_k^i)^{1/2} f^0(t,x,z_{ki})$ for sufficiently large k. From this, from $0 \leq \pi_k^i \leq 1$, and from (5.5.4) we get

$$1 < (\pi_k^i)^{1/2} f^0(t,x,z_{ki}) \leq \pi_k^i f^0(t,x,z_{ki}) \leq y_k^0.$$

Hence

$$(\pi_k^i)^{1/2} < \pi_k^i f^0(t,x,z_{ki}) \leq (\pi_k^i)^{1/2} y_k^0.$$

If we now let $k \to \infty$, we get that

$$\pi_k^i f^0(t,x,z_{ki}) \to 0.$$

From this and the condition $|f^j| \leq Kf^0$, $j = 1,\ldots,n$ we get that $\pi_k^i f^j(t,x,z_{ki}) \to 0$.

In summary, we have shown that there exists a subsequence of $\{\widehat{y}_k\} = \{y_k^0, y_k\}$ and corresponding subsequences of $\{\pi_k\}$ and $\{z_{ki}\}$ such that

$$\widehat{y}_k \to \sum_{i=1}^{s} \pi^i \widehat{f}(t,x,z_i)$$

with $z_i \in \Omega(t)$ and $(\pi^1,\ldots,\pi^s) \in \Pi_s$ where $\pi^1,\ldots,\pi^s$ are the positive components of π. Since $\widehat{y}_k \to \widehat{y}$, we get that $\widehat{y} \in Q_r^+(t,x)$, and so $Q_r^+(t,x)$ is closed.

Example 5.5.6. We give an example of a system in which the weak Cesari property fails to hold, but (5.5.1) and (5.5.2) do hold. This example also shows that the growth condition of Remark 5.5.5 is not a necessary condition for the sets $Q^+(t,x)$ to be closed. Let $x = (x^1,x^2)$, let z be a real number, let $\Omega(t) = \mathbb{R}^1$, let $f^0 \equiv 0$, and let $f(t,x,z) = (z, x^1 z)$. Then (5.5.1) holds with $L(t,z) = |z|$. For each (t,x) in $\mathcal{R}$

$$Q^+(t,x) = \{(\eta,\xi) = (\eta,\xi^1,\xi^2) : \eta \geq 0,\ \xi^1 = z,\ \xi^2 = x^1 z,\ z \in \mathbb{R}\}. \tag{5.5.5}$$

The sets $Q^+(t,x)$ are closed and convex, yet one is not of slower growth than f^0. Also, for each $\delta > 0$

$$\text{cl co } Q^+(N_{\delta x}(t,x)) = \{(\eta,\xi) : \eta \geq 0,\ \xi \in \mathbb{R}^2\}.$$

This and (5.5.5) show that the weak Cesari property fails.

To show that Theorem 5.5.3 can be applied to this problem, let $\{(\psi_k, p_k^1 u_{1k} + p_k^2 u_{2k})\}$ be a relaxed minimizing sequence. Since $Q^+(t,x)$ is convex we may replace the sequence of relaxed controls $\{(p_k^1 u_{1k} + p_k^2 u_{2k})\}$ by a sequence of ordinary controls $\{u_k\}$ with corresponding trajectory $\{\psi_k\}$. Since the function ψ_k are equi-absolutely continuous, it follows from the equation $(\psi_k^1)' = u_k$ that the integrals $\int_{t_0}^{t_1} |u_k| dt$ are bounded by a constant A. Hence

$$\int_{t_0}^{t_1} L(t, u_k(t), dt = \int_{t_0}^{t_1} |u_k(t)| dt \leq A,$$

and (5.5.2) holds.

Proof of Theorem 5.5.3. We first note that conclusion (ii) follows from (i) by virtue of Corollary 4.4.3. Hence it suffices to prove (i).

If $\widehat{f}$ satisfies (5.5.1), then

$$|\widehat{f}_r(t,x,\pi,z) - \widehat{f}_r(t,x',\pi,z)| \leq \sum_{i=1}^{n+2} \pi^i |\widehat{f}(t,x,z_i) - \widehat{f}(t,x',z_i)|$$
$$\leq \sum_{i=1}^{n+2} \pi^i L(t)|x-x'| = L(t)|x-x'|.$$

Hence $\widehat{f}_r$ satisfies (5.5.1) as well as the other hypotheses in Assumption 5.5.1. Let $\widetilde{\Omega}$ be as in the proof of Theorem 5.4.4. Then $\widetilde{\Omega}$ is upper semi-continuous. Hence we may proceed as in the proof of Theorem 5.4.4 and take the relaxed problem to be an ordinary problem with control $v = (p, \overline{v}) = (p^1, \ldots, p^{n+2}, u_1, \ldots, u_{n+2})$.

The proof proceeds exactly as the proof of Theorem 5.4.4 up to and including the definition of λ_j in (5.4.31). The rest of the argument to prove Step 3 proceeds differently. The reader is urged to keep in mind the order in which various subsequences are chosen. □

Define sequences of functions σ_j and θ_j corresponding to ψ_j and λ_j as follows

$$\sigma_j(t) = \sum_{i=1}^{k} \alpha_{ij} f_r(t, \psi(t), v_{n_j+i}) \tag{5.5.6}$$

$$\theta_j(t) = \sum_{i=1}^{k} \alpha_{ij} f_r^0(t, \psi(t), v_{n_j+i}), \tag{5.5.7}$$

where if $t \notin [t_{0q}, t_{1q}]$ we set $\widehat{f}(t, \psi_q, v_q(t)) = 0$.

Let $M_k = \max\{|\widetilde{\psi}_k(t) - \widetilde{\psi}(t)| : t \in \mathcal{I}\}$. Since $\widetilde{\psi}_k$ converges uniformly to $\widetilde{\psi}$ on $\mathcal{I}$, $M_k \to 0$ as $k \to \infty$. Let $\widehat{f}^*_{rq}(t) \equiv \widehat{f}_r(t, \psi(t), v_q(t))$ and let $\widehat{f}_{rq}(t) \equiv \widehat{f}_r(t, \psi_q(t), v_q(t))$. Note that $\widehat{f}^*_{rq}(t) = \widehat{f}_{rq}(t) = 0$ for $t \notin [t_{0q}, t_{1q}]$. Using (5.5.1), (5.5.2), and (5.5.6), we get

$$\begin{aligned}
\int_a^b |\sigma_j - \omega_j| dt &\leq \sum_{i=1}^{k} \alpha_{ij} \int_a^b |f^*_{r,n_j+i} - f_{r,n_j+i}| dt \\
&= \sum_{i=1}^{k} \alpha_{ij} \int_{t_0,n_j+i}^{t_1,n_j+i} |f^*_{r,n_j+i} - f_{r,n_j+i}| dt \\
&\leq \sum_{i=1}^{k} \alpha_{ij} \rho(M_{n_j+i}) \int_{t_0,n_j+i}^{t_1,n_j+i} L(t, u_{n_j+i}(t)) dt \\
&\leq A \sum_{i=1}^{k} \alpha_{ij} \rho(M_{n_j+i}).
\end{aligned}$$

Since $M_k \to 0$ and $\rho(\delta) \to 0$ as $\delta \to 0$ we get that $\sigma_j - \omega_j \to 0$ in $L_1[a,b]$. A similar argument shows that $\theta_j - \lambda_j \to 0$ in $L_1[a,b]$. Hence there exists a subsequence such that

$$\sigma_j(t) - \omega_j(t) \to 0 \quad \text{and} \quad \theta_j(t) - \lambda_j(t) \to 0 \quad \text{a.e.} \tag{5.5.8}$$

We now define λ as in (5.4.30) and show as we did in the paragraph following (5.4.30) that λ is in $L_1[a,b]$ and that (5.4.26) holds.

As in Step 3 of the proof of Theorem 5.4.4, let T' denote the set of points in $[t_0, t_1]$ at which $\lambda(t)$ is finite, $\omega_j(t) \to \psi'(t)$ and at which $v_k(t) \in \widetilde{\Omega}(t)$ for all k. This set has measure $t_1 - t_0$. Let T'' denote the set of points at which (5.5.8) holds. Let $T = T' \cup T''$. Then meas $T = t_1 - t_0$.

Let t be a fixed but arbitrary point of T. Since $\omega_j(t) \to \psi'(t)$, it follows from (5.5.8) that $\sigma_j(t) \to \psi'(t)$. It follows from the definition of λ that there is a subsequence $\{\lambda_j(t)\}$, which in general depends on t such that $\lambda_j(t) \to \lambda(t)$. From (5.5.8) we get that $\theta_j(t) \to \lambda(t)$. By definition, for all t in T, and all j and i

$$\widehat{f}_r(t, \psi(t), v_{n_j+i}(t)) \in Q_r^+(t, \psi(t)).$$

Since $Q_r^+(t, \varphi(t))$ is convex, the points $(\theta_j(t), \sigma_j(t))$ belong to $Q_r^+(t, \varphi(t))$. Since $Q_r^+(t, \varphi(t))$ is also assumed to be closed and $(\theta_j(t), \sigma_j(t)) \to (\lambda(t), \psi'(t))$, we get that $(\lambda(t), \psi'(t)) \in Q_r^+(t, \varphi(t))$. Since t is an arbitrary point of T, we have that $(\lambda(t), \psi'(t)) \in Q_r^+(t, \varphi(t))$ a.e. in $[t_0, t_1]$.

The rest of the proof is exactly the same as in the proof of Theorem 5.4.4.

Theorem 5.5.7. *Let Assumption 5.5.1 hold.*

(i) Let the sets $Q_r^+(t,x)$ be closed. Let $\{\psi_k\}$ be the relaxed trajectories in a minimizing sequence for the relaxed problem such that the trajectories $\{\psi_k\}$ all lie in a compact set and are equi-absolutely continuous. Let all of the control functions $\{u_{k,1}, \ldots, u_{k,n+2}\}$ corresponding to the trajectories $\{\psi_k\}$ all lie in a closed ball of some L_p space $1 \le p \le \infty$. Then the relaxed problem has a solution.

(ii) Let the sets $Q^+(t,x)$ be closed and convex. Then there exists an ordinary control that is a solution of both the relaxed and ordinary problems.

Proof. Since all the controls $\overline{v}_k = \{u_{k,1}, \ldots, u_{k,n+2})$ lie in a closed ball some L_p space and since $p(t) = (p^1(t), \ldots, p^{n+1}(t))$ satisfies $\sum p^i(t) = 1$ and $p^i(t) \ge 0$, it follows that all the controls $v_k = (p_k, \overline{v}_k)$ lie in a closed ball of some L_p space. The sets Q_r^+ are convex. Hence we can view the relaxed problem as an ordinary problem, with all the hypotheses of (i) satisfied. By Corollary 4.4.3 we need only establish (i). We shall do so assuming that the relaxed problem is an ordinary problem, as was done in the proof of Theorem 5.4.4.

The proof proceeds as does the proof of Theorem 5.4.4 through Steps 1 and 2. Step 3 is modified as follows. Since $\widetilde{\psi}_k \to \widetilde{\psi}$ uniformly on $[a,b]$ and all trajectories lie in a compact set, it follows that there exists an $M' > 0$ such that $\|\widetilde{\psi}_k\|_p \le M'$ and $\|\widetilde{\psi}\|_p \le M'$, where $\| \ \|_p$ denotes the $L_p[a,b]$ norm.

Let $\widetilde{v}_k$ denote the extension of v_k from $[t_{0k}, t_{1k}]$ to $[a, b]$ by setting $\widetilde{v}_k(t) = 0$ if $t \notin [t_{0k}, t_{1k}]$. Since by hypothesis the functions v_k lie in a closed ball of radius M in $L_p[t_{0k}, t_{1k}]$ we get that for all k, the functions $v_k = (\widetilde{\psi}_k, \widetilde{v}_k)$ and $w_k = (\widetilde{\psi}, \widetilde{v}_k)$ lie in a ball in $L_p[a, b]$. Also note that $v_k(t) - w_k(t) \to 0$ at all points of $[a, b]$.

Let

$$\widehat{\Delta}_k(t) = \widehat{f}_r(t, \psi(t), v_k(t)) - \widehat{f}_r(t, \psi_k(t), v_k(t)), \tag{5.5.9}$$

where we set $\widehat{\Delta}_k(t) = 0$ if $t \notin [t_{0k}, t_{1k}]$. It is then a consequence of Lemma 5.3.8 with $\xi = (x, \overline{z})$ and $h(t, \xi) = \widehat{f}(t, x, \overline{z})$ that $\widehat{\Delta}_k \to 0$ in measure on $[a, b]$. Since $\widehat{\Delta}_k \to 0$ in measure on $[a, b]$ there exists a subsequence such that

$$\widehat{\Delta}_k(t) \to 0 \quad \text{a.e.} \tag{5.5.10}$$

in $[a, b]$.

The functions ω_j, λ_j, and λ are next defined as in Step 3 of the proof of Theorem 5.4.4 and it is shown that (5.4.28) holds. Sequences $\{\sigma_j\}$ and $\{\theta_j\}$ are defined as in (5.5.6). If, as usual, we denote the first component of $\widehat{\Delta}_k(t)$ by $\Delta_k^0(t)$ and the remaining n components by $\Delta_k(t)$ we get, using (5.5.6), (5.5.9), (5.4.27), and (5.4.29), that

$$\theta_j(t) - \lambda_j(t) = \sum_{i=1}^{k} \alpha_{ij} \Delta^0_{n_j+i}(t)$$

$$\sigma_j(t) - \omega_j(t) = \sum_{i=1}^{k} \alpha_{ij} \Delta_{n_j+i}(t).$$

It then follows from (5.5.10) and Lemma 5.3.7 that (5.5.8) holds. □

The rest of the proof is a verbatim repetition of the last four paragraphs of the proof of Theorem 5.5.3.

5.6 Compact Constraints Revisited

In this section we shall use Theorem 5.4.4 to obtain an existence theorem for problems with compact constraints.

Theorem 5.6.1. *Let Assumption* 5.4.1 *hold, except for (vi), which we replace by the following. The mapping* Ω *from* $\mathcal{R} = \mathcal{I} \times \mathcal{X}$ *to compact subsets* $\Omega(t, x)$ *of* $\mathcal{U}$ *is u.s.c.i. on* $\mathcal{R}$.

(i) *Let the set of admissible relaxed trajectories be non-empty and let all admissible relaxed trajectories lie in a compact set* $\mathcal{R}_0 \subseteq \mathcal{R}$. *Then the relaxed problem has a solution.*

(ii) *Let the sets $Q^+(t,x)$ be convex for all (t,x) in $\mathcal{R}$. Then there exists an ordinary control that is a solution of both the ordinary problem and the relaxed problem.*

Remark 5.6.2. In proving (i) of Theorem 5.6.1 we take the formulation of the relaxed problem to be given by (5.4.1) and (5.4.2), rather than the one given in Section 3.2. In Section 3.5 we showed that these two formulations are equivalent. As already noted several times, the relaxed problem (5.4.1)–(5.4.2) can be viewed as an ordinary problem. Since the sets $Q_r^+(t,x)$ are convex, the relaxed problem, viewed as an ordinary problem satisfies the hypotheses of Theorem 4.4.2, an existence theorem for the ordinary problem. As in previous arguments, we shall assume that the relaxed problem is an ordinary problem. We may also assume without loss of generality that $\beta = 0$, so $f^0 \geq 0$.

Remark 5.6.3. The hypotheses of Theorem 5.6.1 are less stringent than those of Theorem 4.3.5. In Theorem 5.6.1 we do not require $\widehat{f}$ to be Lipschitz in x, whereas we do in Theorem 4.3.5. Also in Theorem 4.3.5 the sets $\Omega(t,x)$ are required to depend on t alone; that is, $\Omega(t,x) = \Omega(t,x')$ for all x, x' in $\mathcal{R}$. In Theorem 5.6.1 the sets $\Omega(t,x)$ can depend on t and x.

Proof of Theorem 5.6.1. By Corollary 4.4.3 we need only establish (i). We prove Theorem 5.6.1 by showing that the hypotheses of Theorem 5.6.1 imply those of Theorem 5.4.4.

We first show that (vi) of Assumption 5.4.1 holds, namely that the mapping $\widetilde{\Omega}$ is upper semi-continuous on $\mathcal{R}$. The set $\widetilde{\Omega}$ is defined following (5.4.5). The upper semi-continuity of $\widetilde{\Omega}$ follows from Lemma 5.2.2 and the assumption that Ω is u.s.c.i. and the sets $\Omega(t,x)$ are compact.

We next show that the trajectories $\{\psi_k\}$ in a minimizing sequence are equi-absolutely continuous. Let

$$\mathcal{D}_r = \{(t,x,\overline{z}) : (t,x) \in \mathcal{R}_0,\ \overline{z} \in \widetilde{\Omega}(t,x)\}. \tag{5.6.1}$$

By Lemma 3.3.11, the set $\mathcal{D}_r$ is compact. Since f_r is continuous on $\mathcal{D}_r$, there exists a constant $A > 0$ such that $|f(t,x,\overline{z})| \leq A$ for all $(t,x,\overline{z})$ in $\mathcal{D}_r$. For each k and almost all $t \in [t_{0k}, t_{1k}]$, we have $(t, \psi_k(t), v_k(t)) \in \mathcal{D}$. Since for a.e. $t \in [t_{0k}, t_{1k}]$

$$\psi_k'(t) = f(t, \psi_k(t), v_k(t)),$$

we have $|\psi_k'(t)| \leq A$, a.e. on $[t_{0,k}, t_{1k}]$. Hence the trajectories $\{\psi_k\}$ are equi-absolutely continuous.

To complete the proof we show that the mapping Q_r^+ satisfies the weak Cesari property at all points of $\mathcal{R}_0$. Let $\mathcal{D}$ be as in (5.4.7). The proof proceeds as does the proof of Lemma 5.4.6 from (5.4.8) to (5.4.16).

Since for each k, $z_{ki} \in \Omega(t, x_{ki})$, the sequence $\{(t, x_{ki}, z_{ki})\}$ is in $\mathcal{D}$ for $i = 1, \ldots, n+2$. From the compactness of $\mathcal{D}$ and from (5.4.13), it follows that there exists a subsequence

$$\{(t, x_{k1}, z_{k1}), \ldots, (t, x_{k,n+2}, z_{k,n+2})\}$$

and points $z_1, \ldots, z_{n+2}$ such that $(t, x, z_i) \in \mathcal{D}$ and

$$(t, x_{ki}, z_{ki}) \to (t, x, z_i) \quad i = 1, \ldots, n+2. \tag{5.6.2}$$

Thus, $z_i \in \Omega(t, x)$.

From (5.4.10), (5.4.13), (5.4.14), (5.4.15), (5.6.2), and the continuity of f we get that

$$y = \lim_{k\to\infty} y_k = \lim_{k\to n} \sum_{i=1}^{n+2} \alpha_{ki} y_{ki} = \sum_{i=1}^{n+2} \alpha_i f(t, x, z_i) \tag{5.6.3}$$

where $z_i \in \Omega(t, x)$. From (5.4.11), (5.4.14), (5.4.16), (5.6.2), and the lower semi-continuity of f^0 we get that

$$\begin{aligned} y^0 &\geq \sum_{i=1}^{n+2} \lim_{k\to\infty} \inf \alpha_{ki} y^0_{ki} \geq \sum_{i=1}^{n+2} \liminf_{k\to\infty} (\alpha_{ki} f^0(t, x_{ki}, z_{ki})) \\ &= \sum_{i=1}^{n+2} \alpha_i f^0(t, x, z_i) \end{aligned} \tag{5.6.4}$$

with $z_i \in \Omega(t, x)$. From (5.6.3) and (5.6.4) we get that $\widehat{y} = (y^0, y)$ is in co $Q^+(t, x)$. But co $Q^+(t, x) = Q^+_r(t, x)$, so $\widehat{y} \in Q^+_r(t, x)$, which shows that Q^+_r has the weak Cesari property at (t, x). □

Chapter 6

The Maximum Principle and Some of Its Applications

6.1 Introduction

In this chapter we state several versions of the maximum principle, corresponding to different hypotheses on the data of the problem. We shall collectively call each of these results, "the maximum principle", and shall use them to characterize the optimal controls in several important classes of problems. The proof of the maximum principle will be given in the next chapter.

In Section 6.2 we use a dynamic programming argument to derive the maximum principle for ordinary problems. Although the arguments are elementary and mathematically correct, the assumptions made rule out most interesting problems. The purpose of this section is to make the maximum principle plausible and to give a geometric interpretation of the theorem. From the view point of logical development Section 6.2 can be omitted, except for the concepts of value function and optimal synthesis, or optimal feedback control, which are introduced in Section 6.2 and used again in Section 6.9.

In Section 6.3 we state the maximum principle for the relaxed problem in Bolza form. The statements of the maximum principle for other formulations of the problem, such as those discussed in Chapter 2, are taken up in the exercises. In special cases of importance more precise characterizations of the optimal pair can often be given. Some of these are also taken up in the exercises. The exercises in this section are an important supplement to the general theory.

In Section 6.4 we use the maximum principle and one of our existence theorems to determine the optimal pair in a specific example. The purpose here is to illustrate how the maximum principle is used and some of the difficulties that one can expect to encounter in large-scale problems.

The remaining sections of the chapter are devoted to applications of the maximum principle to special classes of problems. In Section 6.5 we obtain the first order necessary conditions of the calculus of variations from the maximum principle, both in the classical case and in the case where the functions are in the class $W^{1,1}$. The functions in $W^{1,1}$ are absolutely continuous with square integrable derivatives. In the exercises we take up the relationship between the

classical Bolza problem in the calculus of variations and the control problem. In Section 6.6 we take up control problems that are linear in the state variable. We specialize this in Section 6.7 to linear problems, and further specialize in Section 6.8 to the linear time optimal problem. The standard results for these problems are obtained, whenever possible, as relatively simple consequences of the maximum principle. The power of the maximum principle will be apparent to the reader.

In Section 6.9 we take up the linear plant quadratic cost criterion problem. Here again we obtain the standard characterization of the optimal pair from the maximum principle. We also show that the necessary conditions are sufficient and we obtain the standard synthesis of the optimal control.

6.2 A Dynamic Programming Derivation of the Maximum Principle

Let $\mathcal{R}_1$ be a region of (t, x)-space and let $\mathcal{R}$ be a subregion of $\mathcal{R}_1$ such that the closure of $\mathcal{R}$ is contained in $\mathcal{R}_1$. For each point (τ, ξ) in $\mathcal{R}$ we consider the following problem. Minimize the functional

$$J(\phi, u) = g(t_1, \phi(t_1)) + \int_{\tau}^{t_1} f^0(t, \phi(t), u(t))dt \tag{6.2.1}$$

subject to the state equations

$$\frac{dx}{dt} = f(t, x, u(t)), \tag{6.2.2}$$

control constraints $u(t) \in \Omega(t)$, and end conditions

$$(t_0, \phi(t_0)) = (\tau, \xi) \qquad (t_1, \phi(t_1)) \in \mathcal{T}.$$

We assume that the terminal set $\mathcal{T}$ is a $C^{(1)}$ manifold of dimension r, where $0 < r \leq n$ and that $\mathcal{T}$ is part of the boundary of $\mathcal{R}$. See Figure 6.1. For simplicity we also assume that $\mathcal{T}$ can be represented by a single coordinate patch. That is, we assume that $\mathcal{T}$ consists of all points of the form (t_1, x_1) with

$$t_1 = T(\sigma) \qquad x_1 = X(\sigma), \tag{6.2.3}$$

where T and X are $C^{(1)}$ functions defined on an open parallelepiped Σ in R^r.

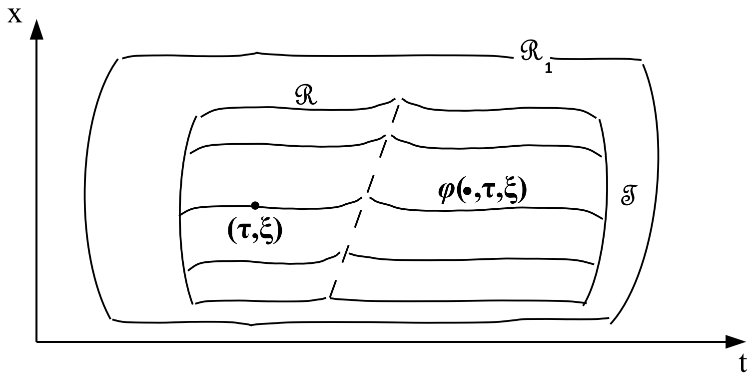

FIGURE 6.1

It is also assumed that the Jacobian matrix of the mapping (6.2.3),

$$\frac{\partial(T,X)}{\partial\sigma} = \begin{pmatrix} \frac{\partial T}{\partial\sigma^1} & \cdots & \frac{\partial T}{\partial\sigma^r} \\ \frac{\partial X^1}{\partial\sigma^1} & \cdots & \frac{\partial X^1}{\partial\sigma^r} \\ \vdots & & \\ \frac{\partial X^n}{\partial\sigma^1} & \cdots & \frac{\partial X^n}{\partial\sigma^r} \end{pmatrix},$$

has rank r at all points of Σ. We assume that the function g in (6.2.1) is defined and $C^{(1)}$ in a neighborhood of $\mathcal{T}$ and that f^0 and f are $C^{(1)}$ functions on $\mathcal{G}_1 = \mathcal{R}_1 \times \mathcal{U}$. Note that the constraint mapping Ω is assumed to be independent of x and to depend only on t.

We assume that for each (τ,ξ) in $\mathcal{R}$ the problem has a unique solution. We denote the unique optimal trajectory for the problem with initial point (τ,ξ) by $\phi(\cdot,\tau,\xi)$. The corresponding unique optimal control is denoted by $u(\cdot,\tau,\xi)$. We assume that the function $u(\cdot,\tau,\xi)$ is piecewise continuous and that at a point of discontinuity t_d the value of $u(\cdot,\tau,\xi)$ is its right-hand limit; thus, $u(t_d,\tau,\xi) = u(t_d+0,\tau,\xi)$. Points (t,x) on the trajectory satisfy the relation $x = \phi(t,\tau,\xi)$. In particular, note that

$$\xi = \phi(\tau,\tau,\xi).$$

The value of the optimal control at time t is $u(t) = u(t,\tau,\xi)$.

For each point (τ,ξ) in $\mathcal{R}$, let $W(\tau,\xi)$ denote the value given to the functional (6.2.1) by the unique optimal pair $(\phi(\cdot,\tau,\xi),\ u(\cdot,\tau,\xi))$. Thus, if $\mathcal{A}(\tau,\xi)$ denotes the set of admissible pairs (ϕ,u) for the problem with initial point (τ,ξ) then

$$W(\tau,\xi) = \min\{J(\phi,u)\colon (\phi,u) \in \mathcal{A}(\tau,\xi)\}. \tag{6.2.4}$$

The function W so defined is called the *value function* for the problem.

Let $\tau_1 > \tau$ and let (τ_1, ξ_1) be a point on the optimal trajectory $\phi(\cdot, \tau, \xi)$. Then $\xi_1 = \phi(\tau_1, \tau, \xi)$. We assert that the optimal pair for the problem starting at (τ_1, ξ_1) is given by $(\phi(\cdot, \tau, \xi),\ u(\cdot, \tau, \xi))$. That is, for $t \geq \tau_1$, $t' = t - \tau_1$,

$$\begin{aligned} \phi(t', \tau_1, \xi_1) &= \phi(t, \tau, \xi) \\ u(t', \tau_1, \xi_1) &= u(t, \tau, \xi). \end{aligned} \tag{6.2.5}$$

In other words, an optimal trajectory has the property that it is optimal for the problem that starts at any point on the trajectory. To see this we write

$$W(\tau, \xi) = \int_{\tau}^{\tau_1} f^{0*}(t, \tau, \xi)dt + \int_{\tau_1}^{t_1} f^{0*}(t, \tau, \xi)dt + g(t_1, \phi(t_1, \tau, \xi)), \tag{6.2.6}$$

where

$$f^{0*}(t, \tau, \xi) = f^0(t, \phi(t, \tau, \xi), u(t, \tau, \xi)). \tag{6.2.7}$$

If $(\phi(\cdot, \tau, \xi), u(\cdot, \tau, \xi))$ were not optimal for the problem initiating at (τ_1, ξ_1), then by (6.2.4) with (τ, ξ) replaced by (τ_1, ξ_1) and by our assumption of uniqueness of optimal pairs, we would have that $W(\tau_1, \xi_1)$ is strictly less than the sum of the last two terms in the right-hand side of (6.2.6). Recall that $u(\cdot, \tau_1, \xi_1)$ is the optimal control for the problem with initial point (τ_1, ξ_1). Hence for a control u defined by

$$u(t) = \begin{cases} u(t, \tau, \xi) & \tau \leq t < \tau_1 \\ u(t, \tau_1, \xi_1) & \tau_1 \leq t \leq t_1 \end{cases}$$

the corresponding trajectory ϕ would be such that $J(\phi, u) < W(\tau, \xi)$, thus contradicting (6.2.4). Hence (6.2.5) holds.

We define a function U on $\mathcal{R}$ as follows

$$U(\tau, \xi) = u(\tau, \tau, \xi).$$

If we set $t = \tau_1$ in the second equation in (6.2.5) and use the definition of U we get that for all $\tau_1 \geq \tau$

$$u(\tau_1, \tau, \xi) = U(\tau_1, \xi_1), \tag{6.2.8}$$

where $\xi_1 = \phi(\tau_1, \tau, \xi)$. Thus, at each point (τ, ξ) in $\mathcal{R}$ the value $U(\tau, \xi)$ of U is the value of the unique optimal control function associated with the unique optimal trajectory through the point. The function U is called the *synthesis of the optimal control or optimal synthesis function.* It is also called the *optimal feedback control.*

Recall that we are assuming here that given a point (τ, ξ) in $\mathcal{R}$, there is a unique optimal pair for the problem with this initial point. In general, there may exist initial points such that there is more than one optimal pair. In this case, the optimal feedback control is multi-valued at this point.

We now suppose that the function W is $C^{(1)}$ on $\mathcal{R}$. We shall derive a

partial differential equation that W must satisfy. Consider a point (τ, ξ) in $\mathcal{R}$ and an interval $[\tau, \tau+\Delta t]$, where $\Delta t > 0$. Let v be a continuous control defined on $[\tau, \tau+\Delta t]$ satisfying $v(t) \in \Omega(t)$. We suppose that Δt is so small that the state equations (6.2.2) with $u(t)$ replaced by $v(t)$ have a solution ψ defined on $[\tau, \tau+\Delta t]$ and satisfying the relation $\psi(\tau) = \xi$. Let $\Delta x = \psi(\tau+\Delta t) - \psi(\tau)$. Thus, the control v transfers the system from ξ to $\xi + \Delta x$ in the time interval $[\tau, \tau+\Delta t]$. For $t \geq \tau + \Delta t$ let us use the optimal control for the problem with initial point $(\tau + \Delta t, \xi + \Delta x)$. Let $\widetilde{u}$ denote the control obtained by using v on $[\tau, \tau+\Delta t]$ and then $u(\cdot, \tau+\Delta t, \xi+\Delta x)$. Let $\widetilde{\phi}$ denote the resulting trajectory. Then $(\widetilde{\phi}, \widetilde{u}) \in \mathcal{A}(\tau, \xi)$ and

$$W(\tau,\xi) \leq J(\widetilde{\phi}, \widetilde{u}) = \int_{\tau}^{\tau+\Delta t} f^0(s, \psi(s), v(s))ds + \int_{\tau+\Delta t}^{t_1} f^{0*}(s, \tau+\Delta t, \xi+\Delta x)ds \\ + g(t_1, \phi(t_1, \tau+\Delta t, \xi+\Delta x)),$$

where f^{0*} is defined in (6.2.7). The sum of the last two terms on the right is equal to $W(\tau+\Delta t, \xi+\Delta x)$. Hence

$$W(\tau+\Delta t, \xi+\Delta x) - W(\tau,\xi) \geq -\int_{\tau}^{\tau+\Delta t} f^0(s, \psi(s), v(s))ds.$$

Since W is $C^{(1)}$ on $\mathcal{R}$ we can apply Taylor's theorem to the left-hand side of the preceding inequality and get

$$W_\tau(\tau,\xi)\Delta t + \langle W_\xi(\tau,\xi), \Delta x\rangle + o(|(\Delta t, \Delta x)|) \geq -\int_{\tau}^{\tau+\Delta t} f^0(s, \psi(s), v(s))ds, \tag{6.2.9}$$

where (W_τ, W_ξ) denotes the vector of partial derivatives of W and $o(|(\Delta t, \Delta x)|)/\ |(\Delta t, \Delta x)| \to 0$ as $|(\Delta t, \Delta x)| \to 0$. From the relation

$$\Delta x/\Delta t = \frac{1}{\Delta t}\int_{\tau}^{\tau+\Delta t} f(s, \psi(s), v(s))ds$$

and the continuity of f, ψ, and v it follows that

$$\lim_{\Delta t\to 0} \frac{\Delta x}{\Delta t} = f(\tau, \psi(\tau), v(\tau)) = f(\tau, \xi, v(\tau)).$$

Therefore, if we divide through by $\Delta t > 0$ in (6.2.9) and then let $\Delta t \to 0$, we get that

$$W_\tau(\tau,\xi) + \langle W_\xi(\tau,\xi), f(\tau,\xi,v(\tau))\rangle \geq -f^0(\tau,\xi,v(\tau)). \tag{6.2.10}$$

If we carry out the preceding analysis with $v(s) = u(s,\tau,\xi)$ on $[\tau, \tau+\Delta t]$, then equality holds at every step of the argument. Therefore, using (6.2.8), we obtain the relation

$$W_\tau(\tau,\xi) = -f^0(\tau,\xi,U(\tau,\xi)) - \langle W_\xi(\tau,\xi), f(\tau,\xi,U(\tau,\xi))\rangle. \tag{6.2.11}$$

We now make the further assumption that the constraint mapping Ω is sufficiently smooth so that for every vector $z \in \Omega(\tau)$ there exists a continuous function v defined on some interval $[\tau, \tau + \Delta t]$, $\Delta t > 0$, with $v(\tau) = z$ and $v(s) \in \Omega(s)$ on $[\tau, \tau + \Delta t]$. In particular, if Ω is a constant mapping, that is, $\Omega(t) = \mathcal{C}$ for all t, then we may take $v(s) = z$ on $[\tau, \tau + \Delta t]$. Under the assumption just made concerning Ω, we can combine (6.2.10) and (6.2.11) to get the relation

$$W_\tau(\tau, \xi) = \max_{z \varepsilon \Omega(\tau)} [-f^0(\tau, \xi, z) - \langle W_\xi(\tau, \xi), f(\tau, \xi, z)\rangle], \tag{6.2.12}$$

with the maximum being attained at $z = U(\tau, \xi)$. Equation (6.2.12) is sometimes called Bellman's equation. Equation (6.2.11) is a Hamilton-Jacobi equation.

Equations (6.2.11) and (6.2.12) can be written more compactly. First define a real valued function H on $\mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \times \mathbb{R}^n$ by the formula

$$H(t, x, z, q^0, q) = q^0 f^0(t, x, z) + \langle q, f(t, x, z)\rangle. \tag{6.2.13}$$

If we now denote a generic point in $\mathcal{R}$ by (t, x) rather than by (τ, ξ) we can write (6.2.11) in terms of H as follows:

$$W_t(t, x) = H(t, x, U(t, x), -1, -W_x(t, x)). \tag{6.2.14}$$

Equation (6.2.12) can be written in the form

$$W_t(t, x) = \max_{z \varepsilon \Omega(t)} H(t, x, z, -1, -W_x(t, x)). \tag{6.2.15}$$

We now suppose that the function W is of class $C^{(2)}$. Under this additional hypothesis we shall derive the Pontryagin Maximum Principle. Let (τ, ξ) again be a fixed point in $\mathcal{R}$. Consider the function F defined on $\mathcal{R}$ by the formula

$$F(x) = W_t(\tau, x) + f^0(\tau, x, U(\tau, \xi)) + \langle W_x(\tau, x), f(\tau, x, U(\tau, \xi))\rangle. \tag{6.2.16}$$

It follows from (6.2.11) that $F(\xi) = 0$. On the other hand, since $U(\tau, \xi) \in \Omega(\tau)$, we obtain the following inequality from (6.2.12) with (τ, ξ) replaced by (τ, x)

$$W_t(\tau, x) \geq -f^0(\tau, x, U(\tau, \xi)) - \langle W_x(\tau, x), f(\tau, x, U(\tau, \xi))\rangle.$$

This says that $F(x) \geq 0$. Hence the function F has a minimum at $x = \xi$. Since W is $C^{(2)}$, F is $C^{(1)}$. Therefore, since ξ is an interior point of the domain of definition of F and F attains its minimum at ξ, we have that $F_x(\xi) = 0$. If we use (6.2.16) to compute the partial derivatives of F with respect to the state variable and then set the partials equal to zero at $x = \xi$, we get that for $i = 1, 2, \ldots, n$,

$$\frac{\partial^2 W}{\partial t \partial x^i} + \partial f^0 \partial x^i + \sum_{j=1}^n \partial^2 W \partial x^i \partial x^j \ f^j + \sum_{j=1}^n \frac{\partial W}{\partial x^j} \frac{\partial f^j}{\partial x^i} = 0, \tag{6.2.17}$$

where the partial derivatives of W are evaluated at (τ, ξ) and the functions f^j and their partial derivatives are evaluated at $(\tau, \xi, U(\tau, \xi))$. Since (τ, ξ) is an arbitrary point in $\mathcal{R}$, it follows that (6.2.17) holds for the arguments (t, x) and $(t, x, U(t, x))$, where (t, x) is any point in $\mathcal{R}$.

Before proceeding with our analysis we introduce some useful terminology.

Definition 6.2.1. If $h\colon (t, x, z) \to h(t, x, z)$ is a function from $\mathcal{G} = \mathcal{R} \times \mathcal{U}$ to $R^k, k \geq 1$, then by the expression "the function h evaluated along the trajectory $\phi(\cdot, \tau, \xi)$" we shall mean the composite function $t \to h(t, \phi(t, \tau, \xi), u(t, \tau, \xi))$. Similarly, if w is a function defined on R, by the expression "the function w evaluated along the trajectory $\phi(\cdot, \tau, \xi)$" we shall mean the composite function $t \to w(t, \phi(t, \tau, \xi))$.

We now let (τ, ξ) be a fixed point in $\mathcal{R}$ and consider the behavior of the partial derivative $W_x = (W_{x^1}, \dots, W_{x^n})$ along the optimal trajectory starting at (τ, ξ). We define a function $\lambda(\cdot, \tau, \xi)\colon t \to \lambda(t, \tau, \xi)$ from $[\tau, t_1]$ to $\mathbb{R}^n$ as follows:

$$\lambda(t, \tau, \xi) = -W_x(t, \phi(t, \tau, \xi)). \tag{6.2.18}$$

Since W is $C^{(2)}$, the function λ is differentiable with respect to t. Using the relation $\phi'(t, \tau, \xi) = f(t, \phi(t, \tau, \xi), u(t, \tau, \xi))$ we get

$$\frac{d\lambda^i}{dt} = -\frac{\partial^2 W}{\partial t \partial x^i} - \sum_{j=1}^{n} \frac{\partial^2 W}{\partial x^i \partial x^j}\, f^j \quad i = 1, \dots, n, \tag{6.2.19}$$

where the partial derivatives of W and the components of f are evaluated along the trajectory $\phi(\cdot, \tau, \xi)$. If we substitute (6.2.19) into (6.2.17) and use (6.2.18) we get

$$\frac{d\lambda^i}{dt} = -\left\{ -\frac{\partial f^0}{\partial x^i} + \sum_{j=1}^{n} \lambda^j\, \frac{\partial f^j}{\partial x^i} \right\} \qquad i = 1, \dots, n.$$

In vector-matrix notation this becomes

$$\frac{d\lambda}{dt} = -\left[-\frac{\partial f^0}{\partial x} + \left(\frac{\partial f}{\partial x}\right)^t \lambda \right], \tag{6.2.20}$$

where $d\lambda/dt$, $\partial f^0/\partial x$ and λ are column vectors and $(\partial f/\partial x)^t$ is the transpose of the matrix whose entry in the i-th row and j-th column is $\partial f^i/\partial x^j$. The partials in (6.2.20) are evaluated along the trajectory $\phi(\cdot, \tau, \xi)$. To summarize, we have shown that associated with the optimal trajectory $\phi(\cdot, \tau, \xi)$ there is a function $\lambda(\cdot, \tau, \xi)$ such that (6.2.20) holds. We point out that since $\partial f^0/\partial x$ and $\partial f/\partial x$ are evaluated along $\phi(\cdot, \tau, \xi)$ they are functions of t on the interval $[\tau, t_1]$. Hence the system (6.2.20) is a system of linear differential equations with time varying coefficients that the function $\lambda(\cdot, \tau, \xi)$ must satisfy. Initial conditions for this system will be discussed below.

In terms of the function H introduced in (6.2.13), Eq. (6.2.20) becomes

$$\lambda'(t,\tau,\xi) = -H_x(t,\phi(t,\tau,\xi),u(t,\tau,\xi),-1,\lambda(t,\tau,\xi)), \tag{6.2.21}$$

where the prime denotes differentiation with respect to time. From the definition of H in (6.2.13) it also follows that

$$\phi'(t,\tau,\xi) = H_q(t,\phi(t,\tau,\xi),u(t,\tau,\xi),-1,\lambda(t,\tau,\xi)). \tag{6.2.22}$$

It follows from (6.2.21) and (6.2.22) that the functions $\phi(\cdot,\tau,\xi)$ and $\lambda(\cdot,\tau,\xi)$ satisfy the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= H_q(t,x,u(t,\tau,\xi),-1,q) \\ \frac{dq}{dt} &= -H_x(t,x,u(t,\tau,\xi),-1,q). \end{aligned} \tag{6.2.23}$$

We can combine (6.2.14) and (6.2.15) and get

$$H(t,x,U(t,x),-1,-W_x(t,x)) = \max_{z\varepsilon\Omega(t)} H(t,x,z,-1,-W_x(t,x)). \tag{6.2.24}$$

Since (6.2.24) holds for all (t,x) in $\mathcal{R}$, it holds along the optimal trajectory $\phi(\cdot,\tau,\xi)$. For points (t,x) on the optimal trajectory $\phi(\cdot,\tau,\xi)$ the relation

$$x = \phi(t,\tau,\xi)$$

holds. From this relation and from (6.2.8) it follows that for such points the relation

$$u(t,\tau,\xi) = U(t,\phi(t,\tau,\xi)) \tag{6.2.25}$$

holds. Relation (6.2.18) also holds along the optimal trajectory. Therefore, for all $\tau \le t \le t_1$, where t_1 is the time at which $\phi(\cdot,\tau,\xi)$ hits $\mathcal{T}$,

$$\begin{aligned} &H(t,\phi(t,\tau,\xi),u(t,\tau,\xi),-1,\lambda(t,\tau,\xi)) \\ &\quad = \max_{z\varepsilon\Omega(t)} H(t,\phi(t,\tau,\xi),z,-1,\lambda(t,\tau,\xi)). \end{aligned} \tag{6.2.26}$$

Equation (6.2.23) with boundary conditions $\phi(\tau,\tau,\xi) = \xi$ and boundary conditions on λ to be determined below together with (6.2.26) characterize the optimal trajectory $\phi(\cdot,\tau,\xi)$ and optimal control $u(\cdot,\tau,\xi)$ in the following way. Associated with $\phi(\cdot,\tau,\xi)$ and $u(\cdot,\tau,\xi)$ there is a function $\lambda(\cdot,\tau,\xi)$ such that $\lambda(\cdot,\tau,\xi)$ and $\phi(\cdot,\tau,\xi)$ are solutions of (6.2.23) with appropriate boundary conditions and such that (6.2.26) holds for $\tau \le t \le t_1$. Equations (6.2.23) and their appropriate boundary conditions together with relation (6.2.26) constitute the Pontryagin Maximum Principle under the present hypotheses. They are a set of necessary conditions that an optimal pair must satisfy. A more precise and more general formulation will be given in Section 6.3.

Equations (6.2.23) and (6.2.26) involve the optimal control $u(\cdot,\tau,\xi)$. We

rewrite these equations so as to involve the synthesis U. If we substitute (6.2.25) into (6.2.21) and (6.2.22) we see that $\phi(\cdot, \tau, \xi)$ and $\lambda(\cdot, \tau, \xi)$ satisfy the equations

$$\begin{aligned} \frac{dx}{dt} &= H_q(t, x, U(t, x), -1, q) \\ \frac{dq}{dt} &= -H_x(t, x, U(t, x), -1, q). \end{aligned} \tag{6.2.27}$$

From (6.2.26) we see that $\phi(\cdot, \tau, \xi)$ and $\lambda(\cdot, \tau, \xi)$ also satisfy

$$H(t, x, U(t, x), -1, q) = \max_{z \varepsilon \Omega(t)} H(t, x, z, -1, q). \tag{6.2.28}$$

We now give a geometric interpretation of the maximum principle. To simplify the discussion we assume the problem to be in Mayer form, that is, $f^0 \equiv 0$. The problem in Bolza form can be written as a Mayer problem, as shown in Section 2.4. We assume that the value function W has level surfaces $W(t, x) = \text{const.}$, and that these level surfaces have gradients (W_t, W_x). Suppose we are at a point $(t, x) = (t, \phi(t))$ of an optimal trajectory ϕ and wish to proceed to minimize the payoff. The best choice would be to go in the direction of steepest descent; that is, we should choose a control z_0 that maximizes $-W_t(t, \phi(t)) - \langle W_x(t, \phi(t)), f(t, \phi(t), z)\rangle$. Thus, $z_0 = U(t, \phi(t))$ and

$$\langle -W_x(t, \phi(t)), f(t, \phi(t), z_0)\rangle = \max_z \langle -W_x(t, \phi(t)), f(t, \phi(t), z)\rangle.$$

If we now use $\lambda(t) = -W_x(t, \phi(t))$, we have (6.2.26) for the Mayer problem, where $f^0 \equiv 0$.

We return to the Bolza problem with $f^0 \neq 0$ and derive the "transversality conditions." These are boundary conditions that the value function W and its partial derivatives must satisfy. The transversality conditions are also the boundary conditions that the function $\lambda(\cdot, \tau, \xi)$ must satisfy.

Let (τ_1, ξ_1) be the terminal point of the optimal trajectory for the problem with initial point (τ, ξ). Then there is a point σ_1 in $\sum$ such that $\tau_1 = T(\sigma_1)$ and $\xi_1 = X(\sigma_1)$. Let Γ^i be the curve on $\mathcal{T}$ passing through (τ_1, ξ_1) defined parametrically by the equations

$$\begin{aligned} t_1 &= T(\sigma_1^1, \ldots \sigma_1^{i-1}, \sigma^i, \sigma_1^{i+1}, \ldots \sigma_1^q) \\ x_1 &= X(\sigma_1^1, \ldots \sigma_1^{i-1}, \sigma^i, \sigma_1^{i+1}, \ldots \sigma_1^q), \end{aligned}$$

where σ^i ranges over some open interval (a^i, b^i). The curve Γ^i is obtained by holding all components of the vector σ but the i-th component fixed and letting the i-th component vary over the interval (a^i, b^i). The curve Γ^i is sometimes called the i-th coordinate curve on $\mathcal{T}$.

We now assume that $\mathcal{T}$ is n-dimensional, that each point of $\mathcal{T}$ is the terminal point of a unique trajectory, and that W can be extended to a $C^{(1)}$

function in a neighborhood of $\mathcal{T}$. It follows from (6.2.1) and the definition of W that for (t_1, x_1) in $\mathcal{T}$

$$W(t_1, x_1) = g(t_1, x_1). \tag{6.2.29}$$

It therefore follows that (6.2.29) holds along each $\Gamma^i, i = 1, \ldots, n$. We may therefore differentiate (6.2.29) along Γ^i with respect to σ^i and get that

$$W_t \frac{\partial T}{\partial \sigma^i} + \langle W_x, \frac{\partial X}{\partial \sigma^i} \rangle = g_t \frac{\partial T}{\partial \sigma^i} + \langle g_x, \frac{\partial X}{\partial \sigma^i} \rangle$$

holds along Γ^i. We rewrite this equation as

$$\langle (W_t - g_t,\ W_x - g_x), \left(\frac{\partial T}{\partial \sigma^i}, \frac{\partial X}{\partial \sigma^i} \right) \rangle = 0 \qquad i = 1, \ldots, n. \tag{6.2.30}$$

In particular, (6.2.30) holds at $\sigma^i = \sigma^i_1$. We may therefore take the argument of W_t, W_x, g_t, and g_x to be (τ_1, ξ_1) and the argument of $\partial T / \partial \sigma^i$ and $\partial X / \partial \sigma^i$ to be σ_1. Using (6.2.14) we can rewrite (6.2.30) as

$$\left\langle (H - g_t, W_x - g_x), \quad \left(\frac{\partial T}{\partial \sigma^i}, \frac{\partial X}{\partial \sigma^i} \right) \right\rangle = 0 \qquad i = 1, \ldots, n, \tag{6.2.31}$$

where H is evaluated at $(\tau_1, \xi_1, u(\tau_1, \tau, \xi), -1, -W_x(\tau_1, \xi_1))$.

If we use (6.2.18), Eq. (6.2.31) can be written

$$\left\langle (H - g_t, -\lambda - g_x), \ \left(\frac{\partial T}{\partial \sigma^i}, \frac{\partial X}{\partial \sigma^i} \right) \right\rangle = 0 \qquad i = 1, \ldots, n, \tag{6.2.32}$$

where $\lambda = \lambda(\tau_1, \tau, \xi)$ and H is evaluated at $(\tau_1, \xi_1, u(\tau_1, \tau, \xi_1), -1, \lambda(\tau_1, \tau, \xi))$.

Equations (6.2.32) when written out become

$$\left(H - \frac{\partial g}{\partial t} \right) \ \frac{\partial T}{\partial \sigma^i} - \sum_{j=1}^{n} \left(\lambda^j + \frac{\partial g}{\partial x^j} \right) \ \frac{\partial x^j}{\partial \sigma^i} = 0 \qquad i = 1, \ldots, n.$$

Since $H = -f^0 + \langle \lambda, f \rangle$, we can rewrite this system as follows:

$$-\left(f^0 + \frac{\partial g}{\partial t} \right) \ \frac{\partial T}{\partial \sigma^i} - \sum_{j=1}^{n} \frac{\partial g}{\partial x^j} \frac{\partial x^j}{\partial \sigma^i} = \sum_{j=1}^{n} \left(\frac{\partial x^j}{\partial \sigma^i} - \frac{\partial T}{\partial \sigma^i} \ f^j \right) \lambda^j. \tag{6.2.33}$$

Here the functions $f^0, f^j, \partial g / \partial t$, and $\partial g / \partial x^j$ are evaluated at the end point (τ_1, ξ_1) of the trajectory $\phi(\cdot, \tau, \xi)$. The partial derivatives of T and X are evaluated at σ_1, the point in Σ corresponding to (τ_1, ξ_1). If $\mathcal{T}$ is a q-dimensional manifold, $0 \leq q \leq n$, then (6.2.33) consists of q equations instead of n equations. This does not follow from our arguments here, but will be shown to hold in the next chapter.

Since the unit tangent vector to Γ^i is the unit vector in the direction

of $(\partial T/\partial\sigma^i,\ \partial X/\partial\sigma^i)$, Eq. (6.2.30) states that at (τ_1, ξ_1) the vector $(W_t - g_t, W_x - g_x)$ is either zero or is orthogonal to Γ^i. We assume that orthogonality holds. From (6.2.31) and (6.2.32) we see that this statement is equivalent to the statement that $(H - g_t, W_x - g_x)$ is orthogonal to Γ^i at (τ_1, ξ_1) and to the statement that $(H - g_t, -\lambda - g_x)$ is orthogonal to Γ^i. Since this is true for each coordinate curve Γ^i, $i = 1, \ldots, n$ and since the tangent vectors to the Γ^i at (τ_1, ξ_1) generate the tangent plane to $\mathcal{T}$ at (τ_1, ξ_1), the following statement is true. *The vector* $(W_t - g_t, W_x - g_x)$*, or equivalently the vector* $(H - g_t, W_x - g_x)$*, or equivalently the vector* $(H - g_t, -\lambda - g_x)$ *evaluated at the end point of an optimal trajectory is orthogonal to* $\mathcal{T}$ *at that point.* This is the geometric statement of the transversality condition. The analytic statement consists of Eq. (6.2.30), Eq. (6.2.31), or Eq. (6.2.32).

Equations (6.2.29) and (6.2.30) are the boundary conditions for the partial differential equation (6.2.14).

Equations (6.2.32), or equivalently, (6.2.33) specify the values of $\lambda(\cdot, \tau, \xi)$ at $t = \tau_1$. They therefore furnish the heretofore missing boundary conditions for the systems (6.2.23) and (6.2.27). Note that Eqs. (6.2.32) and (6.2.33) are linear in λ. Thus, λ satisfies a system of linear differential equations with linear boundary conditions. We point out that the system (6.2.23) with boundary conditions $\phi(\tau, \tau, \xi) = \xi$ and (6.2.32) constitute a two-point boundary value problem in that the values of ϕ are specified at the initial time and the values of λ are specified at the terminal time.

Remark 6.2.2. If one solves the partial differential equation (6.2.14) subject to the boundary conditions (6.2.29) and (6.2.30) by the method of characteristics, one finds that the characteristic equations for the problem are those in Eq. (6.2.27). We leave the verification of this to the reader.

6.3 Statement of Maximum Principle

The theorems of this section are statements of the maximum principle under various hypotheses on the data of the problem. Each statement is a set of necessary conditions satisfied by a relaxed pair (ψ^*, μ^*) that solves the relaxed optimal control problem, formulated as Problem 3.2.1, which we restate for the convenience of the reader.

Relaxed Optimal Control Problem: Minimize

$$J(\psi, \mu) = g(e(\psi)) + \int_{t_0}^{t_1} f^0(t, \psi(t), \mu_t)dt$$

subject to

$$\frac{d\psi}{dt} = f(t, \psi(t), \mu_t) \qquad e(\psi) \in \mathcal{B} \qquad \mu_t \in \Omega(t),$$

where $\mu_t \in \Omega(t)$ means that μ_t is concentrated on $\Omega(t)$.

If the constraint sets are not compact, then we can think of relaxed controls as discrete measure controls. Since relaxed controls are also discrete measure controls, we may consider discrete measure controls even when the constraint sets $\Omega(t)$ are compact. *We shall henceforth only consider discrete measure controls*, unless explicitly stated otherwise. We continue to denote such controls by Greek letters such as μ and the corresponding discrete probability measure on $\Omega(t)$ by μ_t.

We now state the assumptions on the data of the problem in the case of compact constraints.

Assumption 6.3.1. (i) The function $\widehat{f} = (f^0, f^1, \ldots, f^n)$ is defined on a set $\mathcal{G}_0 \equiv \mathcal{I}_0 \times \mathcal{X}_0 \times \mathcal{U}_0$, where $\mathcal{I}_0$ is an open interval in $\mathbb{R}$, $\mathcal{X}_0$ is an open interval in $\mathbb{R}^n$, and $\mathcal{U}_0$ is an open interval in $\mathbb{R}^m$.

(ii) For fixed (x, z) in $\mathcal{X}_0 \times \mathcal{U}_0$ the function $\widehat{f}(\cdot, x, z)$ is measurable on $\mathcal{I}_0$.

(iii) For fixed z in $\mathcal{U}_0$ and almost all t in $\mathcal{I}_0$ the function $\widehat{f}(t, \cdot, z)$ is of class $C^{(1)}$ on $\mathcal{X}_0$.

(iv) For almost all t in $\mathcal{I}_0$ and all x in $\mathcal{X}_0$ the function $\widehat{f}(t, x, \cdot)$ is continuous on $\mathcal{U}_0$.

(v) The mapping Ω is a mapping from $\mathcal{I}_0$ to subsets $\Omega(t)$ of $\mathcal{U}_0$.

(vi) For each compact interval $\mathcal{X} \subset \mathcal{X}_0$, each compact interval $\mathcal{I} = [t_0, t_1] \subset \mathcal{I}_0$, and each compact set $Z \subseteq \mathcal{U}_0$, there exists a function $M(\cdot) = M(\cdot, \mathcal{I}, \mathcal{X}, Z)$ defined on $[t_0, t_1]$ such that M is in $L_2[t_0, t_1]$ and for all (t, x) in $[t_0, t_1] \times \mathcal{X}$ and z in Z,

$$|\widehat{f}(t, x, z)| \leq M(t) \qquad |\widehat{f}_x(t, x, z| \leq M(t). \tag{6.3.1}$$

Here $\widehat{f}_x(t, x, z)$ is the $(n+1) \times n$ matrix whose entry in row i column j is $(\partial f^i / \partial x^j)$, $i = 0, 1, \ldots, n$ $j = 1, \ldots, n$.

(vii) The set $\mathcal{B}$ that specifies the end conditions is a bounded $C^{(1)}$ manifold of dimension r, where $0 \leq r \leq 2n + 1$.

Remark 6.3.2. It follows from (iii) of Assumption 6.3.1 that for each compact interval $\mathcal{I} \times \mathcal{X}$ and compact $Z \subseteq \mathcal{U}_0$ there exists a real valued function Λ in $L_2[\mathcal{I}]$ such that for each pair of points x and x' in $\mathcal{X}$

$$|\widehat{f}(t, x, z) - \widehat{f}(t, x', z)| \leq \Lambda(t)|x - x'|, \tag{6.3.2}$$

for all $t \in \mathcal{I}$ and $z \in Z$. To see this note that for each $i = 0, 1, \ldots, n$, the Mean Value Theorem gives

$$|f^i(t, x, z) - f^i(t, x', z| = |\langle f^i_x(t, x + \theta(x - x'), z), (x' - x)\rangle|,$$

where $0 < \theta < 1$. From the Cauchy-Schwarz inequality, from (6.3.1), and the fact that all norms in a euclidean space are equivalent we get that the right side of this equality is less than or equal to $CM(t)|x - x'|$, where M is as in (6.3.1). The inequality (6.3.2) now follows again from the equivalence of norms in a euclidean space.

Remark 6.3.3. If μ is a relaxed control defined on a compact interval $\mathcal{I} \subset \mathcal{I}_0$ and $\mathcal{X}$ is a compact interval contained in $\mathcal{X}_0$, then as in (6.3.1), there exists a real valued function M on $L_2[\mathcal{I}]$ such that

$$|\widehat{f}(t, x, \mu_t)| \leq M(t) \qquad |\widehat{f}_x(t, x, \mu_t)| \leq M(t). \tag{6.3.3}$$

To see this note that

$$\widehat{f}(t, x, \mu_t) = \int_{\Omega(t)} \widehat{f}(t, x, z) d\mu_t.$$

The first inequality in (6.3.3) follows from the first inequality in (6.3.1) and the fact that μ_t is a probability measure. A similar argument establishes the inequality for $\widehat{f}_x$. It follows from Remark 6.3.2 that $\widehat{f}(t, x, \mu_t)$ satisfies a Lipschitz condition as in (6.3.1).

Remark 6.3.4. If instead of (ii) we require $\widehat{f}$ to be continuous on $\mathcal{G}_0$, then for each compact interval $\mathcal{X} \subseteq \mathcal{X}_0$, each compact interval $\mathcal{I} \subseteq \mathcal{I}_0$, and each compact set $Z \subset \mathcal{U}_0$, the first inequality in (6.3.1) holds with $M(t)$ replaced by a constant M. If we further require $\widehat{f}$ to be $C^{(1)}$ in $\mathcal{X}_0$ for fixed (t, z) in $\mathcal{I}_0 \times \mathcal{U}_0$, and require $\widehat{f}_x$ to be continuous on $\mathcal{I} \times \mathcal{X} \times Z$, then the second inequality in (6.3.1) holds with M constant. Under these assumptions the Lipschitz condition (6.3.2) holds with Λ a constant. Analogous statements hold for $\widehat{f}(t, x, \mu_t)$ and $\widehat{f}_x(t, x, \mu_t)$.

The maximum principle is stated most efficiently in terms of a Hamiltonian function H defined on $\mathcal{I}_0 \times \mathcal{X}_0 \times \mathcal{U}_0 \times \mathbb{R} \times \mathbb{R}^n$ by the formula

$$\begin{aligned} H(t, x, z, q^0, q) &= q^0 f^0(t, x, z) + \langle q, f(t, x, z) \rangle \\ &= \sum_{j=0}^{n+1} q^j f^j(t, x, z) = \langle \widehat{f}(t, x, z), \widehat{q} \rangle, \end{aligned} \tag{6.3.4}$$

where $\widehat{q} = (q^0, q)$.

By $H_r(t, x, \mu_t, q^0, q)$ we mean

$$H_r(t, x, \mu_t, q^0, q) = \int_{\Omega(t)} H(t, x, z, q^0, q) d\mu_t = \langle \widehat{q}, \int_{\Omega(t)} \widehat{f}(t, x, z) d\mu_t \rangle. \tag{6.3.5}$$

We use the subscript r to emphasize that we are considering relaxed controls.

The statement of the maximum principle only involves a given relaxed optimal pair. Therefore, to simplify the typography we shall drop the asterisk notation in designating a relaxed optimal pair. Thus, we write (ψ, μ) instead of (ψ^*, μ^*) for a relaxed optimal pair.

Theorem 6.3.5 (Maximum Principle in Integrated Form; Compact Constraints)**.** *Let Assumption 6.3.1 hold, let* $t_0 = 0$*, and let* $t_1 = 1$*. For each* t *in* $[0, 1]$ *let* $\Omega(t)$ *be compact and let there exist a compact set* $Z \subseteq \mathcal{U}_0$ *such that each set* $\Omega(t)$ *is contained in* Z*. Let* (ψ, μ) *be a relaxed optimal pair with* $\psi(t) \in \mathcal{X}_0$ *for all* t *in* $[0, 1]$*. Then there exist a constant* $\lambda^0 \leq 0$ *and an absolutely continuous function* $\lambda = (\lambda^1, \ldots, \lambda^n)$ *defined on* $[0, 1]$ *such that the following hold.*

(i) The vector $\widehat{\lambda}(t) = (\lambda^0, \lambda(t))$ *is never zero on* $[0, 1]$*.*

(ii) For a.e. t *in* $[0, 1]$

$$\begin{aligned} \psi'(t) &= H_{rq}(t, \psi(t), \mu_t, \widehat{\lambda}(t)) \\ \lambda'(t) &= -H_{rx}(t, \psi(t), \mu_t, \widehat{\lambda}(t)). \end{aligned} \tag{6.3.6}$$

(iii) For any relaxed admissible control ν *defined on* $[0, 1]$

$$\int_0^1 H_r(t, \psi(t), \mu_t, \widehat{\lambda}(t))dt \geq \int_0^1 H_r(t, \psi(t), \nu_t, \widehat{\lambda}(t))dt. \tag{6.3.7}$$

(iv) The $2n$ *vector*

$$(-\lambda^0 g_{x_0}(e(\psi)) - \lambda(0),\ -\lambda^0 g_{x_1}(e(\psi)) + \lambda(1))$$

is orthogonal to $\mathcal{B}$ *at* $e(\psi)$*.*

An equivalent formulation of conclusion (iv) is that

$$\langle -\lambda^0 g_{x_0}(e(\psi)) - \lambda(0), dx_0 \rangle + \langle -\lambda^0 g_{x_1}(e(\psi)) + \lambda(1), dx_1 \rangle = 0 \tag{6.3.8}$$

for all tangent vectors (dx_0, dx_1) to $\mathcal{B}$ at $e(\psi)$.

Remark 6.3.6. We emphasize that Theorems 6.3.5–6.3.22 give *necessary conditions that a relaxed optimal pair* must satisfy. Thus, if we know *a priori* that the relaxed optimal pair is an ordinary optimal pair then Theorems 6.3.5–6.3.22 are applicable, and in their statements replace $\psi(t)$ by $\phi(t)$, replace μ_t by $u(t)$, and replace H_r by H. Theorem 4.4.2 and Corollary 4.4.3 give a sufficient condition for a relaxed optimal pair to be an ordinary optimal pair.

If we know that (ϕ, u) is optimal for the ordinary problem, but do not know that it is optimal for the relaxed problem, then we *cannot* apply Theorems 6.3.5–6.3.17. Example 4.4.4 shows that it is possible for the ordinary control problem to have a solution (ϕ, u) and the relaxed problem to have a solution (ψ, u) with $J(\psi, \mu)$ strictly less than $J(\phi, u)$. Our theorems are applicable to (ψ, μ), but not to (ϕ, u). Such cases are pathological.

Remark 6.3.7. From the definition of H_r it follows that the first equation in (6.3.6) can be written as

$$\psi'(t) = f(t, \psi(t), \mu_t).$$

Thus, the first equation in (6.3.6) is a restatement of the fact that (ψ, μ) is an admissible pair. The second equation in (6.3.6) written in component form is the system

$$\lambda^{i'}(t) = -\lambda^0 f^0_{x^i}(t, \psi(t), \mu_t) - \sum_{j=1}^{n} \lambda^j(t) f^j_{x^i}(t, \psi(t), \mu_t) \qquad i = 1, \ldots, n. \quad (6.3.9)$$

Since λ^0 is a constant, $d\lambda^0/dt$. If we introduce an additional coordinate x^0 and set $\widehat{x} = (x_0, x)$, then since $\widehat{f}$ does not depend on x^0, we may write

$$\lambda_0' = -\lambda^0 f^0_{x^0}(t, \psi, \mu_t) - \sum_{j=1}^{n+1} \lambda^j(t) f^j_{x^0}(t, \psi(t), \mu_t).$$

If we adjoin this equation to (6.3.9) we get that $\widehat{\lambda} = (\lambda^0, \lambda(t))$ satisfies

$$\widehat{\lambda}'(t) = -(\widehat{f}_{\widehat{x}}(t, \psi(t), \mu_t))^t \widehat{\lambda}(t), \qquad (6.3.10)$$

where the superscript t denotes transpose and $\widehat{f}_x$ is the matrix whose entry in row i column j is $(\partial f^i/\partial x_j)$, $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, n$. Thus, $\widehat{\lambda}$ satisfies a system of linear homogeneous differential equations.

Remark 6.3.8. Since $(\lambda^0, \lambda(0)) \neq 0$ never vanishes, we may divide through by $|(\lambda^0(0), \lambda(0))|$ in (6.3.8)–(6.3.10) and relabel to obtain $(\lambda^0, \lambda(t))$ such that $|(\lambda^0, \lambda(0))| = 1$ and satisfies (6.3.6)–(6.3.8).

We now take up the statement of the necessary conditions for optimality when the constraint sets are not assumed to be compact. We defined the relaxed problem for the case of non-compact constraints in Section 3.5 and elaborated on this definition in Section 4.4. We now repeat the essence of these discussions.

In Definition 3.5.3, a *discrete measure control* on an interval $[t_0, t_1]$ was defined to be a control μ such that each μ_t has the form

$$\mu_t = \sum_{i=1}^{n+2} p^i(t) \delta_{u_i(t)},$$

where each p^i is a nonnegative measurable function, $\sum p^i(t) = 1$, and each u_i is a measurable function with range in $\mathcal{U}$. Also, a discrete measure control is a control μ such that for each t in $[t_0, t_1]$ the measure μ_t is a convex combination of Dirac measures.

If the constraint sets $\Omega(t)$ are compact and (ψ, μ) is an admissible relaxed

pair, then by Theorem 3.2.11 there exists a discrete measure control $\widetilde{\mu}$ such that

$$\widehat{\psi}'(t) = \widehat{f}(t, \psi(t), \mu_t) = \widehat{f}(t, \psi(t), \widetilde{\mu}_t),$$

where $\widehat{f} = (f^0, f^1, \ldots, f^n)$. Thus, for every admissible pair $(\widehat{\psi}, \mu)$, there exists a discrete measure control $\widetilde{\mu}$ such that $(\psi, \widetilde{\mu})$ is admissible and $J(\psi, \mu) = J(\psi, \widetilde{\mu})$. Therefore, in considering necessary conditions if the constraint sets $\Omega(t)$ are compact we need only consider discrete measure controls.

If μ_t is a Dirac measure concentrated at $u(t)$, then $H_r(t, x, \mu_t, q^0, q) = H(t, x, u(t), q^0, q)$. In general, if μ_t is a discrete measure control then

$$\begin{aligned} H_r(t, x, \mu_t, q^0, q) &= \langle \widehat{q}, \widehat{f}(t, x, \mu_t) \rangle \\ &= \sum_{j=0}^{n+1} \left(\sum_{k=1}^{n+2} p^k(t) f^j(t, x, u_k(t)) q^j \right) \\ &= \sum_{k=1}^{n+2} p^k(t) \left(\sum_{j=0}^{n+1} q^j f^j(t, x, u_k(t)) \right) \\ &= \sum_{k=1}^{n+2} p^k(t) H(t, x, u_k(t), q^0, q). \end{aligned} \tag{6.3.11}$$

Thus, the relaxed Hamiltonian is a convex combination of ordinary Hamiltonians.

From (6.3.11) we get that (6.3.10) can be written as

$$\widehat{\lambda}'(t) = -\sum_{i=1}^{n+2} p^i(t) H_{\widehat{x}}(t, \psi(t), u_i(t), \widehat{\lambda}). \tag{6.3.12}$$

Theorem 6.3.9 (Maximum Principle in Integrated Form)**.** *Let Assumption 6.3.1 hold. Let $t_0 = 0$ and let $t_1 = 1$. Let $\mathcal{Y}$ denote the union of the sets $\Omega(t)$ as t ranges over $[0,1]$, and let $\mathcal{Y}$ be unbounded. Let there exist a positive integer K such that for $k > K$ and all $t \in [0,1]$, $\Omega_k(t) = (cl\ \Omega(t)) \cap (cl\ \mathcal{B}(0,k))$ is not empty, where cl denotes closure and $\mathcal{B}(0,k)$ is the ball in $\mathbb{R}^m$ of radius k centered at the origin. Let the mapping $\Omega_k \colon t \to \Omega_k(t)$ be u.s.c.i. on $[0,1]$. Let (ψ, μ) be a relaxed optimal pair, with $\psi(t) \in \mathcal{X}_0$ for all $t \in [0,1]$ and where μ is a discrete measure control with*

$$\mu_t = \sum_{i=1}^{n+2} p^i(t) \delta_{u_i(t)}.$$

Let there exist a function M in $L_1[0,1]$ such that $|\widehat{f}(t, \psi(t), z)| \leq M(t)$ a.e. for all z in $\mathcal{Y}$. For each $i = 1, \ldots, n+2$ let the function $t \to |f_x(t, \psi(t), u_i(t))|$ be integrable on $[0,1]$. For each compact set $\mathcal{X} \subset \mathcal{X}_0$ let there exist a function Λ in $L_1[0,1]$ such that for a.e. $t \in [0,1]$ and all x, x' in $\mathcal{X}$

$$|\widehat{f}(t, x, u_i(t)) - \widehat{f}(t, x', u_i(t))| \leq \Lambda(t)|x - x'|. \tag{6.3.13}$$

Then the conclusion of Theorem 6.3.5 holds.

Definition 6.3.10. A triple $(\psi, \mu, \widehat{\lambda})$ that satisfies the conclusions of Theorems 6.3.5 or Theorem 6.3.9 will be called an *extremal.* The pair (ψ, μ) will be called an *extremal pair* and $\widehat{\lambda}$ a *multiplier.*

Definition 6.3.11. A vector $(t, \psi(t), \mu_t, \widehat{\lambda}(t))$, where $(\psi, \mu, \widehat{\lambda})$ is an extremal will be called an *extremal element* and we shall write

$$\pi(t) = (t, \psi(t), \mu_t, \widehat{\lambda}(t)).$$

If we strengthen the hypotheses of Theorems 6.3.5 and 6.3.9 by requiring the mapping Ω to be a constant mapping, we obtain a pointwise statement equivalent to (6.3.7) that is more useful than (6.3.7) in finding extremals.

Theorem 6.3.12 (Pointwise Maximum Principle)**.** *Let Assumption* 6.3.1 *hold with* $\Omega(t) = \mathcal{C}$*, a fixed set in* $\mathfrak{U}_0$*, for all* t *in* $\mathcal{I}_0$*. Let* $t_0 = 0$ *and* $t_1 = 1$*. Let* (ψ, μ) *be an optimal relaxed pair with* $\psi(t) \in \mathcal{X}_0$ *for all* $t \in [0, 1]$ *and with*

$$\mu_t = \sum_{i=1}^{n+2} p^i(t)\delta_{u_i(t)},$$

where $u_1, \ldots, u_{n+2}$ *are measurable functions on* $[t_0, t_1]$ *with* $u_i(t) \in \mathcal{C}$*, a.e. For* $i = 1, \ldots, n+2$ *let* $P_i = \{t \colon p^i(t) > 0\}$*. If* $\mathcal{C}$ *is unbounded, let the following additional hypotheses hold:*

(i) There exists a function M *in* $L[0, 1]$ *such that for all* $z \in \mathcal{C}$*,* $|\widehat{f}(t, \psi(t), z)| \leq M(t)$*.*

(ii) For each $i = 1, \ldots, n+2$*, the function* $t \to |f_x(t, \psi(t), u_i(t))|$ *is integrable on* $[0, 1]$

(iii) The Lipschitz condition (6.3.13) *holds. Let*

$$M(t, x, \widehat{q}) = \sup\{z \in \mathcal{C} \colon H(t, x, z, \widehat{q})\}.$$

Then:

(i) There exists a constant $\lambda^0 \leq 0$ *and an absolutely continuous vector* $\lambda = (\lambda^1, \ldots, \lambda^n)$ *defined on* $[t_0, t_1]$ *such that* $\widehat{\lambda}(t) = (\lambda^0, \lambda^1(t), \ldots, \lambda^n(t))$ *never vanishes, and*

$$\begin{aligned} \psi'(t) &= H_{rq}(t, \psi(t), \mu_t, \widehat{\lambda}(t)) \\ \lambda'(t) &= -H_{rx}(t, \psi(t), \mu_t, \widehat{\lambda}(t)). \end{aligned} \tag{6.3.14}$$

(ii) Not all of the sets P_i *have measure zero. If meas* $(P_k) \neq 0$*, then for almost all* t *in* P_k

$$M(t, \psi(t), \widehat{\lambda}(t)) = H(t, \psi(t), u_k(t), \widehat{\lambda}(t)). \tag{6.3.15}$$

(iii) The transversality condition (6.3.8) holds.

Remark 6.3.13. For each k such that meas $P_k \neq 0$,

$$p^k(t)H(t,\psi(t),u_k(t),\widehat{\lambda}(t)) = p^k(t)M(t,\psi(t),\widehat{\lambda}(t)) \tag{6.3.16}$$

for a.e. t in P_k. If $t \notin P_k$, then $p^k(t) = 0$, so (6.3.16) holds in $[t_0, t_1]P_k$. If meas $P_k = 0$, then $p^k(t) = 0$ a.e. and (6.3.16) again holds. Hence

$$H_r(t,\psi(t),\mu_t,\widehat{\lambda}(t)) = \sum_{k=1}^{n+2} p^k(t)H(t,\psi(t),u_k(t),\widehat{\lambda}(t)) = M(t,\psi(t),\widehat{\lambda}(t)). \tag{6.3.17}$$

Corollary 6.3.14. *Let (ψ,μ) be an optimal relaxed pair defined on $[0,1]$ and let ν be a discrete measure control on $[0,1]$ with*

$$\nu_t = \sum_{i=1}^{n+2} w^i(t)\delta_{v_i(t)} \quad \sum_{i=1}^{n+2} w^i(t) = 1, \ w^i \geq 0 \quad v_i(t) \in \mathcal{C}.$$

Then

$$H_r(t,\psi(t),\mu_t,\widehat{\lambda}(t)) \geq H_r(t,\psi(t),\nu_t,\widehat{\lambda}(t)) \qquad a.e. \tag{6.3.18}$$

Proof. Since each $v_i(t) \in \mathcal{C}$, we have

$$M(t,\psi(t),\widehat{\lambda}(t)) \geq H(t,\psi(t),v_i(t),\widehat{\lambda}(t)).$$

Hence

$$M(t,\psi(t),\widehat{\lambda}(t)) \geq \sum_{i=1}^{n+2} w^i(t)H(t,\psi(t),v_i(t),\widehat{\lambda}(t)) = H_r(t,\psi(t),\nu_t,\widehat{\lambda}(t)).$$

The inequality (6.3.18) now follows from (6.3.17). □

Remark 6.3.15. Inequality (6.3.18) is the pointwise version of the maximum principle for the relaxed problem. If (6.3.18) holds, then so does (6.3.7).

Remark 6.3.16. The proof of the corollary shows that for (6.3.18) to hold we need not suppose that ν_t is the value at t of a discrete measure control ν. Inequality (6.3.18) will hold if we take ν_t to be a discrete measure on $\mathcal{C}$ given by

$$\nu_t = \sum_{i=1}^{n+2} w^i(t)\delta_{\nu_i(t)},$$

where $\nu_i(t) \in \mathcal{C}$.

Theorem 6.3.17. *Let (ii), (iii), (iv), and (vi) of Assumption 6.3.1 be replaced by the following assumptions. The functions $\widehat{f}$, $\widehat{f}_t$, and $\widehat{f}_x$ are continuous on $\mathcal{I}_0 \times \mathcal{X}_0 \times \mathcal{U}_0$ and for fixed z in $\mathcal{U}_0$, the function $\widehat{f}(\cdot,\cdot,z)$ is of class $C^{(1)}$ on $\mathcal{I}_0 \times \mathcal{X}_0$. Let $t_0 = 0$ and let $t_1 = 1$. Let (ψ, μ) be a relaxed optimal pair with $\psi(t) \in \mathcal{X}_0$ for all t in $[0,1]$ and with μ a discrete measure control with*

$$\mu_t = \sum_{i=1}^{n+2} p^i(t)\delta_{u_i(t)}.$$

Let $\Omega(t) = \mathcal{C}$. If $\mathcal{C}$ is not compact, let the following hold.

(i) Each u_i is bounded.

(ii) There exists a function M in $L_1[0,1]$ such that

$$|\widehat{f}(t,\psi(t),z)| \le M(t) \quad \textit{a.e. for all } z \textit{ in } \mathcal{C}. \tag{6.3.19}$$

(iii) For each compact set $\mathcal{X} \subseteq \mathcal{X}_0$ there exists a function Λ in $L_1[0,1]$ such that for all (t,x) and (t',x') in $[0,1] \times \mathcal{X}$, and $i = 1,\dots,n+2$

$$|\widehat{f}(t,x,u_i(t)) - \widehat{f}(t',x',u_i(t))| \le \Lambda(t)(|t-t'| + |x-x'|). \tag{6.3.20}$$

Then the conclusions of Theorem 6.3.12 and Corollary 6.3.14 hold. Furthermore, there exists an absolutely continuous function h defined on $[0,1]$ such that

$$h(t) = H_r(t,\psi(t),\mu_t,\widehat{\lambda}(t)) \qquad \textit{a.e.} \tag{6.3.21}$$

and

$$h'(t) = H_{rt}(t,\psi(t),\mu_t,\widehat{\lambda}(t)) \qquad \textit{a.e.} \tag{6.3.22}$$

Definition 6.3.18. A discrete measure control μ is *piecewise continuous* if each of the functions $p^1,\dots,p^{n+2}$, $u_1,\dots,u_{n+2}$ is piecewise continuous.

Corollary 6.3.19. *If μ is piecewise continuous and $\mathcal{C}$ is closed, then the mapping $t \to H_r(t,\psi(t),\mu_t,\widehat{\lambda}(t))$ is absolutely continuous and the derivative of this mapping is $t \to H_{rt}(t,\psi(t),\mu_t,\widehat{\lambda}(t))$, for all t in $[0,1]$.*

Remark 6.3.20. Combining (6.3.21) and (6.3.22) gives

$$H_r(t,\psi(t),\mu_t,\widehat{\lambda}(t)) = \int_0^t H_{rt}(s,\psi(s),\mu_s,\widehat{\lambda}(s))ds + C \qquad \text{a.e.} \tag{6.3.23}$$

If the control problem is autonomous, that is, $\widehat{f}$ is independent of t, then $H_r(t,\psi(t),\,\mu_t,\widehat{\lambda}(t)) = c$ a.e.

In the corollary we identified the function $t \to H_r(t,\psi(t),\mu_t,\widehat{\lambda}(t))$ with $h(t)$. We shall henceforth always make this identification.

Remark 6.3.21. If $\mathcal{C}$ is compact, then hypotheses (i), (ii), and (iii) are consequences of the compactness of $\mathcal{C}$ and the assumption that $\widehat{f}(\cdot,\cdot,z)$ is of class $C^{(1)}$ on $\mathcal{I}_0 \times \mathcal{X}_0$. Thus, if $\mathcal{C}$ is compact hypotheses (i), (ii), and (iii) are not needed. Also, if $\mathcal{C}$ is compact, since $\widehat{f}$ and $\widehat{f}_x$ are assumed to be continuous on $\mathcal{I}_0 \times \mathcal{X}_0 \times \mathcal{U}_0$, assumptions (ii), (iii), (iv), and (vi) of Assumption 6.3.1 hold.

In Theorems 6.3.5–6.3.17 we assumed that the initial time t_0 and terminal time t_1, were fixed. We now remove this assumption. This will only affect the transversality condition.

Theorem 6.3.22. *In each of Theorems 6.3.5–6.3.17 let the assumption that t_0 and t_1 are fixed be removed. Let all the other assumptions of these theorems hold and in Theorems 6.3.5–6.3.12 let the following additional assumptions be made. For each z in $\mathcal{U}_0$ let $\widehat{f}(\cdot,\cdot,z)$ be of class $C^{(1)}$ on $\mathcal{I}_0 \times \mathcal{X}_0$ and let $\widehat{f}_t$, $\widehat{f}_x$ and $\widehat{f}$ be continuous on $\mathcal{I}_0 \times \mathcal{X}_0 \times \mathcal{U}_0$. For each compact set $\mathcal{I} \subset \mathcal{I}_0$ and each compact set $\subseteq \mathcal{X}_0$ let there exist a function Λ in $L_1[\mathcal{I}]$ such that for all (t,x) and (t',x') in $\mathcal{I} \times \mathcal{X}$*

$$|f(t,x,u_i(t)) - f(t',x',u_i(t))| \leq \Lambda(t)(|t-t'| + |x-x'|).$$

Then in Theorems 6.3.5–6.3.17 the transversality condition is the following. The $(2n+2)$ dimensional vector

$$\begin{aligned}&(- \lambda^0 g_{t_0}(e(\psi)) + H_r(\pi(t_0)), -\lambda^0 g_{x_0}(e(\psi)) - \lambda(t_0),\\ &\quad - \lambda^0 g_{t_1}(e(\psi)) - H_r(\pi(t_1)), -\lambda^0 g_{x_1}(e(\psi)) + \lambda(t_1)),\end{aligned} \tag{6.3.24}$$

is orthogonal to $\mathcal{B}$ at $e(\psi)$, where $\pi(t)$ is as in Definition 6.3.11. The other conclusions of Theorems 6.3.5–6.3.17 are unchanged.

Remark 6.3.23. The additional assumptions on $\widehat{f}(\cdot,\cdot,z)$ in Theorems 6.3.5–6.3.12 and the assumption on the Lipschitz condition in Theorems 6.3.9–6.3.12 are both present in the assumptions of Theorem 6.3.17.

Remark 6.3.24. Since the transversality condition only involves the nature of $\mathcal{B}$ in a neighborhood of the end point of an optimal trajectory, there is no loss of generality in assuming that $\mathcal{B}$ can be represented by a single coordinate patch. We thus assume that $\mathcal{B}$ is the image of an open parallelepiped Σ in $\mathbb{R}^q$ under a mapping

$$t_0 = T_0(\sigma) \quad x_0 = X_0(\sigma) \quad t_1 = T_1(\sigma) \quad x_1 = X_1(\sigma), \tag{6.3.25}$$

where the functions T_i and $X_i, i = 0,1$ are $C^{(1)}$ on Σ and the $(2n+2) \times r$ Jacobian matrix

$$(T_{0\sigma}\, X_{0\sigma}\, T_{1\sigma}\, X_{1\sigma})^t,$$

where the superscript t denotes transpose, has rank r everywhere on Σ.

Remark 6.3.25. If (dt_0, dx_0, dt_1, dx_1) denotes an arbitrary tangent vector to $\mathcal{B}$ at $e(\psi)$, then the transversality condition says that the vector in (6.3.24) is orthogonal to (dt_0, dx_0, dt_1, dx_1). Hence the inner product of (6.3.24) with an arbitrary tangent vector (dt_0, dx_0, dt_1, dx_1) is zero. Thus,

$$\begin{aligned}&[-\lambda^0 g_{t_0} + H_r(\pi(t_0)]dt_0 + \langle -\lambda^0 g_{x_0} - \lambda(t_0), dx_0\rangle \\ &\quad + [-\lambda^0 g_{t_1} - H_r(\pi(t_1)]dt_1 + \langle -\lambda^0 g_{x_1} + \lambda(t_1), dx_1\rangle = 0,\end{aligned} \tag{6.3.26}$$

for all tangent vectors (dt_0, dx_0, dt_1, dx_1), where the partials of g are evaluated at $e(\psi)$.

Since $\mathcal{B}$ has dimension r, the vector space of tangent vectors to $\mathcal{B}$ at $e(\psi)$ is r-dimensional. We therefore need only require that (6.3.26) holds for the r tangent vectors in a basis for the tangent space.

In the parametric equations (6.3.25) defining $\mathcal{B}$, let $\overline{\sigma}$ be the parameter value corresponding to the point $e(\psi)$ in $\mathcal{B}$. Let Γ_i be the i-th coordinate curve in $\mathcal{B}$, obtained by fixing $\sigma_j, j \neq i$ at $\overline{\sigma}_j$ and letting σ_i vary. The unit tangent vector to this curve at $e(\psi)$ is

$$c_i \left(\frac{\partial T_0}{\partial \sigma^i}(\overline{\sigma}), \frac{\partial X_0(\overline{\sigma})}{\partial \sigma^i}, \frac{\partial T_i(\overline{\sigma})}{\partial \sigma^i}, \frac{\partial X_i(\overline{\sigma})}{\partial \sigma^i} \right).$$

Therefore, we may replace (6.3.26) by a set of r homogeneous linear equations for $\lambda^0, \lambda(t_0)$, and $\lambda(t_1)$

$$\begin{aligned}&[-\lambda^0 g_{t_0} + H_r(\pi(t_0))]\frac{\partial T_0(\overline{\sigma})}{\partial \sigma^i} + \langle -\lambda^0 g_{x_0} - \lambda(t_0), \frac{\partial X_0(\overline{\sigma})}{\partial \sigma^i}\rangle \\ &\quad + [-\lambda^0 g_{t_1} - H_r(\pi(t_1))]\frac{\partial T_1(\overline{\sigma})}{\partial \sigma^i} + \langle -\lambda^0 g_{x_1} + \lambda(t_1), \frac{\partial X_1(\overline{\sigma})}{\partial \sigma^i}\rangle = 0,\end{aligned}$$

where $i = 1, \ldots, r$. Written out, this system of equations is:

$$\begin{aligned}&-\lambda^0 \Bigg[\left(\frac{\partial g}{\partial t_0} - \overline{f}^0(t_0) \right) \frac{\partial T_0}{\partial \sigma^i} + \left(\frac{\partial g}{\partial t_1} + \overline{f}^0(t_1) \right) \frac{\partial T_1}{\partial \sigma^i} \\ &+ \sum_{j=1}^{n} \left(\frac{\partial g}{\partial x_0^j} \frac{\partial X_0^j}{\partial \sigma^i} + \frac{\partial g}{\partial x_1^j} \frac{\partial X_1^j}{\partial \sigma^i} \right) \Bigg] \\ &\quad = \sum_{j=1}^{n} \lambda^j(t_0) \left(\frac{\partial X_0^j}{\partial \sigma^i} - \overline{f}^j(t_0) \frac{\partial T_0}{\partial \sigma^i} \right) + \sum_{j=1}^{n} \lambda^j(t_1) \left(\overline{f}^j(t_1) \frac{\partial T_1}{\partial \sigma^i} - \frac{\partial X_1^j}{\partial \sigma^i} \right),\end{aligned} \tag{6.3.27}$$

$i = 1, \ldots, r$, where the partials of T_0, T_1, X_0, and X_1 are evaluated at $\overline{\sigma}$, the partials of g at $e(\psi)$ and $\overline{f}^j(t) = f^j(t, \psi(t), \mu_t)$, $j = 1, \ldots, n$. Thus, the *end conditions of $\widehat{\lambda}$ satisfy a system of linear homogeneous algebraic equations.*

Remark 6.3.26. We generalize the transversality condition as follows. We do not assume that the end set $\mathcal{B}$ is a $C^{(1)}$ manifold of dimension r and that the

end point $e(\psi)$ is interior to $\mathcal{B}$. Instead we assume that the end point $e(\psi)$ is in $\mathcal{B}$ and at $e(\psi)$ there is a set V of $(2n+2)$ dimensional vectors (dt_0, dx_0, dt_1, dx_1) with the following property. For each vector (dt_0, dx_0, dt_1, dx_1) there is a $C^{(1)}$ curve

$$t_0 = T_0(\sigma) \quad x_0 = X_0(\sigma) \quad t_1 = T_1(\sigma) \quad x_1 = X_1(r)$$

defined for $0 \leq \sigma \leq \sigma_0$ such that the points $(T_0(\sigma), X_0(\sigma),\ T_1(\sigma), X_1(\sigma))$ are in $\mathcal{B}$ and $e(\psi) = (T_0(0), X_0(0), T_1(0), X_1(0))$. Under these assumptions (6.3.26) holds with the equality replaced by ≥ 0. To establish this one needs to slightly modify the arguments used to establish (6.3.26).

We leave it as an exercise to show that if $\mathcal{B}$ is a $C^{(1)}$ manifold and $e(\psi)$ is interior to $\mathcal{B}$, then (6.3.27) with $=$ replaced by $\leq$ can be deduced from the assertion in the preceding paragraph.

In Remark 6.3.6 we indicated how the conclusions of Theorems 6.3.5–6.3.22 should be modified if an ordinary pair (ϕ, u) is known *a priori* to be a relaxed optimal pair. We now carry out this modification for Theorems 6.3.12–6.3.22.

Theorem 6.3.27. *Let the hypotheses of Theorem 6.3.12 hold. Let the ordinary admissible pair (ϕ, u) be a solution of the relaxed problem. Then $J(\phi, u)$ is a solution of the ordinary problem, and the following hold.*

(i) There exists an absolutely continuous function $\widehat{\lambda} = (\lambda^0, \lambda) = (\lambda^0, \lambda^1, \ldots, \lambda^n)$ defined on $[t_0, t_1]$ such that λ^0 is either identically minus one or zero, and $(\lambda^0, \lambda(t)) \neq (0, 0)$ for all t.

(ii) The functions ϕ and λ satisfy

$$\begin{aligned} \phi'(t) &= H_q(t, \phi(t), u(t), \widehat{\lambda}(t)) \\ \lambda'(t) &= -H_x(t, \phi(t), u(t), \widehat{\lambda}(t)). \end{aligned} \tag{6.3.28}$$

(iii) For all z in $\mathcal{C}$

$$M(t, \phi(t), \widehat{\lambda}(t)) = H(t, \phi(t), u(t), \widehat{\lambda}(t)) \geq H(t, \phi(t), z, \widehat{\lambda}(t)). \tag{6.3.29}$$

(iv) The $2n$-vector

$$(-\lambda^0 g_{x_0}(e(\phi)) - \lambda(0),\ -\lambda^0 g_{x_1}(e(\phi)) + \lambda(1)) \tag{6.3.30}$$

is orthogonal to $\mathcal{B}$ at $e(\phi)$. If the hypotheses of Theorem 6.3.22 hold, then:

(v) The $(2n+2)$ vector

$$\begin{aligned} &(-\lambda^0 g_{t_0}(e(\phi)) + H(\pi(t_0)), -\lambda^0 g_{x_0}(e(\phi)) - \lambda(t_0), \\ &-\lambda^0 g_{t_1}(e(\phi)) - H(\pi(t_1)), -\lambda^0 g_{x_1}(e(\phi)) + \lambda(t_1)) \end{aligned} \tag{6.3.31}$$

is orthogonal to $\mathcal{B}$ at $e(\phi)$).

If the hypotheses of Theorem 6.3.17 hold, then:

(vi) There exists a constant c such that

$$H(t,\phi(t),u(t),\widehat{\lambda}(t)) = c + \int_0^t H_t(s,\phi(s),u(s),\widehat{\lambda}(s))ds \qquad a.e. \qquad (6.3.32)$$

Proof. That (ϕ, u) is a solution of the ordinary problem was shown in Theorem 4.4.2. The statement that λ^0 is either -1 or 0 follows from $\lambda^0 \leq 0$ and Exercise 6.3.30. All other statements follow from Theorems 6.3.12, 6.3.17, and 6.3.22 and the fact that for an ordinary control v, we have $v(t) = \nu_t = \delta_{v(t)}$ and

$$H_r(t,x,\nu_t,\widehat{q}) = H(t,x,v(t),\widehat{q}).$$

□

The following example shows that, in general, the maximum principle is not a sufficient condition for optimality.

Example 6.3.28. Let

$$f^0(t,x,z) = az^2 - 4bxz^3 + 2btz^4 \quad a > 0 \quad b > 0.$$

Let $\Omega(t,x) = \mathbb{R}^1$, let the state equation be $dx/dt = z$ and let $\mathcal{B}$ consist of a single point $(t_0,x_0,t_1,x_1) = (0,0,1,0)$. The relaxed problem is to minimize

$$J(\psi,\mu) = \int_0^1 f^0(t,\psi(t),\mu_t)dt.$$

We shall exhibit an ordinary admissible pair (ϕ, u) and a multiplier $\widehat{\lambda}(t)$ that is extremal, is such that $J(\phi, u) = 0$, and a sequence (ϕ_n, u_n) of ordinary admissible pairs such that $J(\phi_n, u_n) \to -\infty$. Since $\inf\{J(\psi,\mu)\colon (\psi,\mu)$ admissible$\} \leq \inf\{J(\phi,u)\colon (\phi,u)$ admissible$\}$ we will have shown that neither the relaxed nor ordinary problem has a solution, even though $(\phi, u, \widehat{\lambda})$ is extremal.

We need to find a $(\phi, u, \widehat{\lambda})$ that satisfies the conclusions of Theorem 6.3.27. The function H in this problem is given by

$$H(t,x,z,\widehat{q}) = q^0(az^2 - 4bxz^3 + 2btz^4) + qz.$$

Equations (6.3.28) become

$$\begin{aligned} \phi'(t) &= u(t) \\ \lambda'(t) &= \lambda^0 4bu(t)^3. \end{aligned} \qquad (6.3.33)$$

Since $\mathcal{B}$ is a point, the transversality conditions give no information about $\lambda^0, \lambda(0)$ or $\lambda(1)$.

Let ϕ, u, and λ be identically zero on $[0,1]$ and let $\lambda^0 = -1$. Then (6.3.28) is satisfied and

$$H(t,\phi(t),z,\widehat{\lambda}(t)) = -z^2(a + 2btz^2).$$

Since $a > 0$, $b > 0$ and $0 < t < 1$, we have that $M(t, \phi(t), \widehat{\lambda}(t)) = 0$ and $H(t, \phi(t), u(t), \widehat{\lambda}(t)) = 0$. Hence (6.3.29) holds, and so $(\phi, u, \widehat{\lambda})$ is extremal.

On the other hand, let $0 < k < 1$ and for $n = 2, 3, \ldots$

$$u_n(t) = \begin{cases} nt & 0 \leq t < 1/n \\ -nk/(n-1) & 1/n \leq t \leq 1 \end{cases}$$

and let ϕ_n be the corresponding trajectory. It is a straightforward calculation, which we leave to the reader, to show that $J(\phi_n, u_n) \to -\infty$ as $n \to \infty$.

Exercise 6.3.29. Show that $J(\phi_n, u_n) \to -\infty$ as $n \to \infty$.

Exercise 6.3.30. (a) Show that if $\lambda^0 \neq 0$, then we may take $\lambda^0 = -1$.

(b) Let $\mathcal{J}_0$ be a point and let $\mathcal{J}_1$ be an n-dimensional manifold of class $C^{(1)}$. Show that if an extremal trajectory is not tangent to $\mathcal{J}_1$, then $\lambda^0 \neq 0$.

(c) Show that in this case if $\lambda^0 = -1$, then $\lambda(t_1)$ is unique.

Exercise 6.3.31 (Isoperimetric Constraints)**.** In the control problem suppose that we impose the additional constraints

$$\int_{t_0}^{t_1} h^i(t, \phi(t), u(t)) = c^i \qquad i = 1, \ldots, p$$

on admissible pairs (ϕ, u), where $c = (c^1, \ldots, c^p)$ is a given constant and $h = (h^1, \ldots, h^p)$ has the same properties as the function $\widehat{f}$ in Theorem 6.3.9. For the relaxed version of the control problem, the additional constraints on relaxed admissible pairs take the form

$$\int_{t_0}^{t_1} h^i(t, \psi(t), \mu_t) dt = c^i \qquad i = 1, \ldots, p. \tag{6.3.34}$$

In Section 2.5 we showed how to transform the problem with isoperimetric constraints into a control problem without these constraints. Use this transformation and Theorem 6.3.9 to determine a maximum principle for the relaxed problem with the additional constraints in Eq. (6.3.34). If h is assumed to have the properties of $\widehat{f}$ in Theorem 6.3.12 and $\Omega(t) = \mathcal{C}$ for $t_0 \leq t \leq t_1$, use the transformation and Theorem 6.3.12 and Corollary 6.3.14 to determine a maximum principle for this relaxed problem.

Exercise 6.3.32 (Parameter Optimization)**.** Consider the problem of control and parameter optimization described in Section 2.5. Let $\widehat{f}$ be defined and continuous on $\mathcal{I}_0 \times \mathcal{X}_0 \times \mathcal{U}_0 \times \mathcal{W}_0$, where $\mathcal{I}_0, \mathcal{X}_0, \mathcal{U}_0$ are as in Assumption 6.3.1 and $\mathcal{W}_0$ is an open interval in $\mathbb{R}^k$. For fixed z in $\mathcal{U}_0$, the function $\widehat{f}$ is of class $C^{(1)}$ on $\mathcal{I}_0 \times \mathcal{X}_0 \times \mathcal{W}_0$. For each compact interval $\mathcal{X} \subseteq \mathcal{X}_0$, compact interval

$\mathcal{W} \subseteq \mathcal{W}_0$ and all z in $\mathcal{U}_0$ there exists a function $M(\cdot) = M(\cdot, \mathcal{X}, \mathcal{W})$ defined on $[t_0, t_1]$ such that M is in $L_2[t_0, t]$ and

$$|\widehat{f}(t, x, w, u(t)| \leq M(t) \qquad |\widehat{f}_t(t, x, w, u(t)| \leq M(t)$$
$$|\widehat{f}_x(t, x, w, u(t)| \leq M(t) \qquad |\widehat{f}_w(t, x, w, u(t)| \leq M(t).$$

Let (iv) and (v) of Assumption 6.3.1 hold.

Use the transformation in Section 2.5 to obtain a maximum principle analogous to Theorem 6.3.9 for the relaxed parameter optimization problem.

6.4 An Example

In this section we shall illustrate how the maximum principle and the existence theorems are used to find an optimal control. The example is also useful in pointing out the difficulties to be encountered in dealing with more complicated systems. The reader will note that all of the information in the maximum principle is used to solve the problem. In this example, there is an optimal ordinary control pair (ϕ, u). In Chapter 8, several other examples are analyzed in detail. For some of those examples, the optimal controls are relaxed.

We consider a simplified version of the Production Planning Problem 1.2. We assume that the rate of production function F is the identity function, as is the social utility function U. Thus, $F(x) = x$ and $U(v) = v$. We assume that there is no depreciation of capital, so $\delta = 0$, and we do not "discount the future," so $\gamma = 0$. If we take $\alpha = 1$ and denote the capital at time t as $\phi(t)$ rather than $K(t)$, then the production planning problem can be written as: Minimize

$$J(\phi, u) = -\int_0^T (1 - u(s))\phi(s)ds \tag{6.4.1}$$

subject to

$$\frac{dx}{dt} = u(t)x \qquad x_0 = c \tag{6.4.2}$$

$$0 \leq u(t) \leq 1 \qquad x \geq 0, \tag{6.4.3}$$

where $c > 0, T$ is fixed, and the terminal state x_1 is nonnegative, but otherwise arbitrary.

In the control formulation $\Omega(t) = [0, 1]$ for all t and

$$\mathcal{B} = \{(t_0, x_0, t_1, x_1) \colon t_0 = 0, \quad x_0 = c, \quad t_1 = T, \quad x_1 \geq 0\}.$$

Hence $\mathcal{B}$ and Ω satisfy the hypothesis of Theorem 4.4.2. Also $f^0(t, x, z) =$

$-(1-z)x$ and $f(t,x,z) = zx$, so that f^0 and f satisfy the hypotheses of Theorem 4.4.2. It follows from the state equations and the initial condition (6.4.2) that for any control u satisfying (6.4.3), the corresponding trajectory will satisfy

$$c \leq \phi(t) \leq ce^t. \tag{6.4.4}$$

Hence the constraint $x \geq 0$ is always satisfied and so can be omitted from further consideration. Since $t_1 = T$, (6.4.4) shows that all trajectories lie in a compact set. The sets $Q^+(t,x)$ in this problem are given by

$$\begin{aligned} Q^+(t,x) &= \{(y^0,y) \colon y^0 \geq (z-1)x, \quad y = zx, \quad 0 \leq z \leq 1\} \\ &= \{(y^0,y) \colon y^0 \geq y - x, \quad 0 \leq y \leq x\}. \end{aligned}$$

For each (t,x) the set $Q^+(t,x)$ is closed and convex. Thus, by Theorem 4.4.2 an optimal relaxed pair exists, and is an ordinary pair (ϕ,u). Hence we can use Theorem 6.3.27 to determine (ϕ,u). In the process we shall show that (ϕ,u) is unique.

The function $\widehat{\lambda}$ of Theorem 6.3.27 and the optimal pair (ϕ,u) satisfy

$$\begin{aligned} \phi'(t) &= u(t)\phi(t) \\ \lambda'(t) &= \lambda^0(1-u(t)) - \lambda(t)u(t) \end{aligned} \tag{6.4.5}$$

and the inequality

$$(\lambda^0 + \lambda(t))\phi(t)u(t) \geq (\lambda^0 + \lambda(t))\phi(t)z$$

for all z in $[0,1]$ and almost all t in $[0,T]$.

Since $\mathcal{B}$ consists of a fixed initial point and a terminal set $\mathfrak{J}_1$ given by $t = T$, any tangent vector to $\mathcal{B}$ at any point of $\mathcal{B}$ is a scalar multiple of $(0,0,0,1)$. Therefore, the transversality condition reduces to the condition

$$\lambda(T) = 0 \tag{6.4.6}$$

in the present case. Since $\widehat{\lambda}(t) = (\lambda^0, \lambda(t))$ is never zero, it follows that $\lambda^0 \neq 0$. It then follows from Exercise 6.3.30 that we may take $\lambda^0 = -1$. Hence from (6.4.5), we get that ϕ, u and $\widehat{\lambda}$ satisfy

$$\begin{aligned} \phi'(t) &= u(t)\phi(t) \\ \lambda'(t) &= -(1-u(t)) - \lambda(t)u(t) \end{aligned} \tag{6.4.7}$$

and the inequality

$$(-1 + \lambda(t))\phi(t)u(t) \geq (-1 + \lambda(t))\phi(t)z \tag{6.4.8}$$

for all $0 \leq z \leq 1$ and a.e. t in $[0,T]$.

It follows from (6.4.7) that ϕ and λ are solutions of the system of equations

$$\frac{dx}{dt} = u(t)x \tag{6.4.9}$$

$$\frac{dq}{dt} = -(1-u(t)) - u(t)q.$$

The boundary conditions are given by (6.4.6) and $\phi(0) = c$. Thus, we know the initial value ϕ and the terminal value of λ. Not knowing the values of both ϕ and λ at the same point makes matters difficult, as we shall see.

Since $c > 0$, it follows from (6.4.4) that $\phi(t) > 0$ for all t. Thus, although we do not know $\phi(T)$, the value of ϕ at T, we do know that $\phi(T) > 0$.

From (6.4.8) we see that for a.e. $t, z = u(t)$ maximizes the expression

$$(-1 + \lambda(t))\phi(t)z \tag{6.4.10}$$

subject to $0 \le z \le 1$. Therefore, the sign of the coefficient of z in (6.4.10) is most important. If this coefficient is > 0, then $z = 1$ maximizes; if this coefficient is < 0, then $z = 0$ maximizes. Since $\phi(t) > 0$ for all t, the determining factor is $(-1 + \lambda(t))$.

Let $\phi(T) = \xi$. Since $\xi > 0$ and since $\lambda(T) = 0$ the coefficient of z at $t = T$ in (6.4.10) is negative. Moreover, by the continuity of ϕ and λ there exists a maximal interval of the form $(T - \delta, T]$ such that the coefficient of z on this interval is negative. Hence $z = 0$ maximizes (6.4.10) for all t in this interval. Hence $u(t) = 0$ for all $T - \delta < t \le T$.

At the initial point $t = 0$ no such analysis can be made since $\lambda(0)$ and therefore the sign of the coefficient of z is unknown. This suggests that we attempt to work backward from an arbitrary point $\xi > 0$ on the terminal manifold $t = T$, as was done in the preceding paragraph.

We have already noted that there exists a maximal interval $(T - \delta, T]$ on which $u(t) = 0$. This is the maximal interval with right-hand end point T on which $-1 + \lambda(t) < 0$. We now determine δ. From the first equation in (6.4.9) with $u(t) = 0$ and from the assumption $\phi(T) = \xi$ we see that $\phi(t) = \xi$ on this interval. From the second equation in (6.4.9) with $u(t) = 0$ and from (6.4.6) we see that $\lambda(t) = (T - t)$ on this interval. Therefore, if $T > 1$ it follows that $-1 + \lambda(t) < 0$ for $T - 1 < t \le T$ and that $-1 + \lambda(T - 1) = 0$. Thus, $\delta = 1$. In summary, we have established that if $T > 1$ then on the interval $(T - 1, T]$,

$$u(t) = 0 \qquad \phi(t) = \xi \qquad \lambda(t) = T - t.$$

If we define $u(T - 1) = 0$, then the preceding hold on the interval $[T - 1, T]$.

If $T \le 1$, then $u(t) = 0$, $\phi(t) = \xi$, where $\xi = c$ is the optimal pair. It is clear from the construction that the optimal pair is unique. Thus if the "planning horizon" (the value of T) is too short, the optimal policy is to consume. If you know that you are going to die tomorrow, live it up today.

We now return to the case in which $T > 1$ and determine (ϕ, u) to the left of $T - 1$. The reader is advised to graph the functions ϕ, u, and λ on the interval $[T - 1, T]$ and to complete the graph as the functions are being determined to the left of $T - 1$. We rewrite the second equation in (6.4.9) as

$$\frac{dq}{dt} = -1 + (1 - q)u(t). \tag{6.4.11}$$

We consider this differential equation for λ on the interval $[0, T-1]$. Since λ is continuous on $[0, T]$ we have the terminal condition $\lambda(T-1) = 1$ from the discussion in the next to the last paragraph.

Since $\lambda(T-1) = 1$ and $0 \le u(t) \le 1$, it follows from (6.4.11) and the continuity of λ that on an interval $(T-1-\delta_1, T-1]$ we have $\lambda'(t) < 0$. Hence λ is increasing as we go *backward* in time on this interval. Hence $\lambda(t) > 1$ for t in some interval $(T-1-\delta_1, T-1)$. Since $\phi(t) > 0$ for all t it follows that $z = 1$ maximizes (6.4.10) on $(T-1-\delta_1, T-1)$. Hence $u(t) = 1$ on this interval and Eq. (6.4.9) becomes

$$\frac{dx}{dt} = x \qquad \phi(T-1) = \xi$$
$$\frac{dq}{dt} = -q \qquad \lambda(T-1) = 1.$$

Hence

$$\lambda(t) = \exp(T-1-t) \qquad \phi(t) = \xi \exp(t - T + 1) \tag{6.4.12}$$

on the interval $(T-1-\delta_1, T-1)$. But then $\lambda(t) > 1$ for all $0 \le t < T-1$ so that $(T-1-\delta_1, T-1) \equiv (0, T-1)$. Therefore, on $[0, T-1)$, λ and ϕ are given by (6.4.12) and $u(t) = 1$.

At $t = 0$ we have $\phi(0) = \xi \exp(-T+1)$. We require that $\phi(0) = c$. Hence $\xi = c \exp(T-1)$, and so

$$\phi(t) = ce^t$$

on the interval $0 \le t \le T-1$.

The pair (ϕ, u) that we have determined is an extremal pair. From the procedure used to determine (ϕ, u) it is clear that (ϕ, u) is unique. Therefore, since we know that an optimal pair exists and must be extremal, it follows that (ϕ, u) is indeed optimal. Moreover, it is unique. We point out that although the existence theorem guaranteed the existence of a measurable optimal control, the application of the maximum principle yielded a control that was piecewise continuous.

The procedure used in the preceding example, which is sometimes called "backing out from the target," is one that can often be applied. It illustrates the difficulties arising because the value of ϕ is specified at the initial point and the value of λ is specified at the terminal point. In small-scale problems that can be attacked analytically, one can proceed backward from an arbitrary terminal point and adjust the constants of integration to obtain the desired initial point. In large-scale problems, or problems that must be solved with a computer, this is not so easy. We shall not pursue these matters here.

6.5 Relationship with the Calculus of Variations

In this section we investigate the relationship between the maximum principle and the first order necessary conditions in the calculus of variations. We show, in detail, how the classical first order necessary conditions for the simple problem in the calculus of variations can be obtained from the maximum principle. In Exercise 6.5.1 we ask the reader to derive the first order necessary conditions for the problem of Bolza from the maximum principle. In Exercise 6.5.2 we ask the reader to derive the maximum principle for a certain class of problems from the results stated in Exercise 6.5.1.

In Chapter 2, Section 2.6, we showed that the simple problem in the calculus of variations can be formulated as a control problem as follows. Minimize

$$J(\phi) = \int_{t_0}^{t_1} f^0(t, \phi(t), u(t))dt$$

subject to

$$\frac{d\phi}{dt} = u(t) \qquad e(\phi) \in \mathcal{B} \qquad \Omega(t,x) = \mathcal{U},$$

where $\mathcal{B}$ is a given set in $\mathbb{R}^{2n+2}$, $e(\phi)$ denotes the end point $(t_0, \phi(t_0), t_1, \phi(t_1))$ of ϕ, and $\mathcal{U}$ is an open set in $\mathbb{R}^n$. We shall assume that f^0 is of class $C^{(1)}$ on $\mathcal{G} = \mathcal{R} \times \mathcal{U}$, that $\mathcal{B}$ is an r-dimensional manifold of class $C^{(1)}$ in $\mathbb{R}^{2n+2}$, $0 \leq r \leq 2n+1$, and that g is identically zero. We also assume that there exists a function M in $L_1[\mathcal{I}]$ such that $|f^0(t,x,z)| \leq M(t)$ for all (t,x,z) in $\mathcal{G}$. We assume that the relaxed problem has a solution ψ, which is an ordinary function ϕ with ϕ' bounded, and shall show that Theorem 6.3.27 reduces to the usual first order necessary conditions that a minimizing curve must satisfy.

The function H in the present case is given by the formula

$$H(t,x,z,\widehat{q}) = q^0 f^0(t,x,z) + \langle q, z \rangle.$$

Let ϕ be a solution of the variational problem and let ϕ be defined on an interval $[t_0, t_1]$. Then $(\phi, u) = (\phi, \phi')$ is a solution of the corresponding control problem. The pair (ϕ, u) therefore satisfies the conditions of Theorem 6.3.27. Thus, there exists a scalar $\lambda^0 \leq 0$ and an absolutely continuous vector function $\lambda = (\lambda^1, \ldots, \lambda^n)$ defined on $[t_0, t_1]$ such that $(\lambda^0, \lambda(t)) \neq 0$ for all t in $[t_0, t_1]$ and such that for a.e. t in $[t_0, t_1]$

$$\phi'(t) = -H_q(\pi(t)) = u(t) \tag{6.5.1}$$

$$\lambda'(t) = -H_x(\pi(t)) = -\lambda^0 f^0_x(t, \phi(t), u(t)), \tag{6.5.2}$$

where $\pi(t)$ is as in Definition 6.3.11, and

$$H(t, \phi(t), z, \widehat{\lambda}(t)) = \lambda^0 f^0(t, \phi(t), z) + \langle \lambda(t), z \rangle \tag{6.5.3}$$

is maximized over $\mathcal{U}$ at $z = u(t)$. Moreover, the vector

$$(H(\pi(t_0)), -\lambda(t_0), -H(\pi(t_1)), \lambda(t_1)) \tag{6.5.4}$$

is orthogonal to $\mathcal{B}$ at $e(\phi)$. (Recall that $g \equiv 0$.)

We assert that $\lambda^0 \neq 0$. For if $\lambda^0 = 0$, then from (6.5.3) we get that for a.e. t in $[t_0, t_1]$

$$\langle \lambda(t), u(t) \rangle \geq \langle \lambda(t), z \rangle$$

for all z in $\mathcal{U}$. This says that for a.e. t the linear function $z \to \langle \lambda(t), z \rangle$ is maximized at some point $u(t)$ of the open set $\mathcal{U}$. This can only happen if $\lambda(t) = 0$. But then $(\lambda^0, \lambda(t)) = 0$, which cannot be.

Since $\lambda^0 \neq 0$ we have that $\lambda^0 = -1$ in (6.5.1) to (6.5.4). From (6.5.2) we get that

$$\lambda(t) = \int_{t_0}^{t} f_x^0(s, \phi(s), u(s))ds + c \tag{6.5.5}$$

for some constant vector c. From (6.5.3) we get that the mapping $H(t, \phi(t), z, -1, \lambda(t))$ is maximized at $z = u(t)$. Since $\mathcal{U}$ is open, $u(t)$ is an interior point of $\mathcal{U}$. Since the mapping $z \to H(t, \phi(t), z, -1, \lambda(t))$ is differentiable, the derivative is zero at $z = u(t)$. Thus,

$$H_z(t, \phi(t), u(t), -1, \lambda(t)) = 0,$$

and therefore

$$\lambda(t) = f_z^0(t, \phi(t), u(t)) \tag{6.5.6}$$

for a.e. t in $[t_0, t_1]$.

From (6.5.1), (6.5.5), and (6.5.6) we get that

$$f_z^0(t, \phi(t), \phi'(t)) = \int_{t_0}^{t} f_x^0(s, \phi(s), \phi'(s))ds + c \qquad \text{a.e.} \tag{6.5.7}$$

Equation (6.5.7) is sometimes called the *Euler equation in integrated form* or the *du-Bois Reymond equation.* In the elementary theory it is assumed that ϕ' is piecewise continuous. Equation (6.5.7) then holds everywhere and the function f_z^{0*} defined by the formula

$$f_z^{0*}(t) = f_z^0(t, \phi(t), \phi'(t)) \tag{6.5.8}$$

is *continuous even at corners* of ϕ. By a corner of ϕ we mean a point at which ϕ' has a jump discontinuity. This result is known as the *Weierstrass-Erdmann corner condition.*

From (6.5.7) we get that

$$\frac{d}{dt}(f_z^{0*}) = f_x^{0*} \qquad \text{a.e.,} \tag{6.5.9}$$

where f_z^{0*} is defined by (6.5.8) and f_x^{0*} denotes the mapping $t \to f_x^{0*}(t, \phi(t),$

$\phi'(t))$. Equation (6.5.9) is the *Euler equation.* If we assume that ϕ' is piecewise continuous, then (6.5.9) holds between corners of ϕ.

We next discuss the transversality condition. If in (6.5.4) we take $\lambda^0 = -1$ and use (6.5.1) and (6.5.6), then (6.5.4) becomes

$$(-f^{0_*}(t_0) + \langle f_z^{0_*}(t_0), \phi'(t_0)\rangle, -f_z^{0_*}(t_0), f^{0_*}(t_1) - \langle f_z^{0_*}(t_1), \phi'(t_1)\rangle, f_z^{0_*}(t_1)), \tag{6.5.10}$$

where $f^{0_*}(t) = f^0(t, \phi(t), \phi'(t))$ and $f_z^{0_*}$ is given by (6.5.8). The transversality condition now states that (6.5.10) is orthogonal to $\mathcal{B}$ at $e(\phi)$.

If we take $\lambda^0 = -1$ and use (6.5.1) and (6.5.6), then the statement that $z = u(t)$ maximizes (6.5.3) over $\mathcal{U}$ becomes the following statement. For almost all t in $[t_0, t_1]$ and all z in $\mathcal{U}$

$$\begin{aligned} &- f^0(t, \phi(t), \phi'(t)) + \langle f_z^0(t, \phi(t), \phi'(t)), \phi'(t)\rangle \\ &\quad \geq -f^0(t, \phi(t), z) + \langle f_z^0(t, \phi(t), \phi'(t)), z\rangle. \end{aligned} \tag{6.5.11}$$

If we introduce the function $\mathcal{E}$ defined by

$$\mathcal{E}(t, x, z, y) = f^0(t, x, z) - f^0(t, x, y) - \langle f_z^0(t, x, y), z - y\rangle$$

then (6.5.11) is equivalent to the statement that

$$\mathcal{E}(t, \phi(t), z, \phi'(t)) \geq 0 \tag{6.5.12}$$

for almost all t and all z in $\mathcal{U}$. The inequality (6.5.12) is known as the *Weierstrass condition.* Note that the left-hand side of (6.5.12) consists of the first order terms in the Taylor expansion of the mapping $z \to f^0(t, \phi(t), z)$ about the point $z = \phi'(t)$.

If we assume that ϕ' is piecewise continuous, then (6.5.12) will certainly hold between corners. If $t = \tau$ is a corner, then by letting $t \to \tau + 0$ and $t \to \tau - 0$ we get that (6.5.12) holds for the one-sided limits obtained by letting $t \to \tau \pm 0$.

We now suppose that for fixed (t, x) in $\mathcal{R}$ the function f^0 is $C^{(2)}$ on $\mathcal{U}$. Hence for each t the mapping $z \to H(t, \phi(t), z, \widehat{\lambda}(t))$ is $C^{(2)}$ on $\mathcal{U}$. For a.e. t this function is maximized at $z = u(t)$, which since $\mathcal{U}$ is open must be an interior point of $\mathcal{U}$. Therefore, the quadratic form determined by the matrix $H_{zz}(t, \phi(t), u(t), \widehat{\lambda}(t))$ is negative semi-definite for almost all t. But

$$H_{zz}(t, \phi(t), u(t), \widehat{\lambda}(t)) = -f_{zz}^0(t, \phi(t), \phi'(t)).$$

Hence the quadratic form determined by $f_{zz}^0(t, \phi(t), \phi'(t))$ is positive semi-definite for almost all t. Thus, for all $\eta = (\eta^1, \ldots, \eta^n) \neq 0$ and a.e. t

$$\langle \eta, f_{zz}^0(t, \phi(t), \phi'(t))\eta\rangle \geq 0. \tag{6.5.13}$$

This is known as *Legendre's condition.*

We continue to assume that for fixed (t, x) in $\mathcal{R}$ the function f^0 is $C^{(2)}$ on

$\mathcal{U}$. Let ϕ' be piecewise continuous. We say that ϕ is *non-singular* if for all t in $[t_0, t_1]$ such that t is not a corner, the matrix $f^0_{zz}(t, \phi(t), \phi'(t))$ is non-singular. We shall show that *if ϕ is non-singular, then ϕ is $C^{(2)}$ between corners.* This result is known as *Hilbert's differentiability theorem.*

Let τ be a point in $[t_0, t_1]$ that is not a corner. Consider the following system of equations for the n-vector w:

$$f^0_z(t, \phi(t), w) - \int_{t_0}^{t} f^0_x(s, \phi(s), \phi'(s))ds - c = 0, \tag{6.5.14}$$

where c is as in (6.5.7). From (6.5.7) we see that at $t = \tau$, $w = \phi'(\tau)$ is a solution. Moreover, by hypothesis the matrix $f^0_{zz}(\tau, \phi(t), \phi'(\tau))$ is non-singular. This matrix is the Jacobian matrix for the system of equations (6.5.14). Therefore by the implicit function theorem there exists a unique $C^{(1)}$ solution, say ω, of the system (6.5.14) on an interval $(\tau - \delta, \tau + \delta)$ that does not include any corners of ϕ. On the other hand, from (6.5.7) we see that ϕ' is a solution on $(\tau - \delta, \tau + \delta)$. Therefore, $\phi' = \omega$. Hence ϕ' is $C^{(1)}$ and ϕ is $C^{(2)}$ on $(\tau - \delta, \tau + \delta)$. Since τ is an arbitrary point between corners we get that ϕ is $C^{(2)}$ between corners.

If we now assume further that f^0 is of class $C^{(2)}$ on $\mathcal{G} = \mathcal{R} \times \mathcal{U}$ and that ϕ is of class $C^{(2)}$ between corners, we may use the chain rule in (6.5.9) and get that between corners

$$f^{0*}_{zz}\phi'' + f^{0*}_{zx}\phi' + f^{0*}_{zt} - f^{0*}_{x} = 0,$$

where the asterisk indicates that the functions are evaluated at $(t, \phi(t), \phi'(t))$ and the functions ϕ' and ϕ'' are evaluated at t.

Exercise 6.5.1. Consider the problem of Bolza as formulated in Chapter 2, Section 2.6. We now assume that the function $F = (F^1, \ldots, F^\mu)$ that defines the differential equation side conditions (2.6.2) is given by

$$F^i(t, x, x') = x'^{i} - G^i(t, x, \widetilde{x}') \qquad i = 1, \ldots, \mu, \tag{6.5.15}$$

where $\widetilde{x}' = (x'^{\mu+1}, \ldots, x'^{n})$. We assume the functions f^0 and F are of class $C^{(1)}$ on $\mathcal{G} = \mathcal{R} \times \mathcal{U}$. Note that because of (6.5.15) this amounts to assuming that the function $G = (G^1, \ldots, G^\mu)$ is of class $C^{(1)}$ on an appropriate region of $(t, x, \widetilde{x}')$-space. The set $\mathcal{B}$ is assumed to be a $C^{(1)}$ manifold of dimension r, where $0 \le r \le 2n + 1$ and the function g is assumed to be $C^{(1)}$ in a neighborhood of $\mathcal{B}$. Let $\widehat{\rho} = (\rho^0, \rho) = (\rho^0, \rho^1, \ldots, \rho^\mu)$ and let

$$\begin{aligned} L(t, x, x', \widehat{\rho}) &= \rho^0 f^0(t, x, x') + \sum_{i=1}^{\mu} \rho^i F^i(t, x, x') \\ &= \rho^0 f^0(t, x, x') + \langle \rho, F(t, x, x') \rangle. \end{aligned}$$

Show that under these assumptions if ϕ is a piecewise $C^{(1)}$ minimizing function for the relaxed problem of Bolza, then the following conditions hold.

(a) (Lagrange Multiplier Rule) There exists a constant ψ^0 that is either 0 or -1 and an absolutely continuous function $\psi = (\psi^1, \ldots, \psi^\mu)$ defined on the interval $[t_0, t_1]$ such that for all $t, (\psi^0, \psi(t)) \neq 0$ and

$$L_{x'}(t, \phi(t), \phi'(t), \widehat{\psi}(t)) = \int_{t_0}^{t} L_x(s, \phi(s), \phi'(s), \widehat{\psi}(s))ds + c,$$

where c is an appropriate constant vector and $\widehat{\psi}(t) = (\psi^0, \psi(t))$. Moreover, if

$$P(t) = (t, \phi(t), \phi'(t), \widehat{\psi}(t)) \quad \text{and} \quad A = L - \langle x', L_{x'} \rangle,$$

then the $(2n+2)$-vector

$$\begin{aligned}(-A(P(t_0)) - \psi^0 g_{t_0}(e(\phi)), \qquad & - L_{x'}(P(t_0)) - \psi^0 g_{x_0}(e(\phi)), \\ A(P(t_1)) - \psi^0 g_{t_1}(e(\phi)), \qquad & L_{x'}(P(t_1)) - \psi^0 g_{x_1}(e(\phi)))\end{aligned}$$

is orthogonal to $\mathcal{B}$ at $e(\phi)$.

The last statement is the transversality condition.

(b) (Weierstrass Condition) If

$$\mathcal{E}(t, x, x', X', \widehat{\rho}) = L(t, x, X', \widehat{\rho}) - L(t, x, x', \widehat{\rho}) - \langle F_{x'}(t, x, x'), X' - x' \rangle,$$

then for all X' and almost all t

$$\mathcal{E}(t, \phi(t), \phi'(t), X', \widehat{\psi}(t)) \geq 0,$$

where ϕ and $\widehat{\psi}$ are as in (a).

(c) (Clebsch Condition) The inequality

$$\langle \eta, L_{x'x'}(t, \phi(t), \phi'(t), \widehat{\psi}(t))\eta \rangle \geq 0$$

holds for almost all t and all $\eta = (\eta^1, \ldots, \eta^n) \neq 0$ such that

$$L_{x'}(t, \phi(t), \phi'(t), \widehat{\psi}(t))\eta = 0.$$

Exercise 6.5.2. Consider the control problem with control constraints as in Chapter 2, Section 2.6 and assume that the *constraint qualification* (2.6.6) holds. Assume that the terminal set $\mathcal{B}$ is a $C^{(1)}$ manifold of dimension r, where $0 \leq r \leq 2n+1$. Assume that the functions g, f^0, f, and R are of class $C^{(1)}$ on their domains of definition. Assuming that the results of Exercise 6.5.1 hold for the problem of Bolza in the calculus of variations, prove the following theorem.

Let (ϕ, u) be an optimal control defined on an interval $[t_0, t_1]$. Then there exists a constant λ_0 that is either zero or -1, an absolutely continuous function $\lambda = (\lambda^1, \ldots, \lambda^n)$ defined on $[t_0, t_1]$, and a measurable function $\nu = (\nu^1, \ldots, \nu^r)$ defined on $[t_0, t_1]$ such that the following hold.

(i) The vector $\widehat{\lambda}(t) = (\lambda^0, \lambda(t))$ never vanishes and $\nu(t) \le 0$ a.e.

(ii) For a.e. t in $[t_0, t_1]$

$$\phi'(t) = H_q(t, \phi(t), u(t), \widehat{\lambda}(t))$$

$$\lambda'(t) = -H_x(t, \phi(t), u(t), \widehat{\lambda}(t)) - R_x(t, \phi(t), u(t))\nu(t)$$
$$H_z(t, \phi(t), u(t), \widehat{\lambda}(t)) + R_z(t, \phi(t), u(t))\nu(t) = 0$$
$$\nu^i(t)R^i(t, \phi(t), u(t)) = 0 \qquad i = 1, \ldots, r.$$

(iii) For almost all t in $[t_0, t_1]$

$$H(t, \phi(t), u(t), \widehat{\lambda}(t)) \ge H(t, \phi(t), z, \widehat{\lambda}(t))$$

for all z satisfying $R(t, \phi(t), z) \ge 0$.

(iv) The transversality condition as given in Theorem 6.3.22 holds.

(v) At each t in $[t_0, t_1]$ let $\widehat{R}$ denote the vector formed from R by taking those components of R that vanish at $(t, \phi(t), u(t))$. Then for almost all t

$$\langle e, (H(\pi(t)) + \langle \nu(t), R(t, \phi(t), u(t)) \rangle)_{zz} e \rangle \ge 0$$

for all non-zero vectors $e = (e^1, \ldots, e^m)$ such that $\widehat{R}_z(t, \phi(t), u(t))e = 0$.

6.6 Systems Linear in the State Variable

In this section we apply the maximum principle to the problem of minimizing

$$J(\phi, u) = g(e(\phi)) + \int_{t_0}^{t_1} (\langle a_0(s), \phi(s) \rangle + h^0(s, u(s)))ds \qquad (6.6.1)$$

subject to

$$\frac{dx}{dt} = A(t)x + h(t, u(t)), \qquad (6.6.2)$$

control constraints Ω and terminal constraints $\mathcal{B}$. The following assumption will be made in this section.

Assumption 6.6.1. (i) The constraint mapping Ω is a constant map; that is, $\Omega(t) = \mathcal{C}$ for all t, where $\mathcal{C}$ is a fixed compact set in $\mathbb{R}^m$.

(ii) The set $\mathcal{B}$ is a compact $C^{(1)}$ manifold of dimension r, $0 \le r \le 2n + 1$.

(iii) The vector functions a_0 and $\widehat{h} = (h^0, h^1, \ldots, h^n)$ and the matrix function A are continuous on an interval $\mathcal{I}_0$.

(iv) The function g is $C^{(1)}$ in a neighborhood of $\mathcal{B}$ in $\mathbb{R}^{2n+1}$.

As noted in the first sentence in the proof of Theorem 4.7.8, *we can assume without loss of generality that* $f^0 \equiv 0$. The functional J is then given by

$$J(\phi, u) = g(e(\phi)). \tag{6.6.3}$$

We henceforth assume that J is given by (6.6.3).

By Theorem 4.7.8 an optimal relaxed control exists for the problem of minimizing (6.6.1) subject to (6.6.2), control constraint Ω, and terminal set $\overline{\mathcal{B}}$, where the data of the problem satisfy Assumption 6.6.1. Moreover, the optimal relaxed control is an ordinary control. Before we use Theorem 6.3.27 to investigate the form of the optimal pair, we recall some properties of the solutions of a system of linear homogeneous differential equations. We consider the linear system

$$\frac{dx}{dt} = A(t)x \tag{6.6.4}$$

and the system

$$\frac{dx}{dt} = -A(t)^t x, \tag{6.6.5}$$

where the superscript t denotes transpose. The system (6.6.5) is said to be adjoint to (6.6.4). The matrix A is assumed to be measurable on some interval $\mathcal{I}_0$.

Lemma 6.6.2. *Let $\Phi(t)$ denote the fundamental matrix of solutions of* (6.6.4) *such that $\Phi(t_0) = I$, where I is the identity matrix and $t_0 \in \mathcal{I}_0$. Let $\Psi(t)$ denote the fundamental matrix of solutions of* (6.6.5) *such that $\Psi(t_0) = I$. Then*

$$\Phi^{-1}(t) = \Psi(t)^t. \tag{6.6.6}$$

Proof. By the rule for differentiating a product and (6.6.4) and (6.6.5) we get that

$$(\Psi^t\Phi)' = (\Psi^t)'\Phi + \Psi^t\Phi' = (-\Psi^t A)\Phi + \Psi^t(A\Phi) = 0.$$

Hence $\Psi^t\Phi = C$, where C is a constant matrix. But $\Phi^t(t_0)\Phi(t_0) = I \cdot I = I$, so C is the identity matrix, and the conclusion follows. $\square$

In this problem, the function H is given by

$$H(t, x, z, \widehat{q}) = \langle q, A(t)x\rangle + \langle q, h(t, z)\rangle. \tag{6.6.7}$$

Therefore, if a^{ij} denotes the element in the i-th row and j-th column of A, we get that

$$H_{x^j}(t, x, z, \widehat{q}) = \sum_{i=1}^{n} q^i a^{ij}(t).$$

The second equation in (6.3.28) therefore becomes

$$\frac{dq}{dt} = -A(t)^t q, \tag{6.6.8}$$

which is the same as (6.6.5).

Let λ be a solution of (6.6.8) satisfying the initial condition

$$\lambda(t_0) = \eta. \tag{6.6.9}$$

Let Ψ denote the fundamental matrix solution of (6.6.8), which satisfies $\Psi(t_0) = I$. Thus,

$$\Psi'(t) = -A(t)^t \Psi(t) \qquad \Psi(t_0) = I.$$

Then

$$\lambda(t) = \Psi(t)\eta. \tag{6.6.10}$$

By (6.6.6), Eq. (6.6.10) can also be written as

$$\lambda(t) = (\Phi^{-1})^t(t)\eta.$$

Assumption 6.6.3. (i) The functions a_0 and $\widehat{h}$ and the matrix A are $C^{(1)}$ on $\mathcal{I}_0$.

(ii) The end point $e(\phi)$ is interior to $\mathcal{B}$ and at $e(\phi)$ the vector

$$(g_{t_0}, g_{x_0}, g_{t_1}, g_{x_1}) \tag{6.6.11}$$

is neither zero nor orthogonal to $\mathcal{B}$.

If we set

$$f^*(t) = A(t)\phi(t_i) + h(t, u(t)),$$

then $\langle \lambda, f^* \rangle$ is absolutely continuous, and the transversality condition states that the vector

$$(\langle \lambda(t_0), f^*(t_0)\rangle - \lambda^0 g_{t_0},\ \ -\lambda(t_0) - \lambda^0 g_{x_0},\ \ -\langle \lambda(t_1), f^*(t_1)\rangle - \lambda^0 g_{t_1},\ \ \lambda(t_1) - \lambda^0 g_{x_1}),$$

where all the partial derivatives of g are evaluated at $e(\phi)$, is orthogonal to $\mathcal{B}$ at $e(\phi)$. By virtue of (6.6.9) and (6.6.10) this vector can also be written as

$$(\langle \eta, f^*(t_0)\rangle - \lambda^0 g_{t_0},\ \ -\eta - \lambda^0 g_{x_0},\ \ -\langle \Psi(t_1)\eta, f^*(t_1)\rangle - \lambda^0 g_{t_1},\ \ \Psi(t_1)\eta - \lambda^0 g_{x_1}). \tag{6.6.12}$$

By (6.6.6) this vector can also be written as

$$\begin{aligned}(\langle \eta, f^*(t_0)\rangle - \lambda^0 g_{t_0}, -\eta - \lambda^0 g_{x_0}, -\langle (\Phi^{-1}(t_1))^t \eta, f^*(t_1)\rangle \\ -\lambda^0 g_{t_1}, (\Phi^{-1}(t))^t \eta - \lambda^0 g_{x_1}).\end{aligned} \tag{6.6.13}$$

We now show that $\eta \neq 0$. For if $\eta = 0$, then by (6.6.10), $\lambda(t) = 0$ on $[t_0, t_1]$. Hence by the maximum principle $\lambda^0 \neq 0$. Also, if $\eta = 0$, Eq. (6.6.12) becomes

$-\lambda^0(g_{t_0}, g_{x_0}, g_{t_1}, g_{x_1})$. Since $\lambda^0 \neq 0$, the transversality condition implies that the vector (6.6.11) is either zero or is orthogonal to $\mathcal{B}$ at $e(\phi)$. This, however, was ruled out.

From (6.6.7) and Theorem 6.3.27(iii), we have that

$$\langle \lambda(t), h(t, u(t)) \rangle \geq \langle \lambda(t), h(t, z) \rangle \tag{6.6.14}$$

for a.e. t in $[t_0, t_1]$ and all z in $\mathcal{C}$. From (6.6.10) and (6.6.6) we get that

$$\langle \lambda(t), h(t, z) \rangle = \langle \Psi(t)\eta, h(t, z) \rangle = \langle \eta, \Psi^t(t)h(t, z) \rangle = \langle \eta, \Phi^{-1}(t)h(t, z) \rangle.$$

Thus, (6.6.14) is equivalent to the inequality

$$\langle \eta, \Phi^{-1}(t)h(t, u(t)) \rangle \geq \langle \eta, \Phi^{-1}(t)h(t, z) \rangle. \tag{6.6.15}$$

We summarize the preceding discussion in the following theorem, which gives the maximum principle for systems linear in the state variable.

Theorem 6.6.4. *(i) Let Assumption 6.6.1 hold. Then the relaxed version of the problem "Minimize the functional (6.6.3) subject to (6.6.1), control constraints Ω and terminal set $\mathcal{B}$" has a solution that is an ordinary admissible pair (ϕ, u).*

(ii) Let Φ denote the fundamental matrix solution of (6.6.4) such that $\Phi(t_0) = I$, and let Assumption 6.6.3 hold. Then there exists a nonzero vector η in $\mathbb{R}^n$ and a scalar $\lambda^0 \leq 0$ such that the vector (6.6.13) is orthogonal to $\mathcal{B}$ at $e(\phi)$ and such that for a.e. t in $[t_0, t_1]$

$$\max\{\langle \eta, \Phi^{-1}(t)h(t, z) \rangle : z \in \mathcal{C}\}$$

occurs at $z = u(t)$.

Remark 6.6.5. By virtue of (6.6.6) the quantity to be maximized can also be written as $\langle \eta, \Psi^t(t)h(t, z) \rangle$.

Remark 6.6.6. Note that, in principle, for systems linear in the state variable we only need the initial value η of the function λ in order to determine an extremal trajectory. Of course, the terminal value $\lambda(t_1)$ will also do, since by (6.6.10), $\eta = \Psi^{-1}(t_1)\lambda(t_1)$. To determine η we use the transversality condition. This involves knowing $e(\phi)$ and $u(t_i)$, $i = 0, 1$. Once u is known the variation of parameters formula gives the extremal trajectory.

Exercise 6.6.7. Obtain an analytic formulation of the transversality condition as a system of r linear equations in the unknowns $(\lambda^0, \eta^1, \ldots, \eta^n)$ by setting the inner product of (6.6.9) with each of r linearly independent tangent vectors to $\mathcal{B}$ at $e(\phi)$ equal to zero.

6.7 Linear Systems

A linear system is one in which the function h is given by

$$h(t,z) = B(t)z + d(t), \tag{6.7.1}$$

where B is an $n \times m$ matrix and d is an n-vector. The system (6.6.2) becomes

$$\frac{dx}{dt} = A(t)x + B(t)z + d(t). \tag{6.7.2}$$

For (iii) of Assumption 6.6.1 to hold, B and d must be continuous functions.

The maximum principle for linear systems is an immediate consequence of Theorem 6.6.4.

Theorem 6.7.1. *Let Assumption 6.6.1 hold.*

(i) Then the relaxed version of the problem "Minimize the functional (6.6.3) subject to (6.7.2), control constraints Ω and terminal constraint $\mathcal{B}$" has a solution that is an ordinary admissible pair (ϕ, u).

(ii) Let Φ be the fundamental solution matrix of (6.6.4) satisfying $\Phi(t_0) = I$, and let Assumption 6.6.3 hold. Then there exists a non-zero vector η in $\mathbb{R}^n$ and a scalar $\lambda^0 \le 0$ such that the vector (6.6.13) is orthogonal to $\mathcal{B}$ at $e(\phi)$ and such that for a.e. t in $[t_0, t_1]$

$$\max\{\langle \eta, \Phi^{-1}(t)B(t)z\rangle \colon z \in \mathcal{C}\} \tag{6.7.3}$$

occurs at $z = u(t)$.

Remark 6.7.2. By virtue of (6.6.6) we can write

$$\max\{\langle \eta, \Psi^t(t)B(t)z\rangle \colon z \in \mathcal{C}\}$$

in place of (6.7.3). In either case, we have that $z = u(t)$ maximizes a linear form in z over a set $\mathcal{C}$. To emphasize this we shall let

$$L(t,\eta,z) = \langle \eta, \Phi^{-1}(t)B(t)z\rangle = \langle \eta, \Psi^t(t)B(t)z\rangle. \tag{6.7.4}$$

Let $\psi_i = (\psi_i^1 \ldots \psi_i^n)^t$ denote the function comprising the i-th column of Ψ, let $\eta = (\eta^1, \ldots, \eta^n)$, let $b^j(t)$ denote the j-th column of $B(t)$, and let b^{kj} denote the entry in the k-th row and j-th column of B. Then the coefficient of z^j in (6.7.4) is

$$\sum_{i,k=1}^{n} \eta^i \psi_i^k(t) b^{kj}(t) = \eta \Psi^t(t) b^j(t).$$

In many problems the set $\mathcal{C}$ is the cube given by

$$\mathcal{C} = \{z \colon |z^i| \le 1, i = 1, \ldots, m\}. \tag{6.7.5}$$

In this situation, an optimal control often can be characterized very simply.

Corollary 6.7.3. *Let $\mathcal{C}$ be given by (6.7.5). For each $j = 1, \dots, m$ let*

$$E_j(\eta) = \{t \colon \eta\Psi^t(t)b^j(t) = 0\}$$

have measure zero. Then for almost all t in $[t_0, t_1]$

$$u^j(t) = \textit{signum } \eta\Psi^t(t)b^j(t). \tag{6.7.6}$$

Proof. Since for a.e. $t, u(t)$ maximizes $z \to L(t, \eta, z)$ over $\mathcal{C}$ and since $\eta\Psi^t(t)B^j(t)$ is the coefficient of z^j, the result is immediate. □

Definition 6.7.4. A linear system (6.7.2) is said to be *normal with respect to* $\mathcal{C}$ on an interval $[t_0, t_1]$ if for every non-zero n-vector μ and for a.e. t in $[t_0, t_1]$,

$$\max\{L(t, \mu, z) \colon z \in \mathcal{C}\}$$

occurs at a unique point $z^*(t)$ in $\mathcal{C}$.

Note that whether a system is normal on a given interval is determined by the matrices A and B and by the constraint set $\mathcal{C}$. At the end of this section we shall develop criteria for normality that involve conditions on A, B, and $\mathcal{C}$ that are relatively easy to verify.

If $\mathcal{C}$ is given by (6.7.5), a system is normal if and only if the set $E_j(\mu)$ has measure zero for each μ in $\mathbb{R}^n$ and each $j = 1, \dots, m$. Thus, Corollary 6.7.3 states that if a system is normal with respect to $\mathcal{C}$, where $\mathcal{C}$ is given by (6.7.5), then the optimal control is given by (6.7.6).

We now investigate the structure of an optimal control when $\mathcal{C}$ is a compact convex set. If $\mathcal{C}$ is compact and convex, then by the Krein-Milman Theorem (Lemma 4.7.5) the set of extreme points $\mathcal{C}_e$ of $\mathcal{C}$ is non-empty. The following corollary of Theorem 6.7.1 holds.

Corollary 6.7.5. *Let $\mathcal{C}$ be a compact and convex set and let the system be normal. If u is an optimal control, then $u(t) \in \mathcal{C}_e$ for almost all t.*

Proof. If the conclusion were false, then $u(t) \notin \mathcal{C}_e$ for t in a set E of positive measure. Hence for $t \in E$, there exist points $z_1(t)$ and $z_2(t)$ in $\mathcal{C}$ and real numbers $\alpha(t) > 0$, $\beta(t) > 0$, with $\alpha(t) + \beta(t) = 1$ such that $u(t) = \alpha(t)z_1(t) + \beta(t)z_2(t)$. Since the system is normal, the linear function $L(t, \eta, \cdot)$ achieves its maximum at a unique point $z^*(t)$ for a.e. t in E. By the maximum principle, the maximum is achieved at $u(t)$, so that $z^*(t) = u(t)$. Hence $L(t, \eta, u(t)) > L(t, \eta, z_1(t))$ and $L(t, \eta, u(t)) > L(t, \eta, z_2(t))$ a.e. in E. Therefore,

$$\begin{aligned}
L(t, \eta, u(t)) &= \alpha(t)L(t, \eta, u(t)) + \beta(t)L(t, \eta, u(t)) \\
&> \alpha(t)L(t, \eta, z_1(t)) + \beta(t)L(t, \eta, z_2(t)) \\
&= L(t, \eta, \alpha(t)z_1(t) + \beta(t)z_2(t)) = L(t, \eta, u(t)),
\end{aligned}$$

which is a contradiction. □

Definition 6.7.6. Let $\mathcal{C}$ be a compact polyhedron $\mathcal{P}$ with vertices $e_1, \ldots, e_k$. A control u is said to be polyhedral *bang-bang* on an interval $[t_0, t_1]$ if for a.e. t in $[t_0, t_1], u(t)$ is equal to one of the vertices.

If $\mathcal{C} = \mathcal{P}$, Corollary 6.7.5 can be restated as follows.

Corollary 6.7.7. *Let the system be normal and let the constraint set be a compact polyhedron $\mathcal{P}$. Then any optimal control is polyhedral bang-bang.*

Remark 6.7.8. The bang-bang principle (Theorem 4.7.9) tells us that if u is an optimal control, then there is another optimal control u^* that is bang-bang. The system is not assumed to be normal. Corollary 6.7.5, on the other hand, tells us that if a system is normal and the constraint set is compact and convex, then *any* optimal control *must* be bang-bang.

The preceding results do not guarantee uniqueness of the optimal control for normal systems. The next theorem gives reasonable conditions under which an optimal control is unique.

Theorem 6.7.9. *Let $\mathcal{C}$ be compact and convex, let the system be normal, let $\mathcal{B}$ be a relatively open convex subset of a linear variety in $\mathbb{R}^{2n+2}$, and let g be given by*

$$g(t_0, x_0, t_1, x_1) = g_1(x_0, x_1) + g_2(t_0, t_1), \tag{6.7.7}$$

where g_1 is convex. Let u_1 and u_2 be two optimal controls defined on the same interval $[t_0, t_1]$. Then $u_1 = u_2$ a.e. on $[t_0, t_1]$.

Proof. Let ϕ_1 be the trajectory corresponding to u_1 and let ϕ_2 be the trajectory corresponding to u_2. Define $u_3 = (u_1 + u_2)/2$. Since $\mathcal{C}$ is convex, $u_3(t) \in \mathcal{C}$. Let ϕ_3 be the trajectory corresponding to u_3 that satisfies the initial condition $\phi_3(t_0) = (\phi_1(t_0) + \phi_2(t_0))/2$. Then

$$\begin{aligned}\phi_3(t) &= \Phi(t)\{\phi_3(t_0) + \frac{1}{2}\int_{t_0}^{t} \Phi^{-1}(s)[B(s)(u_1(s) + u_2(s)) + 2d(s)]ds\} \\ &= (\phi_1(t) + \phi_2(t))/2\end{aligned}$$

and $e(\phi_3) = (e(\phi_1) + e(\phi_2))/2$. Since $\mathcal{B}$ is a convex subset of a linear variety, it follows that $e(\phi_3) \in \mathcal{B}$. Hence (ϕ_3, u_3) is an admissible pair.

Let $\mu = \inf\{J(\phi, u) \colon (\phi, u) \text{ admissible}\}$. Then

$$\mu = J(\phi_1, u_1) = J(\phi_2, u_2).$$

Recall that we are assuming that $J(\phi, u)$ is given by (6.6.3). From the definition of μ, from (6.6.3), from (6.7.7), from the convexity of g_1, and the assumption that ϕ_1 and ϕ_2 have the same initial and terminal times we get

$$\mu \leq J(\phi_3, u_3) = g(e(\phi_3)) = g((e(\phi_1) + e(\phi_2))/2) \leq \frac{1}{2}g(e(\phi_1)) + \frac{1}{2}g(e(\phi_2)) = \mu.$$

Thus, $J(\phi_3, u_3) = \mu$, and the pair (ϕ_3, u_3) is optimal. By Corollary 6.7.5, $u_3(t) \in \mathcal{C}_e$ a.e. This contradicts the definition of u_3 unless $u_1 = u_2$ a.e. $\square$

Remark 6.7.10. For problems with t_0 and t_1 fixed, g automatically has the form (6.7.7) with $g_2 \equiv 0$. If we assume that g is a convex function of (t_0, x_0, t_1, x_1), then the assumption that g has the form (6.7.7) can be dropped.

Definition 6.7.11. The linear system (6.7.2) is said to be *strongly normal* on an interval $[t_0, t_1]$ with respect to a constraint set $\mathcal{C}$ if for every non-zero vector μ in $\mathbb{R}^n$, $\max\{L(t, \mu, z) \colon z \in \mathcal{C}\}$ is attained at a unique $z^*(t)$ in $\mathcal{C}$ at all but a finite set of points in $[t_0, t_1]$.

Definition 6.7.12. A control u is said to be piecewise constant on an interval $[t_0, t_1]$ if there exist a finite number of disjoint open subintervals (τ_j, τ_{j+1}) such that the union of the closed subintervals $[\tau_j, \tau_{j+1}]$ is $[t_0, t_1]$ and such that u is constant on each of the open subintervals (τ_j, τ_{j+1}).

The next theorem gives a characterization of the optimal control in strongly normal systems that is of practical significance. Simple criteria for strong normality will be given in Theorem 6.7.14 and its corollaries.

Theorem 6.7.13. *Let (ϕ, u) be an optimal pair and let Assumptions 6.6.1 and 6.6.3 hold. Let the matrix B be continuous and let the constraint set $\mathcal{C}$ be a compact polyhedron $\mathcal{P}$. Let the system (6.7.2) be strongly normal on $[t_0, t_1]$, the interval of definition of (ϕ, u). Then u is piecewise constant on $[t_0, t_1]$ with values in the set of vertices of $\mathcal{P}$.*

Proof. If we remove the points t_0, t_1 and the finite set of points at which the maximum of $L(t, \eta, z)$ is not achieved at a unique $z^*(t)$, we obtain a finite collection of disjoint open intervals (τ_j, τ_{j+1}) such that the union of the closed intervals $[\tau_j, \tau_{j+1}]$ is the interval $[t_0, t_1]$. Let J denote one of the intervals (τ_j, τ_{j+1}). From the assumption of strong normality and the proof of Corollary 6.7.5, it is seen that for each t in J, $u(t)$ is equal to one of the vertices e_i, $i = 1, \ldots, k$, of $\mathcal{P}$. Let M_i denote the set of points t in J at which $u(t) = e_i$. Then not all of the M_i, $i = 1, \ldots, k$ are empty, the sets M_i are pairwise disjoint and $J = \cup M_i$. We now show that if M_i is not empty, then it is open. Let $\tau \in M_i$. Then

$$L(\tau, \eta, e_i) > L(\tau, \eta, e_j) \quad \text{for all } j \neq i. \tag{6.7.8}$$

Since for fixed η, e_i the mapping $t \to L(t, \eta, e_i)$ is continuous, (6.7.8) holds in a neighborhood of τ. Hence all points of this neighborhood are in M_i and hence M_i is open. Since J is connected and since $J = \cup M_j$, where the M_j are open and pairwise disjoint, it follows that for $j \neq i$ the set M_j must be empty. Thus $u(t) = e_i$ in J, and the theorem is proved. $\square$

The conclusion of Theorem 6.7.13 is much stronger than that of Corollary 6.7.7. Here we assert that the optimal control is piecewise constant with values at the vertices $e_1, \ldots, e_k$ of $\mathcal{P}$, while in Corollary 6.7.7 we merely assert that the optimal control is measurable with values at the vertices of $\mathcal{P}$. Of course, the assumptions are more stringent here.

We conclude this section with a presentation of criteria for strong normality.

Theorem 6.7.14. *Let the state equations be given by* (6.7.2)*. Let A be of class $C^{(n-2)}$ on a compact interval $\mathcal{I}$ and let B be of class $C^{(n-1)}$ on $\mathcal{I}$. Let the constraint set be a compact polyhedron $\mathcal{P}$. Let*

$$\begin{aligned} B_1(t) &= B(t) \\ B_j(t) &= -A(t)B_{j-1}(t) + B'_{j-1}(t) \qquad j = 2, \ldots, n. \end{aligned} \tag{6.7.9}$$

If for every vector w in $\mathbb{R}^m$ that is parallel to an edge of $\mathcal{P}$ the vectors

$$B_1(t)w, \; B_2(t)w, \ldots, B_n(t)w \tag{6.7.10}$$

are linearly independent for all t in $\mathcal{I}$, then the system (6.7.2) is strongly normal with respect to $\mathcal{P}$ on $\mathcal{I}$.

Proof. Suppose the conclusion is false. Then there exists a non-zero vector η in $\mathbb{R}^n$ and an infinite set of points E in $\mathcal{I}$ such that for t in E, the maximum over $\mathcal{P}$ of $L(t, \eta, z)$ is not achieved at a unique $z^*(t)$ in $\mathcal{P}$. Since for fixed (t, η) the mapping $z \to L(t, \eta, z)$ is linear and since $\mathcal{P}$ is a compact polyhedron, the maximum over $\mathcal{P}$ of $L(t, \eta, z)$ is attained on some face of $\mathcal{P}$. Since there are only a finite number of faces on $\mathcal{P}$, there exists an infinite set $E_1 \subset E$ and a face $\mathcal{P}_F$ of $\mathcal{P}$ such that for t in E_1, the maximum over $\mathcal{P}$ is attained on $\mathcal{P}_F$. Hence if e_1 and e_2 are two distinct vertices in $\mathcal{P}_F$, $L(t, \eta, e_1) = L(t, \eta, e_2)$ for all t in E_1. Hence if $w = e_1 - e_2$,

$$L(t, \eta, w) = \langle \eta, \Psi^t(t)B(t)w \rangle = 0$$

for all t in E_1. From the first equation in (6.7.9) we get

$$L(t, \eta, w) = \langle \eta, \Psi^t(t)B_1(t)w \rangle = 0 \tag{6.7.11}$$

for all t in E_1.

Since E_1 is an infinite set and $\mathcal{I}$ is compact, E_1 has a limit point τ in $\mathcal{I}$. From (6.7.11) and the continuity of B_1 and Ψ^t we get

$$L(\tau, \eta, w) = \langle \eta, \Psi^t(\tau)B_1(\tau)w \rangle = 0. \tag{6.7.12}$$

By hypothesis, the matrix A is of class $C^{(n-2)}$. Hence the fundamental matrix Ψ of the system adjoint to (6.6.4) is of class $C^{(n-1)}$. Since $B_1 = B$ and B is assumed to be of class $C^{(n-1)}$, it follows from the first equality in (6.7.9) and from (6.7.4) that the mapping $t \to L(t, \eta, w)$ is of class $C^{(n-1)}$ on $\mathcal{I}$. Also,

$$L'(t, \eta, w) = \langle \eta, \Psi^{t'}(t)B_1(t)w \rangle + \langle \eta, \Psi^t(t)B'_1(t)w \rangle.$$

From (6.6.5) we get

$$\Psi^{t'}(t) = -\Psi^t(t)A(t).$$

If we substitute this into the preceding equation we get

$$L'(t, \eta, w) = \langle \eta, \Psi^t(t)(-A(t)B_1(t) + B_1'(t))w \rangle.$$

From the second equation in (6.7.9) we get

$$L'(t, \eta, w) = \langle \eta, \Psi^t(t)B_2(t)w \rangle. \tag{6.7.13}$$

The derivative of a function has a zero between any two zeros of the function. Therefore, $L'(t, \eta, w) = 0$ for all t in an infinite set E_2 having τ as a limit point. From (6.7.13) and the continuity of Ψ^* and B_2 it follows that

$$\langle \eta, \Psi^t(\tau)B_2(\tau)w \rangle = 0.$$

We can proceed inductively in this manner and get

$$\langle \eta, \Psi^t(\tau)B_j(\tau)w \rangle = 0 \qquad j = 1, \ldots, n.$$

Since the n vectors $B_1(\tau)w, \ldots, B_n(\tau)w$ are assumed to be linearly independent, $\eta \neq 0$, and $\Psi^t(\tau)$ is non-singular, this is impossible. This contradiction proves the theorem. □

Corollary 6.7.15. *Let A and B be constant matrices. If for every vector w in $\mathbb{R}^m$ that is parallel to an edge of $\mathcal{P}$, the vectors*

$$Bw, ABw, A^2Bw, \ldots, A^{n-1}Bw$$

are linearly independent, then the system (6.7.2) is strongly normal with respect to $\mathcal{P}$ on $\mathcal{J}$.

The corollary follows from the observation that if A and B are constant matrices, then

$$B_j = (-A)^{j-1}B \qquad j = 1, \ldots, n.$$

If the set $\mathcal{P}$ is a parallelepiped with axes parallel to the coordinate axis, then the only vectors w that we need consider are the standard basis vectors $w_1, \ldots, w_m$ in $\mathbb{R}^m$. Here, w_i is the m-vector whose i-th component is equal to one and other components are all zero. Let b^j denote the j-th column of the matrix B. Then $b^j = Bw_j$, and Corollary 6.7.15 yields the following:

Corollary 6.7.16. *Let A and B be constant matrices and let $\mathcal{P}$ be a parallelepiped with axes parallel to the coordinate axes. Let b^j denote the j-th column of B. For each $j = 1, \ldots, m$, let*

$$b^j, Ab^j, A^2b^j, \ldots, A^{n-1}b^j$$

be linearly independent. Then the system (6.7.2) is strongly normal with respect to $\mathcal{P}$ on $\mathcal{I}$.

6.8 The Linear Time Optimal Problem

In the linear time optimal problem it is required to transfer a given point x_0 to another given point x_1 in minimum time by means of a linear system. More precisely, in the linear time optimal problem it is required to minimize

$$J(\phi, u) = t_1$$

subject to the state equation (6.7.2), constraint condition Ω, and end condition $\mathcal{B}$, where

$$\mathcal{B} = \{(t_0, x_0, t_1, x_1) \colon t_0 = t_0', x_0 = x_0', x_1 = x_1'\},$$

with t_0', x_0', and x_1' given. The function g is now $g(t_1) = t_1$.

If $\Omega(t) = \mathcal{C}$, where $\mathcal{C}$ is a fixed compact convex set, if the system (6.7.2) is normal with respect to $\mathcal{C}$, and Assumptions 6.6.1 and 6.6.3 hold, then by Corollary 6.7.15 an optimal control u exists and has the form $u(t) \in \mathcal{C}_e$ a.e. If $\mathcal{C}$ is a compact polyhedron $\mathcal{P}$, then u is polyhedral bang-bang. If u_1 and u_2 are two optimal controls, then since the problem is one of minimizing t_1, it follows that u_1 and u_2 are both defined on the same interval $[t_0, t_1^*]$, where t_1^* is the minimum time. Hence by Theorem 6.7.9 $u_1 = u_2$ a.e. We summarize this discussion in the following theorem.

Theorem 6.8.1. *In the linear time optimal problem if Assumptions 6.6.1 and 6.6.3 hold and the system is normal with respect to $\mathcal{C}$, then the optimal control u is unique and $u(t) \in \mathcal{C}_e$ a.e.*

There is another class of linear time optimal problems with the property that *extremal controls* are unique. For this class, the arguments used to show uniqueness of extremal controls prove directly, without reference to existence theorems, that an extremal control is unique and is optimal.

Theorem 6.8.2. *Let the system equations be given by* (6.7.2) *with $d \equiv 0$. Let $\mathcal{C}$ be a compact convex set with the origin of $\mathbb{R}^m$ an interior point of $\mathcal{C}$. Let the system be normal with respect to $\mathcal{C}$. Let (ϕ_1, u_1) be an extremal pair for the time optimal problem with terminal state $x_1 = 0$. Let the terminal time at which ϕ_1 reaches the origin be t_1. Let (ϕ_2, u_2) be an admissible pair which transfers x_0 to the origin in time $t_2 - t_0$. Then $t_2 \geq t_1$ with equality holding if and only if $u_1(t) = u_2(t)$ a.e.*

Proof. Suppose there exists a pair (ϕ_2, u_2) for which $t_2 \leq t_1$. From the variation of parameters formula we get

$$0 = \Phi(t_1)\left\{x_0 + \int_{t_0}^{t_1} \Phi^{-1}(s)B(s)u_1(s)ds\right\}$$
$$0 = \Phi(t_2)\left\{x_0 + \int_{t_0}^{t_2} \Phi^{-1}(s)B(s)u_2(s)ds\right\},$$

where Φ is the fundamental matrix for the system (6.6.4) satisfying $\Phi(t_0) = I$. If we multiply the first equation by $\Phi(t_1)^{-1}$ on the left and multiply the second equation by $\Phi(t_2)^{-1}$ on the left we get

$$-x_0 = \int_{t_0}^{t_1} \Phi^{-1}(s)B(s)u_1(s)ds = \int_{t_0}^{t_2} \Phi^{-1}(s)B(s)u_2(s)ds. \tag{6.8.1}$$

Since u_1 is an extremal control there exists a non-zero vector η in $\mathbb{R}^n$ such that for a.e. t in $[t_0, t_1]$, $u_1(t)$ maximizes $L(t, \eta, z)$ over $\mathcal{C}$. If we compute $\langle \eta, -x_0 \rangle$ in (6.8.1) we get

$$\int_{t_0}^{t_2} + \int_{t_2}^{t_1} \langle \eta, \Phi^{-1}(s)B(s)u_1(s) \rangle ds = \int_{t_0}^{t_2} \langle \eta, \Phi^{-1}(s)B(s)u_2(s) \rangle ds.$$

Therefore,

$$\int_{t_0}^{t_2} \{L(s, \eta, u_1(s)) - L(s, \eta, u_2(s))\} ds = -\int_{t_2}^{t_1} L(s, \eta, u_1(s))ds. \tag{6.8.2}$$

Since u_1 is extremal and the system is normal with respect to $\mathcal{C}$, $u_1(t) \in \mathcal{C}_e$ a.e. Since 0 is an interior point of $\mathcal{C}$, $u_1(t) \neq 0$ a.e. and

$$L(t, \eta(t), u_1(t)) > L(t, \eta, 0) = 0.$$

Hence the right-hand side of (6.8.2) is ≤ 0, with equality holding if and only if $t_1 = t_2$. On the other hand, since the system is normal

$$L(t, \eta, u_1(t)) \geq L(t, \eta, u_2(t))$$

for a.e. t, with equality holding if and only if $u_1(t) = u_2(t)$ a.e. Hence the integral on the left in (6.8.2) is ≥ 0 with equality holding if and only if $u_1(t) = u_2(t)$ a.e. Therefore, each side of (6.8.2) is equal to zero. This implies that $t_2 = t_1$ and $u_2 = u_1$ a.e., on $[t_0, t_1]$ and the theorem is proved. □

6.9 Linear Plant-Quadratic Criterion Problem

In the problems studied in this section the state equations are

$$\frac{dx}{dt} = A(t)x + B(t)z + d(t) \tag{6.9.1}$$

and the function f^0 is given by

$$f^0(t, x, z) = \langle x, X(t)x \rangle + \langle z, R(t)z \rangle. \tag{6.9.2}$$

Unless stated otherwise, the following assumption will be in effect throughout this section.

Assumption 6.9.1. (i) The matrices A, B, X, and R in (6.9.1) and (6.9.2) are $C^{(1)}$ on an interval $[a, b]$, as is the function d in (6.9.1).

(ii) For each t in $[a, b]$ the matrix $X(t)$ is symmetric, positive semi-definite and the matrix $R(t)$ is symmetric, positive definite.

(iii) For each t in $[a, b], \Omega(t) = \mathcal{O}$, where $\mathcal{O}$ is a fixed *open* set in $\mathbb{R}^m$.

(iv) The set $\mathcal{B}$ is the n-dimensional manifold consisting of all points (t_0, x_0, t_1, x_1) with (t_0, x_0) fixed and (t_1, x_1) in a specified n-dimensional manifold $\mathcal{J}_1$.

(v) The function $g\colon (t_1, x_1) \to g(t_1, x_1)$ is $C^{(1)}$ on $\mathbb{R}^{n+1}$.

(vi) The controls u are in $L_2[a, b]$.

The problem to be studied is that of minimizing

$$J(\phi, u) = g(t_1, \phi(t_1)) + \frac{1}{2}\int_{t_0}^{t_1} \{\langle \phi(s), X(s)\phi(s)\rangle + \langle u(s), R(s)u(s)\rangle\}\, ds \quad (6.9.3)$$

subject to the state equation (6.9.1), the control constraints Ω, and the terminal (t_0, x_0, t_1, x_1) in $\mathcal{B}$, where the data of the problem satisfy Assumption 6.9.1.

In Exercise 5.4.20, we showed that this problem has a solution (ϕ, u) which is also a solution of the corresponding relaxed problem. Therefore, (ϕ, u) satisfies the conditions of Theorem 6.3.27. We assume the following:

Assumption 6.9.2. The trajectory ϕ is not tangent to $\mathcal{T}_1$ at $(t_1, \phi(t_1))$.

We now characterize optimal pairs by means of the maximum principle. The function H is given by

$$\begin{aligned} H(t, x, z, \widehat{q}) = \frac{q^0}{2}\{\langle x, X(t)x\rangle + \langle z, R(t)z\rangle\} + \langle q, A(t)x\rangle \\ + \langle q, B(t)z\rangle + \langle q, d(t)\rangle. \end{aligned} \quad (6.9.4)$$

Thus,

$$H_x(t, x, z, \widehat{q}) = q^0 X(t)x + A(t)^t q.$$

By (iv) of Assumption 6.9.1, the transversality condition given in Theorem 6.3.22 takes the following form. The vector

$$(-H(\pi(t_1)) - \lambda^0 g_{t_1}, \lambda(t_1) - \lambda^0 g_{x_1}),$$

where the partial derivatives of g are evaluated at $(t_1, \phi(t_1))$, is orthogonal to $\mathcal{T}_1$ at $(t_1, \phi(t_1)$. It follows from (iv) of Assumption 6.9.1, from Assumption 6.9.2, and from Exercise 6.3.30 that $\lambda^0 \neq 0$, and that we may take $\lambda^0 = -1$. With $\lambda^0 = -1$, the vector $\lambda(t_1)$ is unique. The transversality condition now states that the vector

$$(g_{t_1} + f_1^0 - \langle \lambda(t_1), f_1\rangle, g_{x_1} + \lambda(t_1)) \quad (6.9.5)$$

is orthogonal to $\mathcal{T}_1$ at $(t_1, \phi(t_1))$, where the partial derivatives of g are evaluated at $(t_1, \phi(t_1)), f_1^0$ denotes (6.9.2) evaluated at $(t_1, \phi(t_1), u(t_1))$, and f_1 denotes the right-hand side of (6.9.1) evaluated at $(t_1, \phi(t_1), u(t_1))$.

From Theorem 6.3.27 we have that

$$\begin{aligned}\frac{d\phi}{dt} &= A(t)\phi(t) + B(t)u(t) + d(t) \\ \frac{d\lambda}{dt} &= X(t)\phi(t) - A^t(t)\lambda(t)\end{aligned} \tag{6.9.6}$$

and that (6.3.29) holds. From (6.9.4) we see that in the present context, (6.3.29) becomes

$$-\frac{1}{2}\langle u(t), R(t)u(t)\rangle + \langle\lambda(t), B(t)u(t)\rangle \geq -\frac{1}{2}\langle z, R(t)z\rangle + \langle\lambda(t), B(t)z\rangle$$

for all z in $\mathcal{O}$ and almost all t in $[t_0, t_1]$. Thus, for almost every t in $[t_0, t_1]$ the mapping

$$z \to -\frac{1}{2}\langle z, R(t)z\rangle + \langle\lambda(t), B(t)z\rangle \tag{6.9.7}$$

attains its maximum over $\mathcal{O}$ at $z = u(t)$. But $\mathcal{O}$ is open, so the derivative of the mapping (6.9.7) is zero at $z = u(t)$. Hence

$$-R(t)u(t) + B^t(t)\lambda(t) = 0.$$

Since $R(t)$ is non-singular for all t, we get that

$$u(t) = R^{-1}(t)B^t(t)\lambda(t) \qquad \text{a.e.} \tag{6.9.8}$$

Note that since B, R, and λ are continuous, *the optimal control is also continuous.*

If we now substitute (6.9.8) into the first equation in (6.9.6), we get the following theorem from the maximum principle.

Theorem 6.9.3. *Let* (ϕ, u) *be an optimal pair with interval of definition* $[t_0, t_1]$. *Let Assumption* 6.9.2 *hold. Then there exists an absolutely continuous function* $\lambda = (\lambda^1, \ldots, \lambda^n)$ *defined on* $[t_0, t_1]$ *such that* (ϕ, λ) *is a solution of the linear system*

$$\begin{aligned}\frac{dx}{dt} &= A(t)x + B(t)R^{-1}(t)B^t(t)q + d(t) \\ \frac{dq}{dt} &= X(t)x - A^t(t)q\end{aligned} \tag{6.9.9}$$

and such that vector (6.9.5) *is orthogonal to* $\mathcal{T}_1$ *at* $(t_1, \phi(t_1))$. *The optimal control is given by* (6.9.8).

We now specialize the problem by taking $\mathcal{T}_1$ to the hyperplane $t_1 = T$; that is,

$$\mathcal{T}_1 = \{(t_1, x_1)\colon t_1 = T,\ x_1 \text{ free}\}, \tag{6.9.10}$$

and by taking g to be given by

$$g(x_1) = \frac{1}{2}\,\langle x_1, Gx_1\rangle, \tag{6.9.11}$$

where G is a positive semi-definite symmetric matrix. We suppose that $T < b$.

Remark 6.9.4. If (6.9.10) holds, then every tangent vector $(dt_0, dx_0, \ldots, dx_n)$ to $\mathcal{T}_1$ has its first component equal to zero. On the other hand, a tangent vector to the trajectory ϕ has its first component always different from zero. Moreover, it follows from (6.9.1) and the continuity of an optimal control u that the trajectory has a tangent vector at all points. Hence if (6.9.10) holds, then Assumption 6.9.2 is automatically satisfied.

Corollary 6.9.5. *If (6.9.10) and (6.9.11) hold, then ϕ and λ satisfy the system (6.9.16) subject to the boundary conditions*

$$\phi(t_0) = x_0 \qquad \lambda(T) = -G\phi(T). \tag{6.9.12}$$

The first condition is a restatement of the initial condition already imposed. The second follows from the orthogonality of (6.9.5) to $\mathcal{T}_1$ at the terminal point of the trajectory and from (6.9.11).

An admissible pair (ϕ, u) that satisfies the conditions of Theorem 6.9.3 will be called an *extremal pair*. If (6.9.10) and (6.9.11) hold, then an extremal pair satisfies (6.9.12).

In the next theorem we show that if (6.9.10) and (6.9.11) hold, then an extremal pair is unique and must be optimal. This will be done without reference to any existence theorems previously established.

Theorem 6.9.6. *Let (6.9.10) and (6.9.11) hold. Let (ϕ, u) be an extremal pair and let (ϕ_1, u_1) be any other admissible pair. Then $J(\phi_1, u_1) \geq J(\phi, u)$, with equality holding if and only if $u = u_1$. In that event, $\phi = \phi_1$.*

Proof. First note that because the system (6.9.1) is linear and (t_0, x_0) is fixed, if $u = u_1$ then $\phi = \phi_1$. Let

$$\phi_f = \phi(T) \qquad \phi_{1f} = \phi_1(T).$$

Since $X(t)$ is positive semi-definite and $R(t)$ is positive definite for all t and since G is positive semi-definite, we get

$$\begin{aligned} 0 \leq{} & \langle(\phi_{1f} - \phi_f), G(\phi_{1f} - \phi_f)\rangle \\ & + \int_{t_0}^{T} \{\langle(\phi_1 - \phi), X(\phi_1 - \phi)\rangle + \langle(u_1 - u), R(u_1 - u)\rangle\}dt, \end{aligned}$$

with equality holding if and only if $u_1 = u$. Hence

$$0 \leq 2J(\phi_1, u_1) + 2J(\phi, u) - 2\langle\phi_{1f}, G\phi_f\rangle - 2\int_{t_0}^{T} \{\langle\phi_1, X\phi\rangle + \langle u_1, Ru\rangle\}dt,$$

which we rewrite as

$$J(\phi_1, u_1) + J(\phi, u) \geq \langle \phi_{1f}, G\phi_f \rangle + \int_{t_0}^{T} \{\langle \phi_1, X\phi \rangle + \langle u_1, Ru \rangle\} dt. \quad (6.9.13)$$

Since (ϕ, u) is an extremal pair, there is an absolutely continuous vector λ such that λ and ϕ are solutions of (6.9.9) that satisfy (6.9.12) and such that (6.9.8) holds. We now substitute for $X\phi$ in the right-hand side of (6.9.13) from the second equation in (6.9.9) and substitute for u in the right-hand side of (6.9.13) from (6.9.8). We get

$$J(\phi_1, u_1) + J(\phi, u) \geq \langle \phi_{1f}, G\phi_f \rangle + \int_{t_0}^{T} \{\langle \phi_1, \lambda' + A^t\lambda \rangle + \langle u_1, B^t\lambda \rangle\} dt. \quad (6.9.14)$$

The integral on the right in (6.9.14) can be written as

$$\int_{t_0}^{T} \{\langle \phi_1, \lambda' \rangle + \langle A\phi_1 + Bu_1, \lambda \rangle\} dt.$$

Since (ϕ_1, u_1) is admissible we have from (6.9.1) that

$$A\phi_1 + Bu_1 = \phi_1' - d.$$

Substituting this into the last integral gives

$$\int_{t_0}^{T} \{\langle \phi_1, \lambda' \rangle + \langle \phi_1', \lambda \rangle - \langle d, \lambda \rangle\} dt.$$

Therefore, we can rewrite (6.9.14) as follows:

$$J(\phi_1, u_1) + J(\phi, u) \geq \langle \phi_{1f}, G\phi_f \rangle + \langle \phi_{1f}, \lambda(T) \rangle - \langle x_0, \lambda(t_0) \rangle - \int_{t_0}^{T} \langle d, \lambda \rangle dt.$$

If we now use (6.9.12) we get

$$J(\phi_1, u_1) + J(\phi, u) \geq -\langle x_0, \lambda(t_0) \rangle - \int_{t_0}^{T} \langle d, \lambda \rangle dt.$$

Recall that equality holds if and only if $u_1 = u$, in which case $\phi_1 = \phi$. Therefore if we take $u_1 = u$ in the preceding inequality we get

$$2J(\phi, u) = -\langle x_0, \lambda(t_0) \rangle - \int_{t_0}^{T} \langle d, \lambda \rangle dt. \quad (6.9.15)$$

Substituting (6.9.15) into the preceding inequality gives

$$J(\phi_1, u_1) \geq J(\phi, u),$$

with equality holding if and only if $u_1 = u$. □

Exercise 6.9.7. Consider the linear quadratic problem with t_0 and t_1 fixed, $\mathcal{T}_1$ a linear variety of dimension n, and $g\colon x_1 \to g(x_1)$ a convex function. Suppose also that the constraint set $\mathcal{O}$ is convex. Show, without appealing to the maximum principle of Theorem 6.9.3, that if (ϕ, u) is an optimal pair then (ϕ, u) is unique.

The linear plant quadratic criterion problem posed in this section, with $\mathcal{T}_1$ as in (6.9.10) and g as in (6.9.11), admits a very elegant and relatively simple synthesis of the optimal control. The determination of this synthesis will take up the remainder of this section.

For the problem with fixed initial point (τ, ξ), with $a \leq \tau < T$ and $\xi \in \mathbb{R}^n$, it follows from Exercise 5.4.20 that there exists an ordinary optimal pair $(\phi(\cdot, \tau, \xi), u(\cdot, \tau, \xi))$ that is a solution of the ordinary problem. By Theorem 6.9.3 this pair is extremal and for $\tau \leq t \leq T$

$$u(t, \tau, \xi) = R^{-1}(t)B^t(t)\lambda(t, \tau, \xi).$$

It then follows from Theorem 6.9.6 that the optimal pair for the problem with initial point (τ, ξ) is unique. Therefore, as in Section 6.2, we obtain a field $\mathcal{F}$ of optimal trajectories. We obtain a synthesis of the optimal control, or feedback control, U as follows:

$$U(\tau, \xi) = u(\tau, \tau, \xi) = R^{-1}(\tau)B^t(\tau)\lambda(\tau, \tau, \xi). \tag{6.9.16}$$

This holds for all $a \leq \tau < T$ and for all ξ, since we may choose any such (τ, ξ) to be the initial point for the problem.

The feedback law in (6.9.16) is not satisfactory since it requires knowing the value of the adjoint variable, or multiplier, λ at the initial point. If the formalism of Section 6.2 is valid, then we have $W_x(\tau, \xi) = -\lambda(\tau, \tau, \xi)$ and we can write

$$U(\tau, \xi) = -R^{-1}(\tau)B^t(\tau)W_x(\tau, \xi). \tag{6.9.17}$$

This leads us to investigate the value function W for the present problem. We shall proceed formally, as in Section 6.2, assuming that all functions have the required number of derivatives existing and continuous. In this way we shall obtain insights and conjectures as to the structure of the feedback control. We shall then show rigorously, by other methods, that these conjectures are valid.

We henceforth suppose that $d = 0$ in (6.9.1).

The function W satisfies the Hamilton-Jacobi equation (6.2.11), which in the present case becomes

$$W_t = -\frac{1}{2}\langle x, Xx\rangle - \frac{1}{2}\langle U, RU\rangle - \langle W_x, Ax\rangle - \langle W_x, BU\rangle.$$

In this relation and in what follows we shall omit the arguments of the functions involved. Using (6.9.17) we can rewrite this equation as follows:

$$W_t = -\frac{1}{2}\langle x, Xx\rangle - \frac{1}{2}\langle R^{-1}B^tW_x, B^tW_x\rangle - \langle W_x, Ax\rangle + \langle W_x, BR^{-1}B^tW_x\rangle.$$

Hence

$$W_t = -\frac{1}{2}\langle x, Xx\rangle + \frac{1}{2}\langle W_x, BR^{-1}B^tW_x\rangle - \langle W_x, Ax\rangle. \tag{6.9.18}$$

The form of Eq. (6.9.18) leads to the conjecture that there exists a solution of the Hamilton-Jacobi equation (6.2.11) of the form

$$W(t,x) = \frac{1}{2}\langle x, P(t)x\rangle, \tag{6.9.19}$$

where for each t, $P(t)$ is a symmetric matrix. For then

$$W_x = Px \qquad W_t = \frac{1}{2}\langle x, P'(t)x\rangle, \tag{6.9.20}$$

and for proper choice of $P(t)$ we would have a quadratic form in the left equal to a quadratic form on the right.

If we assume a solution of the form (6.9.19), substitute (6.9.20) into (6.9.18), and recall that $P^t = P$, we get

$$\frac{1}{2}\langle x, P'x\rangle = -\frac{1}{2}\langle x, Xx\rangle + \frac{1}{2}\langle x, PBR^{-1}B^tPx\rangle - \langle x, PAx\rangle. \tag{6.9.21}$$

For any matrix M, we can write

$$M = \frac{(M+M^t)}{2} + \frac{(M-M^t)}{2}.$$

Hence

$$\langle x, Mx\rangle = \frac{1}{2}\langle x, (M+M^t)x\rangle \qquad \text{for all } x.$$

If we apply this observation to the matrix PA in (6.9.21) we get

$$\frac{1}{2}\langle x, P'x\rangle = -\frac{1}{2}\langle x, Xx\rangle + \frac{1}{2}\langle x, PBR^{-1}B^tPx\rangle - \frac{1}{2}\langle x, (PA + A^tP)x\rangle.$$

Therefore, if a solution to the Hamilton-Jacobi equation of the form (6.9.19) exists, the matrix P must satisfy the following differential equation:

$$P' = -X + PBR^{-1}B^tP - (PA + A^tP). \tag{6.9.22}$$

Moreover, since

$$W(T, x_1) = g(x_1) = \frac{1}{2}\langle x_1, Gx_1\rangle,$$

it follows from (6.9.19) that the solution of (6.9.22) must satisfy the initial condition

$$P(T) = G. \tag{6.9.23}$$

Equation (6.9.22) is sometimes called the matrix Riccati equation. If a

solution of (6.9.22) satisfying (6.9.23) exists, then from the first relation in (6.9.20) and from (6.9.17) we would expect the optimal synthesis or feedback control law to be given by

$$U(t,x) = -R^{-1}(t)B^t(t)P(t)x. \tag{6.9.24}$$

Note that the control law is linear in x, and its determination merely requires the solution of an ordinary differential equation.

We now show that the state of affairs suggested by the analysis in the last few paragraphs is indeed true.

Theorem 6.9.8. *Let $\mathcal{T}_1$ be as in (6.9.10), let g be as in (6.9.11), let $d = 0$ in (6.9.1), and let the constraint set $\mathcal{O}$ contain the origin. Then the problem of minimizing (6.9.3) subject to (6.9.1), control constraint Ω, and terminal set $\mathcal{B}$, where the data of the problem satisfy Assumption 6.9.1, has an optimal synthesis. This synthesis is given by (6.9.24) and holds for all $a \leq t < T$ and all x in R^n. The matrix $P(t)$ is symmetric for each t and the function P is a solution, defined for all $a \leq t \leq T$ of the matrix Riccati equation (6.9.22) with initial condition (6.9.23).*

Proof. It follows from standard existence and uniqueness theorems for ordinary differential equations that (6.9.22) has a unique solution satisfying (6.9.23) on some interval $(T-\delta, T]$. Note that if P is a solution of (6.9.22) satisfying (6.9.23), then so is P^t. By the uniqueness of solutions we then get that $P = P^t$, so that P is symmetric.

Let τ be any point on $(T-\delta, T)$ and let ξ be any point in $\mathbb{R}^n$. We shall use the solution P obtained in the previous paragraph to construct an extremal for the problem with initial point (τ, ξ). By Theorem 6.9.6 this extremal will be unique and will furnish the minimum for the problem with initial point (τ, ξ).

Consider the linear system

$$\frac{dx}{dt} = A(t)x - B(t)R^{-1}(t)B^t(t)P(t)x \tag{6.9.25}$$

subject to initial conditions $x(\tau) = \xi$. We denote the solution of this system by $\phi(\cdot, \tau, \xi)$. This solution is defined on the interval of definition of P and is unique.

Let

$$\lambda(t, \tau, \xi) = -P(t)\phi(t, \tau, \xi), \qquad \tau \leq t \leq T. \tag{6.9.26}$$

Note that

$$\lambda(T, \tau, \xi) = -P(T)\phi(T, \tau, \xi) = -G\phi(T, \tau, \xi),$$

where the last equality follows from (6.9.23). Thus, λ satisfies (6.9.12).

If we differentiate (6.9.26) and then use (6.9.22) and (6.9.25), we get

$$\frac{d\lambda}{dt} = -\frac{dP}{dt}\phi - P\frac{d\phi}{dt}$$

$$= (X - PBR^{-1}B^tP + PA + A^tP)\phi - P(A - BR^{-1}B^tP)\phi$$
$$= X\phi + A^tP\phi.$$

If we now use (6.9.26) we get

$$\frac{d\lambda}{dt} = X\phi - A^t\lambda.$$

Hence by Theorem 6.9.3 and Corollary 6.9.5, $\phi(\cdot, \tau, \xi)$, and $\lambda(\cdot, \tau, \xi)$ determine an extremal pair $(\phi(\cdot, \tau, \xi), u(\cdot, \tau, \xi))$ with

$$u(t, \tau, \xi) = R^{-1}(t)B^t(t)\lambda(t, \tau, \xi).$$

It now follows from Theorem 6.9.6 that this extremal pair is the unique optimal pair for the problem.

From the definition of λ in (6.9.26), and the last equation it follows that

$$u(t, \tau, \xi) = -R^{-1}(t)B^t(t)P(t)\phi(t, \tau, \xi).$$

Therefore, since $\phi(\tau, \tau, \xi) = \xi$,

$$u(\tau, \tau, \xi) = -R^{-1}(\tau)B^t(\tau)P(\tau)\xi.$$

Since (τ, ξ) is an arbitrary point in $(T - \delta, T) \times \mathbb{R}^n$ and since the optimal pair from (τ, ξ) is unique we obtain a synthesis of the optimal control by setting

$$U(\tau, \xi) = u(\tau, \tau, \xi).$$

Hence the optimal synthesis in $(T - \delta, T) \times \mathbb{R}^n$ can be written as

$$U(t, x) = -R^{-1}(t)B^t(t)P(t)x,$$

where we have written a generic point as (t, x) instead of (τ, ξ). This, however, is precisely the relation (6.9.24).

We now show that the solution P of (6.9.22) with initial condition (6.9.23) is defined on the entire interval $[a, T]$. It will then follow that (6.9.24) holds for all $a \leq t \leq T$ and all x in $\mathbb{R}^n$.

Let us now suppose that $\delta > 0$ is such that $(T - \delta, T]$ is the maximal interval with T as the right-hand end point on which the solution P is defined. From the standard existence theorem in the theory of ordinary differential equations and from the form of Eq. (6.9.22) it follows that $P(t)$ must be unbounded as $t \to T - \delta$ from the right. We shall show that if $T - \delta \geq a$, then P is bounded as $t \to T - \delta$ from the right. From this it will, of course, follow that P is defined on $[a, T]$ and (6.9.25) holds for all $a \leq t \leq T$ and x in $\mathbb{R}^n$.

From the existence theorem for linear quadratic problems (Exercise 5.4.20) and from Theorem 6.9.6 it follows that for all (τ, ξ) in $[a, T] \times \mathbb{R}^n$ the function

$$W(\tau, \xi) = J(\phi(\cdot, \tau, \xi), u(\cdot, \tau, \xi))$$

is defined, where $(\phi(\cdot,\tau,\xi), u(\cdot,\tau,\xi))$ is the unique optimal pair for the problem with initial point (τ,ξ). The function W so defined is called the *value function* or *value.* Let $\widetilde{\phi}(\cdot,\tau,\xi)$ denote the trajectory for the problem corresponding to the control $\widetilde{u}$, where $\widetilde{u}(t) = 0$ on $[\tau, T]$. Then

$$0 \leq W(\tau,\xi) \leq J(\widetilde{\phi}(\cdot,\tau,\xi), \widetilde{u}), \tag{6.9.27}$$

where the leftmost inequality follows from (ii) of Assumption 6.9.1 and from (6.9.11). From (6.9.1) with $d = 0$ we see that

$$\widetilde{\phi}(t,\tau,\xi) = \Phi(t,\tau)\xi,$$

where $\Phi(\cdot,\tau)$ is the fundamental matrix for the system $dx/dt = A(t)x$ satisfying $\Phi(\tau,\tau) = I$. Therefore, from (6.9.3) and (6.9.11), we get

$$J(\widetilde{\phi}(\cdot,\tau,\xi), \widetilde{u}) = \langle \Phi(T,\tau)\xi, G\Phi(T,\tau)\xi\rangle + \int_\tau^T \langle \Phi(s,\tau)\xi, X(s)\Phi(s,\tau)\xi\rangle ds.$$

From this it follows that given a compact set $\mathcal{X}$ in $\mathbb{R}^n$, there exists a constant M, depending on $\mathcal{X}$ such that for all $a \leq \tau \leq T$ and all ξ in $\mathcal{X}$

$$J(\widetilde{\phi}(\cdot,\tau,\xi), \widetilde{u}) \leq M.$$

Combining this inequality with (6.9.27) shows that given a compact set $\mathcal{X}$ in $\mathbb{R}^n$, there exists a constant M, depending on $\mathcal{X}$ such that for all $a \leq \tau \leq T$ and all ξ in $\mathcal{X}$

$$0 \leq W(\tau,\xi) \leq M. \tag{6.9.28}$$

In (6.9.15), which was derived in the course of proving Theorem 6.9.6, we have an expression for $J(\phi(\tau,\xi), u(\tau,\xi))$, and hence for $W(\tau,\xi)$. Since $d = 0$ in the present discussion, we have from (6.9.15)

$$W(\tau,\xi) = -\frac{1}{2}\langle \xi, \lambda(\tau,\tau,\xi)\rangle. \tag{6.9.29}$$

Here λ is the adjoint function, or multiplier function, for the problem with initial point (τ,ξ). It is not assumed here that λ is given by (6.9.26).

We now consider points (τ,ξ) such that $T - \delta < \tau \leq T$. For such points (6.9.26) holds. Thus,

$$\lambda(\tau,\tau,\xi) = -P(\tau)\phi(\tau,\tau,\xi) = -P(\tau)\xi.$$

Substituting this into (6.9.29) gives

$$W(\tau,\xi) = \frac{1}{2}\langle \xi, P(\tau)\xi\rangle, \tag{6.9.30}$$

which is valid for $T - \delta < \tau \leq T$ and all ξ. From this and from (6.9.28) we get that for all $T - \delta \leq \tau \leq T$ and all ξ in a compact set $\mathcal{X}$,

$$0 \leq \frac{1}{2}\langle \xi, P(\tau)\xi\rangle \leq M.$$

Hence $P(\tau)$ must be bounded on $T - \delta \leq \tau \leq T$, and the theorem is proved. $\square$

In the course of proving Theorem 6.9.8 we also proved the following.

Corollary 6.9.9. *The value function W is given by* (6.9.30) *for all $a \le \tau \le T$ and ξ in $\mathbb{R}^n$.*

Chapter 7

Proof of the Maximum Principle

7.1 Introduction

In this chapter we prove Theorems 6.3.5 through 6.3.22 and their corollaries. Theorem 6.3.5 will be proved by a penalty function method, which we outline here. For simplicity, let f be a real valued differentiable function defined on an open set $\mathcal{X}$ in $\mathbb{R}^n$. Consider the unconstrained problem of minimizing f on $\mathcal{X}$. If f attains a minimum at a point x_0 in $\mathcal{X}$, then the necessary condition $df(x_0) = 0$ holds, where df is the differential of f. This condition is obtained by making a perturbation $x_0 + \varepsilon\delta x$, where δx is arbitrary but fixed, and ε is sufficiently small so that $x_0 + \varepsilon\delta x$ is in $\mathcal{X}$. Then since f is differentiable and attains a minimum at x_0,

$$f(x_0 + \varepsilon\delta x) - f(x_0) = df(x_0)\varepsilon\delta x + \theta(\varepsilon\delta x) \geq 0,$$

where $\theta(\varepsilon\delta x)/(\varepsilon\delta x) \to 0$ as $\varepsilon \to 0$. In the rightmost inequality if we first divide through by $\varepsilon > 0$ and then let $\varepsilon \to 0$, we get $df(x_0)\delta x \geq 0$. If we divide through by $\varepsilon < 0$ and then let $\varepsilon \to 0$, we get $df(x_0)\delta x \leq 0$. Since δx is arbitrary, we get $df(x_0) = 0$.

Now consider the constrained problem of minimizing f over those points in $\mathcal{X}$ that satisfy the constraint $g(x) = 0$, where g is a differentiable function defined on $\mathcal{X}$. Again let f attain its minimum at x_0 for the constrained problem. The perturbation δx is now not arbitrary but must be such that for sufficiently small ε not only must $x_0 + \varepsilon\delta x$ be in $\mathcal{X}$ but it must satisfy $g(x_0 + \varepsilon\delta x) = 0$. Thus, the argument used in the unconstrained problem, which must hold for arbitrary δx, fails and other arguments must be used.

In the penalty function method one considers a sequence of unconstrained problems: Minimize

$$F(x, K_n) = f(x) + K_n(g(x))^2 \qquad x \in \mathcal{X}, \tag{7.1.1}$$

where $K_n \to \infty$ as $n \to \infty$. Conditions are placed on f and g such that the following occur. For each n, the problem of minimizing $F(x, K_n)$ has a solution x_n, at which point the unconstrained necessary condition $dF(x_n, K_n) = 0$ holds. As $K_n \to \infty$, the ever increasing penalty forces the x_n to converge to a point x_0 that satisfies the constraint $g(x_0) = 0$ and satisfies $f(x_0) \leq f(x)$ for

all x such that $g(x) = 0$. As $n \to \infty$ the necessary condition $dF(x_n, K_n) = 0$ converges to the necessary condition for the constrained problem. We shall carry out this program for the relaxed optimal control problem.

Our proof of Theorem 6.3.5 is suggested by E. J. McShane's penalty method proof [59] of the necessary conditions for the following finite dimensional problem.

Problem 7.1.1. Let X_0 be an open convex set in $\mathbb{R}^n$. Let $f, \mathbf{g}$, and $\mathbf{h}$ be C^1 functions with domain X_0 and ranges in $\mathbb{R}^1$, $\mathbb{R}^m$, and $\mathbb{R}^k$, respectively. Let

$$X = \{\mathbf{x} : \mathbf{x} \in X_0,\ \mathbf{g}(\mathbf{x}) \leq \mathbf{0},\ \mathbf{h}(\mathbf{x}) = \mathbf{0}\}.$$

Minimize f over X.

The problem is often stated as:

$$\begin{aligned}&\text{Minimize } f(\mathbf{x})\\ &\text{Subject to: } \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \qquad \mathbf{h}(\mathbf{x}) = \mathbf{0}.\end{aligned}$$

If the constraints $\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$ and $\mathbf{h}(\mathbf{x}) = \mathbf{0}$ are absent, the problem becomes an unconstrained problem. From elementary calculus we have that a necessary condition for a point $\mathbf{x}_*$ to be a solution is that all first-order partial derivatives of f vanish at $\mathbf{x}_*$.

The following notation will be used. If f is a real valued function and $\mathbf{g} = (g_1, \ldots, g_m)$ is a vector valued function that is differentiable at $\mathbf{x}_0$, then

$$\nabla f(\mathbf{x}_0) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}_0),\ \frac{\partial f}{\partial x_2}(\mathbf{x}_0), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x}_0)\right)$$

and

$$\nabla \mathbf{g}(\mathbf{x}_0) = \begin{pmatrix} \nabla g_1(\mathbf{x}_0) \\ \nabla g_2(\mathbf{x}_0) \\ \vdots \\ \nabla g_m(\mathbf{x}_0) \end{pmatrix} = \left(\frac{\partial g_i(\mathbf{x}_0)}{\partial x_j}\right).$$

By $B(\mathbf{x}, \varepsilon)$ we shall mean the open ball of radius ε centered at $\mathbf{x}$. A point $\mathbf{x}$ such that $\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$ and $\mathbf{h}(\mathbf{x}) = \mathbf{0}$ will be called a *feasible point*. To keep the essentials of our proof of Theorem 6.3.5 from being obscured by the technical requirements imposed by the infinite dimensionality of our problem, we present in the next section, McShane's proof of the necessary conditions for a solution to Problem 7.1.1.

7.2 Penalty Proof of Necessary Conditions in Finite Dimensions

Theorem 7.2.1. *Let $\mathbf{x}_*$ be a solution of Problem 7.1.1. Then there exists a real number $\lambda_0 \geq 0$, a vector $\boldsymbol{\lambda} \geq \mathbf{0}$ in $\mathbb{R}^m$, and a vector $\boldsymbol{\mu}$ in $\mathbb{R}^k$ such that*

(i) $(\lambda_0, \boldsymbol{\lambda}, \boldsymbol{\mu}) \neq \mathbf{0}$

(ii) $\langle \boldsymbol{\lambda}, \mathbf{g}(\mathbf{x}_*) \rangle = 0$ *and*

(iii) $\lambda_0 \nabla f(\mathbf{x}_*) + \boldsymbol{\lambda}^t \nabla \mathbf{g}(\mathbf{x}_*) + \boldsymbol{\mu}^t \nabla \mathbf{h}(\mathbf{x}_*) = \mathbf{0}$.

Vectors $(\lambda_0, \boldsymbol{\lambda}, \boldsymbol{\mu})$ having the properties stated in the theorem are called multipliers, as are the components of these vectors.

Remark 7.2.2. The necessary condition asserts that $\lambda_0 \geq 0$ and not the stronger statement $\lambda_0 > 0$. If $\lambda_0 > 0$, then we may divide through by λ_0 in (ii) and (iii) and then relabel $\boldsymbol{\lambda}/\lambda_0$ and $\boldsymbol{\mu}/\lambda_0$ as $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, respectively, and thus obtain statements (i) through (iii) with $\lambda_0 = 1$. In the absence of further conditions that would guarantee that $\lambda_0 > 0$, we cannot assume that $\lambda_0 = 1$. Theorems with conditions guaranteeing $\lambda_0 > 0$ are called Karush-Kuhn-Tucker or KKT theorems; those with $\lambda_0 \geq 0$ are called Fritz-John or F-J theorems.

Remark 7.2.3. If $\mathbf{g} \equiv \mathbf{0}$, that is, the inequality constraints are absent, Theorem 7.2.1 becomes the Lagrange multiplier rule.

Remark 7.2.4. Since $\boldsymbol{\lambda} \geq \mathbf{0}$ and $\mathbf{g}(\mathbf{x}_*) \leq \mathbf{0}$, condition (ii) is equivalent to

(i)$'$ $\lambda_i g_i(\mathbf{x}_*) = 0 \qquad i = 1, \ldots, m.$

Condition (iii) is a system of n equations

(iii)$'$ $\lambda_0 \frac{\partial f}{\partial x_j}(\mathbf{x}_*) + \sum_{i=1}^m \lambda_i \frac{\partial g_i(\mathbf{x}_*)}{\partial x_j} + \sum_{i=1}^k \mu_i \frac{\partial h_i(\mathbf{x}_*)}{\partial x_j} = 0$, $j = 1, \ldots, n.$

Remark 7.2.5. The necessary conditions are also necessary conditions for a local minimum. For if $\mathbf{x}_*$ is a local minimum, then there exists a $\delta > 0$ such that $f(\mathbf{x}_*) \leq f(\mathbf{x})$ for all $\mathbf{x}$ that are in $B(\mathbf{x}_*, \delta) \cap X_0$ and that satisfy the constraints $\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$ and $\mathbf{h}(\mathbf{x}) = \mathbf{0}$. Thus, $\mathbf{x}_*$ is a global solution to the problem in which X_0 is replaced by $X_0 \cap B(\mathbf{x}_*, \delta)$.

Proof of Theorem 7.2.1. Let E denote the set of indices such that $g_i(\mathbf{x}_*) = 0$ and let I denote the set of indices such the $g_i(\mathbf{x}_*) < 0$. By $\mathbf{g}_I$ we mean the vector consisting of those components of $\mathbf{g}$ whose indices are in I. The vector $\mathbf{g}_E$ has similar meaning. To simplify the notation we assume that $E = \{1, 2, \ldots, r\}$ and that $I = \{r+1, \ldots, m\}$. Since E or I can be empty, we have $0 \leq r \leq m$.

By a translation of coordinates, we may assume that $\mathbf{x}_* = \mathbf{0}$ and that $f(\mathbf{x}_*) = 0$.

Let ω be a function from $(-\infty, \infty)$ to $\mathbb{R}^1$ such that: (i) ω is strictly increasing on $(0, \infty)$, (ii) $\omega(u) = 0$ for $u \le 0$, (iii) ω is C^1, and (iv) $\omega'(u) > 0$ for $u > 0$. We want $\omega > 0$ for $u > 0$ as well as convex.

Since $\mathbf{g}$ is continuous and X_0 is open, there exists an $\varepsilon_0 > 0$ such that $B(\mathbf{0}, \varepsilon_0) \subset X_0$ and $\mathbf{g}_I(\mathbf{x}) < \mathbf{0}$ for $\mathbf{x} \in B(\mathbf{0}, \varepsilon_0)$.

Define a *penalty function* F as follows:

$$F(\mathbf{x}, p) = f(\mathbf{x}) + \|\mathbf{x}\|^2 + p\left\{\sum_{i=1}^{r} \omega(g_i(\mathbf{x})) + \sum_{i=1}^{k} [h_i(\mathbf{x})]^2\right\}, \tag{7.2.1}$$

where $\mathbf{x} \in X_0$ and p is a positive integer. We assert that for each ε satisfying $0 < \varepsilon < \varepsilon_0$, there exists a positive integer $p(\varepsilon)$ such that for $\mathbf{x}$ with $\|\mathbf{x}\| = \varepsilon$,

$$F(\mathbf{x}, p(\varepsilon)) > 0. \tag{7.2.2}$$

We prove the assertion by assuming it to be false and arriving at a contradiction. If the assertion were false, then there would exist an ε', with $0 < \varepsilon' < \varepsilon_0$, such that for each positive integer p there exists a point $\mathbf{x}_p$ with $\|\mathbf{x}_p\| = \varepsilon'$ and $F(\mathbf{x}_p, p) \le 0$. Hence, from (7.2.1)

$$f(\mathbf{x}_p) + \|\mathbf{x}_p\|^2 \le -p\left\{\sum_{i=1}^{r} \omega(g_i(\mathbf{x}_p)) + \sum_{i=1}^{k} [h_i(\mathbf{x}_p)]^2\right\}. \tag{7.2.3}$$

Since $\|\mathbf{x}_p\| = \varepsilon'$ and since $S(\mathbf{0}, \varepsilon') = \{\mathbf{y} : \|\mathbf{y}\| = \varepsilon'\}$ is compact, there exist subsequences, which we relabel as $\mathbf{x}_p$ and a point $\mathbf{x}_0$ with $\|\mathbf{x}_0\| = \varepsilon'$ such that $\mathbf{x}_p \to \mathbf{x}_0$. Since $f, \mathbf{g}$, and $\mathbf{h}$ are continuous,

$$f(\mathbf{x}_p) \to f(\mathbf{x}_0) \qquad \mathbf{g}_E(\mathbf{x}_p) \to \mathbf{g}_E(\mathbf{x}_0) \qquad \mathbf{h}(\mathbf{x}_p) \to \mathbf{h}(\mathbf{x}_0).$$

Therefore, if in (7.2.3) we divide through by $-p$ and then let $p \to \infty$, we get

$$0 \ge \sum_{i=1}^{r} \omega(g_i(\mathbf{x}_0)) + \sum_{i=1}^{k} [h_i(\mathbf{x}_0)]^2 \ge 0.$$

Hence for each $i = 1, \ldots, r$ we have $g_i(\mathbf{x}_0) \le 0$, and for each $i = 1, \ldots, k$ we have $h_i(\mathbf{x}_0) = 0$. Since $\|\mathbf{x}_0\| = \varepsilon' < \varepsilon_0$, we have $g_I(\mathbf{x}_0) < \mathbf{0}$. Thus, $\mathbf{x}_0$ is a feasible point and so

$$f(\mathbf{x}_0) \ge f(\mathbf{0}) = 0. \tag{7.2.4}$$

On the other hand, from (7.2.3) and from $\|\mathbf{x}_p\| = \varepsilon'$ we get that $f(\mathbf{x}_p) \le -(\varepsilon')^2$, and so

$$f(\mathbf{x}_0) \le -(\varepsilon')^2 < 0,$$

which contradicts (7.2.4). This proves the assertion.

For each ε in $(0, \varepsilon_0)$ the function $F(\ , p(\varepsilon))$ is continuous on the closed ball

$\overline{B(\mathbf{0},\varepsilon)}$. Since $\overline{B(\mathbf{0},\varepsilon)}$ is compact, $F(\ ,p(\varepsilon))$ attains its minimum on $\overline{B(\mathbf{0},\varepsilon)}$ at some point $\mathbf{x}_\varepsilon$ with $\|\mathbf{x}_\varepsilon\| \le \varepsilon$.

Since $F(\mathbf{x}, p(\varepsilon)) > 0$ for $\mathbf{x}$ with $\|\mathbf{x}\| = \varepsilon$, and since $F(\mathbf{0}, p(\varepsilon)) = f(\mathbf{0}) = 0$, it follows that $F(\ ,p(\varepsilon))$ attains its minimum on $\overline{B(\mathbf{0},\varepsilon)}$ at an *interior* point $\mathbf{x}_\varepsilon$ of $\overline{B(\mathbf{0},\varepsilon)}$. Hence,

$$\frac{\partial F}{\partial x_j}\,(\mathbf{x}_\varepsilon, p(\varepsilon)) = 0 \qquad j = 1,\ldots,n.$$

Calculating $\partial F/\partial x_j$ from (7.2.1) gives:

$$\begin{aligned}\frac{\partial f}{\partial x_j}\,(\mathbf{x}_\varepsilon) + 2(x_\varepsilon)_j + \sum_{i=1}^{r}\ p(\varepsilon)\omega'(g_i(\mathbf{x}_\varepsilon))\frac{\partial g_i}{\partial x_j}(\mathbf{x}_\varepsilon)&\\ + \sum_{i=1}^{k}\ 2p(\varepsilon)h_i(\mathbf{x}_\varepsilon)\frac{\partial h_i}{\partial x_j}(\mathbf{x}_\varepsilon) &= 0\end{aligned} \tag{7.2.5}$$

for $j = 1,\ldots,n$.

Define:

$$\begin{aligned}
&L(\varepsilon) = 1 + \sum_{i=1}^{r}\ [p(\varepsilon)\omega'\ (g_i(\mathbf{x}_\varepsilon))]^2 + \sum_{i=1}^{k}\ [2p(\varepsilon)h_i(\mathbf{x}_\varepsilon)]^2\\
&\lambda_0(\varepsilon) = 1/\sqrt{L(\varepsilon)}\\
&\lambda_i(\varepsilon) = p(\varepsilon)\omega'(g_i(\mathbf{x}_\varepsilon))/\sqrt{L(\varepsilon)} \qquad i = 1,\ldots,r\\
&\lambda_i(\varepsilon) = 0 \qquad i = r+1,\ldots,m\\
&\mu_i(\varepsilon) = 2p(\varepsilon)h_i(\mathbf{x}_\varepsilon)/\sqrt{L(\varepsilon)} \qquad i = 1,\ldots,k.
\end{aligned}$$

Note that

$$\begin{aligned}&\text{(i)}\ \ \lambda_0(\varepsilon) > 0 \qquad \text{(ii)}\ \ \lambda_i(\varepsilon) \ge 0\ \ i = 1,\ldots,r\\ &\text{(iii)}\ \ \lambda_i(\varepsilon) = 0;\ i = r+1,\ldots,m \qquad \text{(iv)}\ \ \|(\lambda_0(\varepsilon),\ \boldsymbol{\lambda}(\varepsilon),\ \boldsymbol{\mu}(\varepsilon))\| = 1,\end{aligned} \tag{7.2.6}$$

where $\boldsymbol{\lambda}(\varepsilon) = (\lambda_1(\varepsilon),\ldots,\lambda_m(\varepsilon))$ and $\boldsymbol{\mu}(\varepsilon) = (\mu_1(\varepsilon),\ldots,\mu_k(\varepsilon))$.

If we divide through by $\sqrt{L(\varepsilon)}$ in (7.2.5), we get

$$\lambda_0(\varepsilon)\frac{\partial f}{\partial x_j}\,(\mathbf{x}_\varepsilon) + \frac{2(x_\varepsilon)_j}{\sqrt{L(\varepsilon)}} + \sum_{i=1}^{r}\lambda_i(\varepsilon)\ \frac{\partial g_i(\mathbf{x}_\varepsilon)}{\partial x_j}\ + \sum_{i=1}^{k}\ \mu_i(\varepsilon)\frac{\partial h_i(\mathbf{x}_\varepsilon)}{\partial x_j} = 0. \tag{7.2.7}$$

Now let $\varepsilon \to 0$ through a sequence of values ε_k. Then since $\|\mathbf{x}_\varepsilon\| < \varepsilon$, we have that

$$\mathbf{x}_{\varepsilon_k} \to \mathbf{0}. \tag{7.2.8}$$

Since the vectors $(\lambda_0(\varepsilon),\ \boldsymbol{\lambda}(\varepsilon),\ \boldsymbol{\mu}(\varepsilon))$ are all unit vectors (see Eq. (7.2.6)), there exists a subsequence, that we again denote as ε_k, and a unit vector $(\lambda_0, \boldsymbol{\lambda}, \boldsymbol{\mu})$ such that

$$(\lambda_0(\varepsilon_k),\ \boldsymbol{\lambda}(\varepsilon_k),\ \boldsymbol{\mu}(\varepsilon_k)) \to (\lambda_0, \boldsymbol{\lambda}, \boldsymbol{\mu}). \tag{7.2.9}$$

Since $(\lambda_0, \boldsymbol{\lambda}, \boldsymbol{\mu})$ is a unit vector, it is different from zero.

From Eqs. (7.2.7), (7.2.8), and (7.2.9) and the continuity of the partials of $f, \mathbf{g}$, and $\mathbf{h}$, we get

$$\lambda_0 \frac{\partial f}{\partial x_j}(\mathbf{0}) + \sum_{i=1}^{r} \lambda_i \frac{\partial g_i(\mathbf{0})}{\partial x_j} + \sum_{i=1}^{k} \mu_i \frac{\partial h_i(\mathbf{0})}{\partial x_j} = 0. \tag{7.2.10}$$

From (7.2.6) and (7.2.9) we see that $\lambda_i \geq 0$, $i = 0, 1, \ldots, r$ and that $\lambda_i = 0$ for $i = r+1, \ldots, m$. Thus, $\lambda_0 \geq 0$ and $\boldsymbol{\lambda} \geq \mathbf{0}$. Since $g_i(\mathbf{0}) = 0$ for $i = 1, \ldots, r$ and $\lambda_i = 0$ for $i = r+1, \ldots, m$, we have that $\lambda_i g_i(\mathbf{0}) = 0$ for $i = 1, \ldots, m$. Also, we can take the upper limit in the second term in (7.2.10) to be m and write

$$\lambda_0 \frac{\partial f}{\partial x_j}(\mathbf{0}) + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(\mathbf{0})}{\partial x_j} + \sum_{i=1}^{k} \mu_i \frac{\partial h_i(\mathbf{0})}{\partial x_j} = 0.$$

This completes the proof of the theorem. □

7.3 The Norm of a Relaxed Control; Compact Constraints

In Chapter 3, for problems with compact constraint sets $\Omega(t)$ contained in a fixed compact set Z, we defined the notion of a relaxed control μ on a compact interval I. In Eqs. (3.3.1) and (3.3.2) we pointed out that a relaxed control μ determines a continuous linear transformation L_μ from $C(I \times Z)$ to $\mathbb{R}^n$ by the formula

$$L_\mu(g) = \int_I \left(\int_Z g(t, z) d\mu_t \right) dt \qquad g \in C(I \times Z). \tag{7.3.1}$$

Moreover, $\|L\mu\| = |I|$ where $|I|$ denotes the length of I, and from the Riesz representation theorem we have that

$$\|L_\mu\| = \|\nu\|_{var} = |\nu|(I \times Z),$$

where ν is the measure $d\mu_t dt$ on $I \times Z$ and $|\nu|$ denotes the total variation measure of ν. We could therefore define the norm of μ, denoted by $\|\mu\|$ to be the total variation measure of ν. For our purposes it is more useful to define $\|\mu\|$ to be $\|L_\mu\|$.

Definition 7.3.1. The *norm* of a relaxed control μ, denoted by $\|\mu\|$ is the norm of L_μ, the continuous linear transformation that μ determines by (7.3.1). Thus, $\|\mu\| = \|L_\mu\| = |I|$, where $|I|$ denotes the length of I.

In the proof of Theorem 6.3.5 we shall consider certain linear combinations of relaxed controls. Let I be a compact real interval, let Ω be a mapping from I to compact subsets $\Omega(t)$ of a fixed compact set Z in $\mathbb{R}^m$. Let μ_1 and μ_2 be relaxed controls such that for each t in I, μ_{1t} and μ_{2t} are concentrated on $\Omega(t)$. Then for any real numbers α and β by $\mu = \alpha\mu_1 + \beta\mu_2$ we mean the mapping:

$$t \to \mu_t \equiv \alpha\mu_{1t} + \beta\mu_{2t}.$$

The mapping μ defines a continuous linear transformation L_μ from $C(I \times Z)$ to $\mathbb{R}^n$ by the formula

$$L_\mu(g) = \alpha \int_I \int_{\Omega(t)} g(t,z)d\mu_{1t}dt + \beta \int_I \int_{\Omega(t)} g(t,z)d\mu_{2t}dt \tag{7.3.2}$$

for all g in $C(I \times Z)$.

If $\alpha\mu_1 + \beta\mu_2$ is a convex combination of μ_1 and μ_2 (i.e., $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$), then μ is again a relaxed control and $\|\mu\|$ is defined. Otherwise, we must define what is meant by the norm of $\alpha\mu_1 + \beta\mu_2$.

Definition 7.3.2. If $\mu = \alpha\mu_1 + \beta\mu_2$, where μ_1 and μ_2 are relaxed controls on I such that for each t in I, μ_{1t} and μ_{2t} are concentrated on $\Omega(t)$, then

$$\|\mu\| \equiv \|\alpha\mu_1 + \beta\mu_2\| \equiv \|L_\mu\|,$$

where L_μ is given by (7.3.2).

Note that this definition is consistent with Definition 7.3.1.

We conclude with two known results from functional analysis involving norms and their interpretation in our context.

Lemma 7.3.3. *Let $\{L_n\}$ be a sequence of continuous linear transformations and L a continuous linear transformation from $C(I \times Z)$ to $\mathbb{R}^n$ such that $\|L_n - L\| \to 0$ as $n \to \infty$. Then L_n converges weak-* to L.*

Proof. For arbitrary g in $C(I \times Z)$

$$|L_n(g) - L(g)| \leq \|L_n - L\|\|g\|.$$

Since $\|L_n - L\| \to 0$, the result follows. □

Corollary 7.3.4. *If $\{\mu_n\}$ is a sequence of relaxed controls and μ is a relaxed control such that $\|\mu_n - \mu\| \to 0$, then μ_n converges weakly to μ.*

Proof. By Lemma 7.3.3 the sequence $\{L_n\}$ of continuous linear transformations corresponding to $\{\mu_n\}$ converges weak-* to the continuous linear transformation L corresponding to μ. The weak convergence of μ_n to μ follows from Remark 3.3.13. □

The converse of Lemma 7.3.3 is false. We have instead that the norm is lower semi-continuous with respect to weak-* convergence.

Lemma 7.3.5. *Let $\mathcal{B}$ be a Banach space and $\mathcal{B}^*$ the space of continuous linear transformations from $\mathcal{B}$ to $\mathbb{R}^n$. Let $\{L_n\}$ be a sequence of elements in $\mathcal{B}^*$ converging weak-* to an element L in $\mathcal{B}^*$. Then*

$$\liminf_{n\to\infty} \|L_n\| \geq \|L\|, \tag{7.3.3}$$

where $\|\ \|$ denotes the norm in $\mathcal{B}^*$.

Proof. From the identity $L = (L - L_n) + L_n$ we get that for each g in $\mathcal{B}$

$$|L(g)| \leq |L(g) - L_n(g)| + \|L_n\|\|g\|.$$

Hence

$$\begin{aligned} |L(g)| &\leq \lim_{n\to\infty} |L(g) - L_n(g)| + (\liminf_{n\to\infty} \|L_n\|)\|g\| \\ &= (\liminf_{n\to\infty} \|L_n\|)\|g\|, \end{aligned}$$

the first limit being equal to zero since L_n converges weak-* to L. From this (7.3.3) follows. $\square$

Corollary 7.3.6. *Let $\{\mu_n\}$ be a sequence of relaxed controls converging weakly to a relaxed control μ. Then $\liminf_{n\to\infty} \|\mu_n\| \geq \|\mu\|$.*

7.4 Necessary Conditions for an Unconstrained Problem

In this section we derive two necessary conditions satisfied by a local minimizer for the unconstrained problem of minimizing

$$J(\psi) = \gamma(e(\psi)) + \int_{t_0}^{t_1} G(t, \psi(t), \psi'(t))dt \tag{7.4.1}$$

over an appropriate set T of functions ψ, where G has a special form.

Classical derivations of necessary conditions assume that a minimum exists in the class of piecewise smooth functions. Tonelli [86], in consonance with results of existence theorems, derived the necessary conditions for a minimum in the class of absolutely continuous functions. An account of these results in English is given in Cesari [27]. For our purposes we need to consider minima in the class of absolutely continuous functions with derivatives in L_2 for a special form of G. The following assumption will be adequate for our purposes and will allow us to derive the necessary conditions using essentially the same arguments as those used in the classical case.

Assumption 7.4.1. (i) The function G is given by

$$G(t, x, x') = A\langle x', x'\rangle + B\langle x', a(t, x)\rangle + b(t, x), \tag{7.4.2}$$

where $t \in \mathcal{I}$, a compact real interval containing $[t_0, t_1]$, $x \in \mathbb{R}^n$ and $x' \in \mathbb{R}^n$.

(ii) The functions a and b are measurable on $\mathcal{I}$ for fixed x and are $C^{(1)}$ on $\mathbb{R}^n$ for fixed $t \in \mathcal{I}$.

(iii) For each compact interval $\mathcal{X}$ in $\mathbb{R}^n$ there exists a function M in $L_2[\mathcal{I}]$ such that

$$\begin{aligned} |a(t, x)| &\leq M(t) \qquad & |b(t, x)| &\leq M(t) \\ |a_x(t, x)| &\leq M(t) \qquad & |b_x(t, x)| &\leq M(t) \end{aligned} \tag{7.4.3}$$

for a.e. t in $\mathcal{I}$ and all x in $\mathcal{X}$.

(iv) The function γ is $C^{(1)}$ in a neighborhood of a manifold $\overline{\mathcal{B}}$, where $\overline{\mathcal{B}}$ is the closure of a $C^{(1)}$ manifold $\mathcal{B}$ of dimension $0 \leq r < 2n + 2$ in $\mathcal{I} \times \mathbb{R}^n \times \mathcal{I} \times \mathbb{R}^n$. As usual, we denote points in $\overline{\mathcal{B}}$ by (t_0, x_0, t_1, x_1).

(v) The set of functions T on which we are minimizing (7.4.1) are those functions ψ whose graphs are in $\mathcal{I} \times \mathcal{X}$, where $\mathcal{X}$ is a compact set in $\mathbb{R}^n$ and that have the following properties. The end points $e(\psi)$ are in $\overline{\mathcal{B}}$, the functions ψ are absolutely continuous on $[t_0, t_1]$ and have derivatives ψ' in $L_2[t_0, t_1]$.

Remark 7.4.2. As a consequence of (iii) and (v) of Assumption 7.4.1 the integral in (7.4.1) with G as in (7.4.2) exists for all ψ in T.

Among the several established necessary conditions that a minimizing function ψ_0 must satisfy are the Euler equations and the transversality conditions which the end points of ψ_0 must satisfy. We now derive these necessary conditions for the problem of minimizing (7.4.1) over a subset of T with G as in (7.4.2).

The next two lemmas are needed in the derivation of the Euler equations.

Lemma 7.4.3. *Let G satisfy Assumption 7.4.1. Then for fixed functions ψ and η, where ψ and η are absolutely continuous and have derivatives ψ' and η' in $L_2[t_0, t_1]$ the function*

$$I(\theta) = \int_{t_0}^{t_1} G(t, \psi(t) + \theta\eta(t), \psi'(t) + \theta\eta'(t))dt$$

is defined on an interval $(-\delta, \delta)$ and is differentiable with respect to θ. Moreover, if we define

$$G(t, \theta) \equiv G(t, \psi(t) + \theta\eta(t), \psi'(t) + \theta\eta'(t))$$

and define $G_x(t,\theta)$ and $G_{x'}(t,\theta)$ similarly, then $\mathcal{I}$ is differentiable with derivative $\mathcal{I}'(\theta)$ given by

$$I'(\theta) = \int_{t_0}^{t_1} [\langle G_x(t,\theta), \eta(t)\rangle + \langle G_{x'}(t,\theta), \eta'(t)\rangle]dt. \tag{7.4.4}$$

Proof. That $I(\theta)$ exists follows from (i) through (iii) of Assumption 7.4.1. To show that $I'(\theta)$ exists and is given by (7.4.4), we first write

$$\sigma^{-1}[I(\theta+\sigma) - I(\theta)] = \int_{t_0}^{t_1} \sigma^{-1}[G(t,\theta+\sigma) - G(t,\theta)]dt \quad \sigma \neq 0.$$

From the Mean Value Theorem we get that there exists a $\overline{\sigma}$, where $0 < \overline{\sigma} < \sigma$, such that

$$\sigma^{-1}[I(\theta+\sigma) - I(\theta)] = \int_{t_0}^{t_1} [\langle G_x(t,\theta+\overline{\sigma}), \eta(t)\rangle + \langle G_{x'}(t,\theta+\overline{\sigma}), \eta'(t)\rangle]dt.$$

From (iii) of Assumption 7.4.1 we get that for all $0 < \sigma < \overline{\sigma}$ the integrand is bounded by a fixed integrable function. If we now let $\sigma \to 0$, then the existence of $I'(\theta)$ and Eq. (7.4.4) follow from (ii) of Assumption 7.4.1 and the Dominated Convergence Theorem. □

Lemma 7.4.4. *Let h be a function in $L_2[t_0,t_1]$ with range in $\mathbb{R}^n$ such that for every absolutely continuous function η from $[t_0,t_1]$ to $\mathbb{R}^n$ with η' in $L_2[t_0,t_1]$ and $\eta(t_0) = \eta(t_1) = 0$ we have*

$$\int_{t_0}^{t_1} \langle h(t), \eta'(t)\rangle dt = 0. \tag{7.4.5}$$

Then $h(t) =$ constant a.e. on $[t_0,t_1]$.

Proof. Let

$$c = (t_1 - t_0)^{-1} \int_{t_0}^{t_1} h(t)dt$$

and let

$$\eta(t) = \int_{t_0}^{t} [h(s) - c]ds. \tag{7.4.6}$$

Then η satisfies the hypotheses of the lemma. From (7.4.5), (7.4.6), and $\eta(t_0) = \eta(t_1) = 0$ we get that

$$0 = \int_{t_0}^{t_1} \langle h(t), h(t) - c\rangle dt = \int_{t_0}^{t_1} \|h(t) - c\|^2 dt,$$

where $\|\ \|$ denotes the L_2 norm. Hence $h = c$, a constant a.e. □

Lemma 7.4.5 (Euler Equation). *Let Assumption 7.4.1 hold and let ψ_1 be a function in the set T defined in (v) of Assumption 7.4.1. For fixed $\varepsilon > 0$ let*

$$T_\varepsilon(\psi_1) = \{\psi \in T\colon \|\psi' - \psi_1'\|^2 < \varepsilon,\ |\psi(t_0) - \psi_1(t_0)|^2 < \varepsilon\}, \tag{7.4.7}$$

where $\| \ \|$ denotes the L_2 norm. Let $\overline{\psi}$ minimize (7.4.1) on the set $T_\varepsilon(\psi_1)$. Let $\overline{G}_x(t) = G_x(t, \overline{\psi}(t), \overline{\psi}'(t))$ and let $\overline{G}_{x'}(t)$ have similar meaning. Then there exists a constant c such that for a.e. t in $[t_0, t_1]$

$$\overline{G}_{x'}(t) = \int_{t_0}^{t} \overline{G}_x(s)ds + c. \tag{7.4.8}$$

Proof. From the Cauchy-Schwarz inequality and (7.4.7) we have that for all ψ in $T_\varepsilon(\psi_1)$

$$\begin{aligned} |\psi(t) - \psi_1(t)| &\le |\psi(t_0) - \psi(t_1)| + \int_{t_0}^{t} |\psi'(s) - \psi_1'(s)|ds \\ &\le \varepsilon^{1/2}(1 + |t_1 - t_0|^{1/2}). \end{aligned}$$

Hence for $\psi \in T_\varepsilon(\psi_1)$ all points $\psi(t)$ with $t_0 \le t \le t_1$ are in a compact set. Consequently, for $\psi \in T_\varepsilon(\psi_1)$ the integral

$$I(\psi) = \int_{t_0}^{t_1} G(t, \psi(t), \psi'(t))dt,$$

which is the integral in (7.4.1), exists. Since $\overline{\psi}$ minimizes (7.4.1) over all ψ in $T_\varepsilon(\psi_1)$, it follows that $\overline{\psi}$ minimizes (7.4.1) over all ψ in $T_\varepsilon(\psi_1)$ with $e(\psi) = e(\overline{\psi})$.

Let η be a function defined on $[t_0, t_1]$ with properties as in Lemma 7.4.3 and with $\eta(t_1) = \eta(t_2) = 0$. Then for real θ with $|\theta|$ sufficiently small, the function $\psi = \overline{\psi} + \theta\eta$ is in $T_\varepsilon(\psi_1)$. Hence $I(\overline{\psi} + \theta\eta)$ has a minimum at $\theta = 0$. By Lemma 7.4.3, the function $\theta \to I(\overline{\psi} + \theta\eta)$ is differentiable with derivative given by (7.4.4). Therefore, since $I(\overline{\psi} + \theta\eta)$ has a minimum at $\theta = 0$

$$\frac{dI}{d\theta}\ (\overline{\psi} + \theta\eta)|_{\theta=0} = 0.$$

From this and (7.4.4) we get that

$$\int_{t_0}^{t_1} [\langle \overline{G}_x, \eta\rangle + \langle \overline{G}_{x'}, \eta'\rangle]dt = 0.$$

Integrating by parts in the first term and using $\eta(t_0) = \eta(t_1) = 0$ gives

$$\int_{t_0}^{t_1} \langle -\int_{t_0}^{t} \overline{G}_x(s)ds + \overline{G}_{x'}(t),\ \eta'(t)\rangle dt = 0.$$

Equation (7.4.8) now follows from Lemma 7.4.4. ∎

Remark 7.4.6. Equation (7.4.8) is the Euler equation in integrated form. From (7.4.8) it follows that $\overline{G}_{x'}$ is absolutely continuous, is differentiable a.e., and that

$$\frac{d}{dt}\,\overline{G}_{x'}|_{t=\tau} = \overline{G}_x(\tau) \qquad \text{a.e.}$$

This equation is usually called the Euler equation.

Remark 7.4.7. In Section 6.5 we derived the Euler equation for the unconstrained problem from the maximum principle. The reasoning involved is not circular, since we will not appeal to that result in our proof of the maximum principle. We have established the Euler equation for a restricted class of unconstrained problems and will use this result in our proof of the maximum principle.

We now take up the transversality condition under the assumption that the initial and terminal times are fixed. For typographic simplicity we take the initial time t_0 to be zero and the terminal time to be one. The set $\mathcal{B}$ thus consists of points $(0, x_0, 1, x_1)$.

Lemma 7.4.8. *Let Assumption 7.4.1 hold. Let $t_0 = 0$ and let $t_1 = 1$. Let $\overline{\psi}$ minimize (7.4.1) over the set $T_\varepsilon(\psi_1)$ defined in (7.4.7) and let*

$$\overline{G}_{x'}(t) = G_{x'}(t, \overline{\psi}(t), \overline{\psi}'(t)).$$

Then for all unit tangent vectors (dx_0, dx_1) to $\mathcal{B}$ at $e(\overline{\psi})$

$$\langle \gamma_{x_0}(e(\overline{\psi})) - \overline{G}_{x'}(0), dx_0\rangle + \langle \gamma_{x_1}(e(\overline{\psi})) + \overline{G}_{x'}(1), dx_1\rangle = 0, \tag{7.4.9}$$

where $\overline{G}_{x'}$ is identified with the absolutely continuous function on the right-hand side of (7.4.8) that equals $\overline{G}_{x'}$ a.e.

Proof. The transversality condition is a consequence of the fact that if the end points of $\overline{\psi}$ are varied along a curve lying in $\mathcal{B}$, then the functions thus obtained cannot give a smaller value to (7.4.1) than $\overline{\psi}$ does.

Because we are assuming that $t_0 = 0$ and $t_1 = 1$, the end point $e(\overline{\psi})$ is $(0, \overline{x}_0, 1, \overline{x}_1)$, where $\overline{x}_0 = \overline{\psi}(0)$ and $\overline{x}_1 = \overline{\psi}(1)$. Let $\xi = (\xi_0, \xi_1)$ be a $C^{(1)}$ function from an interval $|s| \le \delta$ to $\mathbb{R}^{2n}$ that represents a curve lying in $\mathcal{B}$ with

$$\xi(0) = e(\overline{\psi}) = (0, \overline{x}_0, 1, \overline{x}_1), \tag{7.4.10}$$

and with $|\xi(s) - \psi_1(0)|^2 < \varepsilon$. Let

$$P(t,s) = (\xi_0(s) - \overline{x}_0)(1-t) + (\xi_1(s) - \overline{x}_1)t \tag{7.4.11}$$

and let

$$\psi(t,s) = \overline{\psi}(t) + P(t,s) \tag{7.4.12}$$

for $0 \le t \le 1$ and $|s| \le \delta$. Then

$$\psi(0,s) = \xi_0(s) \quad \psi(1,s) = \xi_1(s) \quad \psi(t,0) = \overline{\psi}(t).$$

Thus, the end points of ψ lie in B.

From (7.4.11) and (7.4.12) we get that

$$\begin{aligned}|\psi(0,s)-\psi_1(0)| &\le |\overline{\psi}(0)-\psi_1(0)+|P(0,s)|\\ &= |\overline{\psi}(0)-\psi_1(0)|+|\xi_0(s)-\overline{x}_0|.\end{aligned}$$

Since $\overline{\psi}$ is in $T_\varepsilon(\psi_1)$, the first term in the rightmost expression is $< \sqrt{\varepsilon}$. From the continuity of ξ_0 we get that the second term can be made arbitrarily small by taking $|s|$ sufficiently small. Hence for $|s|$ sufficiently small

$$|\psi(0,s)-\psi_1(0)|^2 < \varepsilon. \tag{7.4.13}$$

From (7.4.12) we get

$$\begin{aligned}\|\psi'(\cdot,s)-\psi_1'\| &\le \|\overline{\psi}'-\psi_1'\|+\|P_t(\cdot,s)\|\\ &\le \|\overline{\psi}'-\psi_1'\|+\|\xi_0(s)-\overline{x}_0\|+\|\xi_1(s)-\overline{x}_1\|.\end{aligned}$$

By an argument similar to the one used to establish (7.4.13), we get that for $|s|$ sufficiently small, all functions $\psi(\cdot,s)$ are in $T_\varepsilon(\psi_1)$.

Since $\overline{\psi}$ minimizes (7.4.1) over $T_\varepsilon(\psi_1)$, since for $|s|$ sufficiently small all functions $\psi(\cdot,s)$ are in $T_\varepsilon(\psi_1)$, and since $\overline{\psi}$ is in $T_\varepsilon(\psi_1)$, the function $\overline{\psi}=\psi(\cdot,0)$ minimizes J over all $\psi(\cdot,s)$ for $|s|$ sufficiently small.

Let

$$I(s)=\int_0^1 G(t,\psi(t,s),\psi'(t,s))dt,$$

where $'$ indicates differentiation with respect to t. Then

$$J(\psi(\cdot,s))=\gamma(\xi(s))+I(s) \tag{7.4.14}$$

and

$$J(\psi(\cdot,s))\ge J(\psi(\cdot,0))=J(\overline{\psi}). \tag{7.4.15}$$

Since ξ_0 and ξ_1 are $C^{(1)}$ on $|s|\le\delta$, it follows from Eqs. (7.4.10) through (7.4.12) that the function $t\to\partial G(t,\psi(t,s),\psi'(t,s))/\partial s$ is bounded on $[0,1]$ by an integrable function. Hence, by the argument used in Lemma 7.4.3 we get that $I'(s)$ exists. Also, since γ is of class $C^{(1)}$, the function $s\to\gamma(\xi(s))$ is $C^{(1)}$. Hence $s\to J(\psi(\cdot,s))$ is differentiable. It then follows from (7.4.15) that

$$\frac{dJ(\psi(\cdot,s))}{ds}|_{s=0}=0. \tag{7.4.16}$$

A straightforward calculation of (7.4.16) gives

$$[d\gamma/ds]_{s=0}+\int_0^1[\langle\overline{G}_x(t),P_s(t,0)\rangle+\langle\overline{G}_{x'}(t),P_{ts}(t,0)\rangle]dt=0, \tag{7.4.17}$$

where $\overline{G}_x(t) = G_x(t, \overline{\psi}(t), \overline{\psi}'(t))$ and $\overline{G}_{x'}(t)$ has similar meaning. The function $\overline{\psi}$ satisfies the Euler equation, so there exists a constant c such that

$$\overline{G}_{x'}(t) = \int_0^t \overline{G}_x(\sigma)d\sigma + c \qquad \text{a.e.}$$

If we set

$$h(t) = \int_0^t \overline{G}_x(\sigma)d\sigma + c,$$

then h is absolutely continuous and $h'(t) = \overline{G}_x(t)$ a.e. We may therefore rewrite (7.4.17) as

$$d\gamma/ds]_{s=0} + \int_0^1 [\langle h'(t), P_s(t,0)\rangle + \langle h(t), P_{ts}(t,0)\rangle]dt = 0. \tag{7.4.18}$$

From (7.4.11) we see that $P_{st}(t,s) = P_{ts}(t,s)$. Hence if we integrate the first term in (7.4.17) by parts we get that

$$d\gamma/ds]_{s=0} + h(t)P_s(t,0]_0^1 = 0.$$

If we take $\overline{G}_{x'}(t) = h(t)$ everywhere, and use (7.4.11), we can write the preceding equality as

$$\langle \gamma_{x_1}(e(\overline{\psi})) + \overline{G}_{x'}(1), \xi_1'(0)\rangle + \langle \gamma_{x_0}(e(\overline{\psi})) - \overline{G}_{x'}(0), \xi_0'(0)\rangle = 0.$$

Since the curve $\xi(s)$ is arbitrary, $(\xi_1'(0), \xi_0'(0))$ is a scalar multiple of an arbitrary unit tangent vector (dx_0, dx_1), and the lemma follows. $\square$

Remark 7.4.9. If $e(\psi_\varepsilon)$ is a point of $\overline{\mathcal{B}}\backslash\mathcal{B}$, then curves $(\xi_0(s), \xi(s))$ emanating from $e(\psi_\varepsilon)$ and lying in $\overline{\mathcal{B}}$ are only defined for $s \geq 0$. Also the tangent vectors $dx_0 = c\xi_0'(s)$ and $dx_1 = c\xi_1'(s)$ are such that their projections from the tangent plane onto $\mathcal{B}$ are directed into $\mathcal{B}$. The proof is then modified to get $I'(0) \geq 0$. From this the conclusion follows that for all tangent vectors (dx_0, dx_1) pointing into $\mathcal{B}$, we have

$$\langle \gamma_{x_0}(e(\overline{\psi})) - \overline{G}_{x'}(0), dx_0\rangle + \langle \gamma_{x_1}(e(\overline{\psi})) + \overline{G}_{x'}(1), dx_1\rangle \geq 0.$$

7.5 The ε-Problem

The proof of Theorem 6.3.5 will be carried out in several steps in this and the next two sections. First, we formulate a set of unconstrained penalized problems depending on a small parameter ε and show that these problems have a solution. This constitutes the "ε-problem." Second, we show that for each ε the ε-problem has an interior minimum. Third, we apply the results

of Section 7.4 and other arguments to obtain necessary conditions satisfied by the solution of the ε-problem. We call these necessary conditions the "ε-maximum principle." This will be done in Section 7.6. Lastly, we let ε tend to zero and show that the solutions of the ε problem tend to the solution of the constrained problem and that the ε-maximum principle tends to the maximum principle of Theorem 6.3.5.

Remark 7.5.1. In the sequel we shall be extracting subsequences of sequences. If $\{\psi_n\}$ is a sequence and $\{\psi_{n_k}\}$ is a subsequence *we shall relabel the subsequence as* $\{\psi_n\}$ without explicitly saying so.

Theorem 6.3.5 involves a relaxed optimal pair. Therefore, for typographic simplicity we designated an optimal relaxed pair by (ψ, μ), without any additional symbols to denote optimality. The proof of Theorem 6.3.5 will involve pairs other than the optimal one. Therefore, we now denote a relaxed optimal pair by (ψ_*, μ_*). To simplify notation we take $t_0 = 0$ and $t_1 = 1$. We suppose that $[0, 1]$, the interval on which (ψ_*, μ_*) is defined, is interior to $\mathcal{I}_0$.

Step I Formulation of the ε-problem. *Without loss of generality we may assume that* $J(\psi_*, \mu_*) = 0$. The graph of ψ_*, that is, $\{(t, \psi_*(t)) \colon t \in [0, 1]\}$ is compact and hence it is a positive distance, say ε_1', from the boundary of $\mathcal{I}_0 \times \mathcal{X}_0$. Let $\varepsilon_1 = \min(1, \varepsilon_1')$. It follows from (6.3.3) that ψ_*' is in $L_2[0, 1]$.

Let $\mathcal{D}$ denote the set of absolutely continuous functions ψ and relaxed controls μ defined on $[0, 1]$ but *not necessarily related by the differential equation* $\psi'(t) = f(t, \psi(t), \mu_t)$, such that the graph of ψ is in $[0, 1] \times \mathcal{X}_0$, the derivative ψ' is in $L_2[0, 1]$, $e(\psi)$ is in $\overline{\mathcal{B}}$, and μ_t is concentrated on $\Omega(t)$ for all $t \in [0, 1]$. Since all points $\{x = \psi(t) \colon 0 \le t \le 1\}$ are in a compact set, it follows from (6.3.3) that the function $t \to f(t, \psi(t), \mu_t)$ is in $L_2[0, 1]$, as is $t \to f^0(t, \psi(t), \mu_t)$. For each $0 < \varepsilon < \varepsilon_1$ and (ψ, μ) in $\mathcal{D}$ let

$$F(\psi, \mu, K, \varepsilon) = g(e(\psi)) + \int_0^1 f^0(t, \psi(t), \mu_t)dt + |\psi(0) - \psi_*(0)|^2 \qquad (7.5.1)$$
$$+ \|\psi' - \psi_*'\|^2 + \varepsilon\|\mu - \mu_*\|_L + K\|\psi' - f(t, \psi(t), \mu_t)\|^2,$$

where $\| \ \|$ denotes the $L_2[0, 1]$ norm and $\| \ \|_L$ denotes the norm defined in Section 7.3. The function F is well defined on $\mathcal{D} \times (0, \varepsilon_1)$.

For each $0 < \varepsilon < \varepsilon_1$, let $\mathcal{D}_\varepsilon$ denote the set of absolutely continuous functions ψ and relaxed controls μ defined on $[0, 1]$ such that ψ' is in $L_2[0, 1]$, μ_t is concentrated on $\Omega(t)$ for all $t \in [0, 1]$ and such that

$$|\psi(0) - \psi_*(0)| \le \varepsilon/2 \qquad \|\psi' - \psi_*'\| \le \varepsilon/2 \qquad e(\psi) \in \overline{\mathcal{B}} \qquad (7.5.2)$$
$$\|\mu - \mu_*\|_L \le \varepsilon.$$

From

$$\psi(t) - \psi_*(t) = \psi(0) - \psi_*(0) + \int_0^t [\psi'(s) - \psi_*'(s)]ds$$

and the Cauchy-Schwarz inequality we get that for ψ in $\mathcal{D}_\varepsilon$

$$|\psi(t) - \psi_*(t)| \leq |\psi(0) - \psi_*(0)| + \|\psi' - \psi'_*\|. \tag{7.5.3}$$

Since the graph of ψ_* is compact and is at distance ε'_1 from the boundary of $\mathcal{I}_0 \times \mathcal{X}_0$, it follows from (7.5.3) and the first two inequalities in (7.5.2) that all points $\psi(t)$, $0 \leq t \leq 1$ are in a fixed compact set $\mathcal{X}$ that is independent of ψ and is contained in $\mathcal{X}_0$. Thus, the graph of ψ is contained in $\mathcal{I}_0 \times \mathcal{X}_0$. Hence $\mathcal{D}_\varepsilon$ is contained in $\mathcal{D}$ and F is defined on $\mathcal{D}_\varepsilon \times (0, \varepsilon)$.

For each $0 < \varepsilon < \varepsilon_1$, *we define the* ε*-problem to be: Minimize* $F(\psi, \mu, K, \varepsilon)$ *over the set* $\mathcal{D}_\varepsilon$.

Step II consists of establishing the following lemma.

Lemma 7.5.2. *For each* $0 < \varepsilon < \varepsilon_1$, *the* ε*-problem has a solution.*

Proof. Let $m_\varepsilon(K) = \inf F(\psi, \mu, K, \varepsilon)$, where the infimum is taken over all (ψ, μ) in $\mathcal{D}_\varepsilon$. In Step I we saw that if (ψ, μ) is in $\mathcal{D}_\varepsilon$, then all points $\psi(t)$, $0 \leq t \leq 1$ are in a fixed compact set $\mathcal{X} \subseteq \mathcal{X}_0$ and independent of ψ. In particular, the points $e(\psi)$ are in a compact subset of $\overline{\mathcal{B}}$. Since g is continuous, g is bounded on $\mathcal{D}_\varepsilon$. Also, since Z is compact and $d\mu_t$ is a probability measure, it follows from (6.3.3) that the integrals

$$\int_0^1 f^0(t, \psi(t), \mu_t)dt = \int_0^1 (\int_Z f^0(t, \psi(t), z)d\mu_t)dt$$

are bounded in $\mathcal{D}_\varepsilon$. Hence $m_\varepsilon(K)$ is finite.

Let $\{(\psi_n, \mu_n)\}$ be a sequence in $\mathcal{D}_\varepsilon$ such that

$$\lim_{n\to\infty} F(\psi_n, \mu_n, K, \varepsilon) = m_\varepsilon(K). \tag{7.5.4}$$

From the identity

$$\psi'(t) = \psi'(t) - \psi'_*(t) + \psi'_*(t)$$

and the Cauchy-Schwarz inequality, we get that for any measurable set $E \subseteq [0, 1]$

$$|\int_E \psi'(t)dt| \leq \|\psi' - \psi'_*\|[\text{meas } E]^{1/2} + |\int_E \psi'_*(t)dt|.$$

Thus, the functions $\{\psi_n\}$ are equi-absolutely continuous. In Step I we showed that all ψ in D_ε are uniformly bounded. It then follows from Lemma 5.3.3 and Ascoli's theorem that there exists a subsequence $\{\psi_n\}$ and an absolutely continuous function ψ_ε such that ψ_n converges uniformly to ψ_ε. Since $\{\psi_n\}$ is in $\mathcal{D}_\varepsilon$, $|\psi_\varepsilon(0) - \psi_*(0)| \leq \varepsilon/2$. Since $e(\psi_n) \in \overline{\mathcal{B}}$, it follows that $e(\psi_\varepsilon) \in \overline{\mathcal{B}}$. Moreover, by Theorem 5.3.5, ψ'_n converges weakly in $L_1[0, 1]$ to ψ'_ε.

Let $\{\mu_n\}$ be a subsequence corresponding to $\{\psi_n\}$. By the discussion in Section 7.3, the inequality $\|\mu_n - \mu_*\|_L \leq \varepsilon$ means that $\|L_n - L_*\| \leq \varepsilon$ where $\{L_n\}$ and L_* are the linear transformations from $C(I \times Z)$ to $\mathbb{R}^n$ determined by $\{\mu_n\}$ and μ_*. Since a closed ball in the dual space of a separable Banach space

is weak-* sequentially compact, there exists a subsequence $\{L_n\}$ and a linear transformation L_ε such that L_n converges weak-* to L_ε, where $\|L_\varepsilon - L_*\| \leq \varepsilon$. It then follows from Corollary 7.3.4 that the sequence $\{\mu_n\}$ of relaxed controls converges weakly to a relaxed control μ_ε with $\|\mu_\varepsilon - \mu_*\|_L \leq \varepsilon$. We then select a subsequence of ψ_n corresponding to $\{\mu_n\}$ so that $(\{\psi_n\}, \{\mu_n\})$ satisfies all the assertions of this paragraph.

Thus far we have shown that $(\psi_\varepsilon, \mu_\varepsilon)$ satisfies all the inequalities in (7.5.2) except $\|\psi'_\varepsilon - \psi'_*\| \leq \varepsilon/2$. From (7.5.2) we have that $\|\psi'_n - \psi'_*\| \leq \varepsilon/2$. Since closed balls in L_2 are weakly compact, there exists a subsequence $\{(\psi_n, \mu_n)\}$ and a function h in $L_2[0,1]$ with $\|h - \psi_*\| \leq \varepsilon/2$ such that $\psi'_n \to h$ weakly in L_2. Therefore, for any bounded measurable function γ,

$$\lim_{n\to\infty} \int_0^1 \psi'_n \gamma dt = \int_0^1 h\gamma dt.$$

Thus, $\psi'_n \to h$ weakly in L_1. But, $\psi'_n \to \psi'_\varepsilon$ weakly in $L_1[0,1]$. Hence, by the uniqueness of weak limits, $\psi'_\varepsilon = h$, and so $\|\psi'_\varepsilon - \psi'_*\| \leq \varepsilon/2$. In summary, we have shown that $(\psi_\varepsilon, \mu_\varepsilon)$ is in $\mathcal{D}_\varepsilon$

It remains to show that $(\psi_\varepsilon, \mu_\varepsilon)$ is a minimizer. By the Cauchy-Schwarz inequality

$$\begin{aligned}\int_0^1 [\psi'_n(t) - f(t, \psi_n(t), \mu_{nt})][\psi'_\varepsilon(t) - f(t, \psi_\varepsilon(t), \mu_{\varepsilon t})]dt \\ \leq \|\psi'_n - f(t, \psi_n(t), \mu_{nt})\| \|\psi'_\varepsilon - f(t, \psi_\varepsilon(t), \mu_{\varepsilon t})\|.\end{aligned}$$

From the uniform convergence of ψ_n to ψ, from the weak L_2 convergence of ψ'_n to ψ'_ε, from the weak convergence of μ_n to μ_ε, and from Lemma 4.3.3, we get that as $n \to \infty$, the integral on the left tends to $\|\psi'_\varepsilon - f(t, \psi_\varepsilon(t), \mu_{\varepsilon t})\|^2$. Hence

$$\liminf_{n\to\infty} \|\psi'_n - f(t, \psi_n(t), \mu_{nt})\| \geq \|\psi'_\varepsilon - f(t, \psi_\varepsilon(t), \mu_{\varepsilon t})\|, \tag{7.5.5}$$

and

$$\lim_{n\to\infty} \int_0^1 f^0(t, \psi_n(t), \mu_{nt})dt = \int_0^1 f^0(t, \psi_\varepsilon(t), \mu_{\varepsilon t})dt. \tag{7.5.6}$$

It follows from Lemma 7.3.5 and Corollary 7.3.6 that

$$\liminf_{n\to\infty} \|\psi'_n - \psi'_*\| \geq \|\psi'_\varepsilon - \psi'_*\| \qquad \liminf_{n\to\infty} \|\mu_n - \mu_*\| \geq \|\mu_\varepsilon - \mu^*\|. \tag{7.5.7}$$

From (7.5.1), from $e(\psi_n) \to e(\psi_\varepsilon)$, the continuity of g, from $\psi_n(0) \to \psi_\varepsilon(0)$, and from (7.5.4) to (7.5.7) we get

$$\begin{aligned}m_\varepsilon(K) &= \lim_{n\to\infty} F(\psi_n, \mu_n, K, \varepsilon) = \liminf_{n\to\infty} F(\psi_n, \mu_n, K, \varepsilon) \\ &\geq F(\psi_\varepsilon, \mu_\varepsilon, K, \varepsilon) \geq m_\varepsilon(K).\end{aligned}$$

Hence $(\psi_\varepsilon, \mu_\varepsilon)$ minimizes, and the lemma is proved. □

Step III $(\psi_\varepsilon, \mu_\varepsilon)$ is an interior minimum. The meaning of this statement is given in Lemma 7.5.4.

Step III is a consequence of the following result.

Lemma 7.5.3. *For each* $0 < \varepsilon < \varepsilon_1$, *there exists a* $K(\varepsilon) > 0$ *such that* $F(\psi, \mu, K(\varepsilon), \varepsilon) > 0$ *for all* (ψ, μ) *in* $\mathcal{D}_\varepsilon$ *that satisfy at least one of the equalities*

$$|\psi(0) - \psi_*(0)| = \varepsilon/2 \quad \|\psi' - \psi'_*\| = \varepsilon/2 \quad \|\mu - \mu_*\|_L = \varepsilon. \tag{7.5.8}$$

Proof. If the conclusion of the lemma were false, then there would exist an $0 < \varepsilon_0 < \varepsilon_1$, a sequence $\{K_n\}$ with $K_n \to +\infty$, and a sequence $\{(\psi_n, \mu_n)\}$ in $\mathcal{D}_{\varepsilon_0}$ whose elements satisfy *at least one of the equalities* in (7.5.8) with $\varepsilon = \varepsilon_0$, and such that

$$\begin{aligned} g(e(\psi_n)) + \int_0^1 f^0(t, \psi_n(t), \mu_{nt})dt + |\psi_n(0) - \psi_*(0)|^2 \\ + \|\psi'_n - \psi'_*\|^2 + \varepsilon\|\mu_n - \mu_*\|_L \leq -K_n\|\psi'_n - f(t, \psi_n, \mu_{nt})\|^2. \end{aligned} \tag{7.5.9}$$

By the argument used in the proof of Lemma 7.5.2, there exists a subsequence $\{(\psi_n, \mu_n)\}$, an absolutely continuous function ψ_0 with derivative ψ'_0 in L_2 and a relaxed control μ_0 such that: (i) ψ_n converges uniformly to ψ_0, (ii) $e(\psi_0) \in \overline{\mathcal{B}}$, (iii) ψ'_n converges weakly in $L_2[0,1]$ to ψ'_0, (iv) μ_n converges weakly to μ_*, (v) $\|\psi'_0 - \psi'_*\| \leq \varepsilon_0/2$, and (vi) $\|\mu_0 - \mu_*\|_L \leq \varepsilon_0$. It follows from the preceding that (ψ_0, μ_0) is in $\mathcal{D}_{\varepsilon_0}$. From the preceding and Lemma 4.3.3 we get that

$$\begin{aligned} \lim_{n\to\infty} \int_0^1 f^0(t, \psi_n(t), \mu_{nt})dt &= \int_0^1 f^0(t, \psi_0(t), \mu_{0t})\ dt \\ \lim_{n\to\infty} \int_\Delta [\psi'_n(t) - f(t, \psi_n(t), \mu_{nt})]dt &= \int_\Delta [\psi'_0(t) - f(t, \psi_0(t), \mu_{0t})]dt, \end{aligned} \tag{7.5.10}$$

where Δ is any measurable subset of $[0,1]$.

All terms on the left in (7.5.9) are bounded by a constant independent of n. Hence if in (7.5.9) we divide both sides by $-K_n$ and then let $n \to \infty$ we get that

$$0 \geq \limsup_{n\to\infty} \|\psi'_n - f(t, \psi_n(t), \mu_{nt})\|^2 \geq 0.$$

From (7.5.10) and the Cauchy-Schwarz inequality we get that

$$\begin{aligned} \left| \int_\Delta [\psi'_0(t) - f(t, \psi_0(t), \mu_{0t})]dt \right| &= \lim_{n\to\infty} \left| \int_\Delta [\psi'_n(t) - f(t, \psi_n(t), \mu_{nt})]dt \right| \\ &\leq \limsup_{n\to\infty} \|\psi'_n - f(t, \psi_n(t), \mu_{nt})\| = 0. \end{aligned}$$

Since Δ is any measurable subset of $[0,1]$ this implies that

$$\psi'_0(t) = f(t, \psi_0(t), \mu_{0t}) \qquad \text{a.e.}$$

Thus, (ψ_0, μ_0) is admissible for the non-penalized problem. Therefore, $J(\psi_0, \mu_0) \geq J(\psi_*, \mu_*) = 0$, and so

$$g(e(\psi_0)) + \int_0^1 f^0(t, \psi_0(t), \mu_{0t})dt \geq 0. \tag{7.5.11}$$

From (7.5.9) and the assumption that at least one of the equalities in (7.5.8) holds, we get that

$$g(e(\psi_n)) + \int_0^1 f^0(t, \psi_n(t), \mu_{nt})dt \leq -\varepsilon^2/4.$$

If we let $n \to \infty$ in this inequality and use (7.5.10), the convergence of ψ_n to ψ, and the continuity of g, we get that

$$g(e(\psi_0)) + \int_0^1 f^0(t, \psi_0(t), \mu_{0t})dt \leq -\varepsilon^2/4,$$

which contradicts (7.5.11). This proves the lemma. □

Lemma 7.5.4. *For each* $0 < \varepsilon < \varepsilon_1$, *let* $K(\varepsilon)$ *be as in Lemma* 7.5.3 *and let* $(\psi_\varepsilon, \mu_\varepsilon)$ *minimize* $F(\psi, \mu, K(\varepsilon), \varepsilon)$ *over* $\mathcal{D}_\varepsilon$. *Then*

$$|\psi_\varepsilon(0) - \psi_*(0)| < \varepsilon/2 \quad \|\psi'_\varepsilon - \psi'_*\| < \varepsilon/2 \quad \|\mu_\varepsilon - \mu_*\|_L < \varepsilon. \tag{7.5.12}$$

Proof. The existence of $(\psi_\varepsilon, \mu_\varepsilon)$ was shown in Lemma 7.5.2. Since (ψ_*, μ_*) is in $\mathcal{D}_\varepsilon$, and since

$$F(\psi_*, \mu_*, K(\varepsilon), \varepsilon) = J(\psi_*, \mu_*) = 0,$$

it follows that $F(\psi_\varepsilon, \mu_\varepsilon, K(\varepsilon), \varepsilon) \leq 0$.

The inequalities in (7.5.12) now follow from Lemma 7.5.3. □

Remark 7.5.5. In Lemma 7.5.3 if $\{\varepsilon_n\}$ is a strictly decreasing sequence such that $0 < \varepsilon_n < \varepsilon_1$ and such that $\varepsilon_n \to 0$, then we can take $\{K(\varepsilon_n)\}$ to be a strictly increasing sequence such that $K(\varepsilon_n) \to +\infty$.

7.6 The ε-Maximum Principle

For each $0 < \varepsilon < \varepsilon_1$ the pair $(\psi_\varepsilon, \mu_\varepsilon)$ minimizes $F(\psi, \mu, K(\varepsilon), \varepsilon)$ over all (ψ, μ) in $\mathcal{D}_\varepsilon$. Hence, if we take $\mu = \mu_\varepsilon$, then ψ_ε minimizes $F(\psi, \mu_\varepsilon, K(\varepsilon), \varepsilon)$ over the set

$$T_{\varepsilon/2}(\psi_*) = \{\psi \in T \colon \|\psi' - \psi'_*\| < \varepsilon/2, \quad |\psi(0) - \psi_*(0)| < \varepsilon/2\},$$

where the set T is defined in (v) of Assumption 7.4.1.

We can write

$$F(\psi, \mu_\varepsilon, K(\varepsilon), \varepsilon) - \|\mu_\varepsilon - \mu_*\|_L = \gamma(e(\psi)) + \int_0^1 G^\varepsilon(t, \psi(t), \psi'(t))dt, \quad (7.6.1)$$

where

$$\gamma(e(\psi)) = g(e(\psi)) + \langle \psi(0) - \psi_*(0), \psi(0) - \psi_*(0) \rangle$$

and

$$\begin{aligned} G^\varepsilon(t, x, x') &= f^0(t, x, \mu_{\varepsilon t}) + \langle x' - \psi'_*(t), x' - \psi'_*(t) \rangle \\ &\quad + K(\varepsilon)\langle x' - f(t, x, \mu_{\varepsilon t}), x' - f(t, x, \mu_{\varepsilon t}) \rangle. \end{aligned} \quad (7.6.2)$$

The function G^ε has the form of G in (7.4.2) and satisfies Assumption 7.4.1 by virtue of the assumptions on $\widehat{f}$ and $\widehat{f}_x$ made in Assumption 6.3.1. Thus, ψ_ε minimizes (7.6.1) with γ and G^ε as in (7.6.2) over all ψ in $T_{\varepsilon/2}(\psi_*)$, and the hypotheses of Lemma 7.4.5 hold with $\psi_* = \psi_1$. Let

$$\overline{G}^\varepsilon_x(t) = G^\varepsilon_x(t, \psi_\varepsilon(t), \psi'_\varepsilon(t)) \quad \overline{G}^\varepsilon_{x'}(t) = G^\varepsilon_{x'}(t, \psi_\varepsilon(t), \psi'_\varepsilon(t)). \quad (7.6.3)$$

Then by Lemma 7.4.5 there exists a constant $c = c(\varepsilon)$ such that for a.e. t in $[0, 1]$

$$\overline{G}^\varepsilon_{x'}(t) = \int_0^t \overline{G}^\varepsilon_x(s)ds + c(\varepsilon). \quad (7.6.4)$$

If we define $\bar{\lambda}(\varepsilon, t)$ to be equal to the right-hand side of (7.6.4), then $\bar{\lambda}(\varepsilon, \cdot)$ is absolutely continuous and

$$\bar{\lambda}(\varepsilon, t) = \overline{G}^\varepsilon_{x'}(t) \qquad \bar{\lambda}'(\varepsilon, t) = \overline{G}^\varepsilon_x(t) \quad \text{a.e.} \quad (7.6.5)$$

Calculating $\overline{G}^\varepsilon_{x'}(t)$ and $\overline{G}^\varepsilon_x(t)$ using (7.6.2) and (7.6.3) gives

$$\begin{aligned} \bar{\lambda}(\varepsilon, t) &= 2(\psi'_\varepsilon(t) - \psi'_*(t)) + 2K(\varepsilon)\ (\psi'_\varepsilon(t) - f(t, \psi_\varepsilon(t), \mu_{\varepsilon t})) \\ \bar{\lambda}'(\varepsilon, t) &= f^0_x(\varepsilon, t) - 2K(\varepsilon) f_x(\varepsilon, t)^T (\psi'_\varepsilon(t) - f(t, \psi_\varepsilon(t), \mu_{\varepsilon t})), \end{aligned} \quad (7.6.6)$$

where

$$\begin{aligned} f^0_x(\varepsilon, t) &= f^0_x(t, \psi_\varepsilon(t), \mu_{\varepsilon t}) \\ f_x(\varepsilon, t) &= f_x(t, \psi_{\varepsilon(t)}, \mu_{\varepsilon t}) = (\partial f^i(t, \psi_\varepsilon(t), \mu_{\varepsilon t})/\partial x^j), \end{aligned}$$

and the superscript T denotes transpose. We now combine the two equations in (7.6.6) to get

$$\bar{\lambda}'(\varepsilon, t) = f^0_x(\varepsilon, t) - f_x(\varepsilon, t)^T \bar{\lambda}(\varepsilon, t) + 2 f_x(\varepsilon, t)^T (\psi'_\varepsilon(t) - \psi'_*(t)). \quad (7.6.7)$$

Let

$$M(\varepsilon) = 1 + |\bar{\lambda}(\varepsilon, 0)| \quad (7.6.8)$$

and let

$$\lambda(\varepsilon, t) = \bar{\lambda}(\varepsilon, t)/M(\varepsilon). \tag{7.6.9}$$

We now divide through by $M(\varepsilon)$ in (7.6.7) and use (7.6.9) to get

$$\lambda'(\varepsilon, t) = M(\varepsilon)^{-1} f_x^0(\varepsilon, t) - f_x(\varepsilon, t)^T \lambda(\varepsilon, t) + 2M(\varepsilon)^{-1} f_x(\varepsilon, t)^T (\psi_\varepsilon'(t) - \psi_x'(t)). \tag{7.6.10}$$

In summary, we have shown that since ψ_ε minimizes (7.6.1) over $T_{\varepsilon/2}(\psi_*)$, there exists an absolutely continuous function $\lambda(\varepsilon, \cdot)$ such that (7.6.10) holds a.e. on $[0, 1]$.

Since ψ_ε minimizes (7.6.1) over $T_{\varepsilon/2}(\psi_*)$, the transversality condition (7.4.9) of Lemma 7.4.8 holds. We suppose the $e(\psi_\varepsilon)$ is an interior point of $\mathcal{B}$. From the first equation in (7.6.2) we get that

$$\begin{aligned} \gamma_{x_0}(e(\psi_\varepsilon)) &= g_{x_0}(e(\psi_\varepsilon)) + 2(\psi_\varepsilon(0) - \psi_*(0)) \\ \gamma_{x_1}(e(\psi_\varepsilon)) &= g_{x_1}(e(\psi_\varepsilon)). \end{aligned}$$

From (7.6.5) we get that $\bar{\lambda}(\varepsilon, 0) = \overline{G}_{x'}^{\varepsilon}(0)$ and $\bar{\lambda}(\varepsilon, 1) = \overline{G}_{x'}^{\varepsilon}(1)$.

Substituting these quantities into (7.4.9), dividing through by $M(\varepsilon)$, and using (7.6.9) gives

$$\begin{aligned} \langle M(\varepsilon)^{-1}(g_{x_0}(e(\psi_\varepsilon)) + 2(\psi_\varepsilon(0) - \psi_*(0))) - \lambda(\varepsilon, 0), dx_0\rangle & \\ + \langle M(\varepsilon)^{-1} g_{x_1}(e(\psi_\varepsilon)) + \lambda(\varepsilon, 1), dx_1\rangle &= 0 \end{aligned} \tag{7.6.11}$$

for all tangent vectors (dx_0, dx_1) to $\mathcal{B}$ at $e(\psi_\varepsilon)$. If $e(\psi_\varepsilon)$ is a boundary point of $\mathcal{B}$, then the equality in (7.6.11) is replaced by ≥ 0 and is required to hold for all tangent vectors (dx_0, dx_1) whose projection from the tangent plane onto $\mathcal{B}$ is directed into $\mathcal{B}$.

We now deduce another necessary condition, which is the analog of the Weierstrass condition in the calculus of variations, by considering changes in the control μ_ε. Let μ be an arbitrary relaxed control. For $-1 \leq \theta \leq 1$ define

$$\mu_\theta = \mu_\varepsilon + \theta(\mu - \mu_\varepsilon) = (1 - \theta)\mu_\varepsilon + \theta\mu.$$

Each μ_θ determines an element L_θ in $C^*(I \times Z)$ in the usual way. It is easily checked that if $0 \leq \theta \leq 1$, *then μ_θ is a relaxed control.* Since $\|\mu_\varepsilon - \mu_*\|_L < \varepsilon$, it follows that there exists a $0 < \theta_0 < 1$ such that if $0 \leq \theta \leq \theta_0$, then $\|\mu_\theta - \mu_*\|_L < \varepsilon$. Thus,

$$\rho_\varepsilon(\theta) = F(\psi_\varepsilon, \mu_\theta, K(\varepsilon), \varepsilon)$$

is defined for all $0 \leq \theta \leq \theta_0$ and has a minimum at $\theta = 0$. Hence if ρ_ε has a right-hand derivative $\rho_\varepsilon'(0+)$ at $\theta = 0$, then we must have

$$\rho_\varepsilon'(0+) \geq 0.$$

We shall show that each of the three terms in the definition of ρ_ε that

involve θ, namely the first integral, $K\|\psi'_\varepsilon - f(t,\psi_\varepsilon(t),\mu_\theta)\|^2$ and $\varepsilon\|\mu_\theta - \mu_*\|_L$ has a right-hand derivative at $\theta = 0$ and shall calculate their right-hand derivatives at $\theta = 0$.

We consider the first integral in the definition of ρ_ε. Let

$$\begin{aligned} f^0(t,\theta) &\equiv f^0(t,\psi_\varepsilon(t),\mu_{\theta t}) \\ &= \int_Z f^0(t,\psi_\varepsilon(t),z)d\mu_{\varepsilon t} + \theta \int_Z f^0(t,\psi_\varepsilon(t),z)(d\mu_t - d\mu_{\varepsilon t}). \end{aligned}$$

Then f^0 has a partial derivative with respect to θ given by

$$\partial f^0/\partial\theta = \int_Z f^0(t,\psi_\varepsilon(t),z)(d\mu_t - d\mu_{\varepsilon t}).$$

Thus, $\partial f^0/\partial\theta$ is bounded by an integrable function, and the function

$$A(\theta) = \int_0^1 f^0(t,\theta)dt,$$

which is the first term in the definition of ρ_ε, is differentiable at $\theta = 0$, with derivative given by

$$A'(0) = \int_0^1 [\partial f^0(t,\theta)/\partial\theta]_{\theta=0}dt = \int_0^1 f^0(t,\psi_\varepsilon(t),\mu_t - \mu_{\varepsilon t})dt.$$

We next consider

$$B(\theta) = K(\varepsilon)\|\psi'_\varepsilon(t) - f(t,\psi_\varepsilon(t),\mu_{\theta t})\|^2.$$

Let

$$p(t,\theta) = K(\varepsilon)\langle(\psi'_\varepsilon(t) - f(t,\psi_\varepsilon(t),\mu_{\theta t})),\ (\psi'_\varepsilon(t) - f(t,\psi_\varepsilon(t),\mu_{\theta t}))\rangle.$$

Then

$$\frac{\partial p}{\partial\theta} = -2K(\varepsilon)\langle(\psi'_\varepsilon(t) - f(t,\psi_\varepsilon(t),\mu_{\theta t})),\ f(t,\psi_\varepsilon(t),\mu_t - \mu_{\varepsilon t})\rangle.$$

Thus, $\partial p/\partial\theta$ is bounded by an integrable function, and so $B(\theta)$ is differentiable at $\theta = 0$, with derivative given by

$$B'(0) = -\int_0^1 \langle 2K(\varepsilon)(\psi'_\varepsilon(t) - f(t,\psi_\varepsilon(t),\mu_{\varepsilon t})),\ f(t,\psi_\varepsilon(t),\mu_\varepsilon - \mu_{\varepsilon t})\rangle dt.$$

Using the first equation in (7.6.6), we rewrite this expression as

$$B'(0) = \int_0^1 \langle -\bar{\lambda}(\varepsilon,t),\ f(t,\psi_\varepsilon(t),\mu_t - \mu_{\varepsilon t})\rangle dt$$

$$+\,2\int_0^1 \langle \psi_\varepsilon'(t) - \psi_*'(t),\ f(t,\psi_\varepsilon(t),\mu_t - \mu_{\varepsilon t})\rangle dt.$$

Lastly, we consider

$$\gamma(\varepsilon,\theta) = \|\mu_\theta - \mu_*\|_L = \|\mu_\varepsilon - \mu_* + \theta(\mu - \mu_\varepsilon)\|_L,\ -1 \le \theta \le 1,$$

where μ is an arbitrary relaxed control with $\mu_\varepsilon \in \Omega(t)$. It follows from the triangle inequality that $\gamma(\varepsilon,\cdot)$ is a convex function on $[-1,1]$. Hence γ has a right-hand derivative $\gamma'(\varepsilon,0+)$ at $\theta = 0$.

We next obtain bounds for $\gamma'(\varepsilon,0+)$. From the three-chord property of convex functions we get that for $0<\theta<\theta_0$,

$$\gamma(\varepsilon,0) - \gamma(\varepsilon,-1) \le \frac{\gamma(\varepsilon,\theta)-\gamma(\varepsilon,0)}{\theta} \le \gamma(\varepsilon,1) - \gamma(\varepsilon,0).$$

Hence

$$\|\mu_\varepsilon - \mu_*\|_L - \|2\mu_\varepsilon - \mu_* - \mu\|_L \le \gamma'(\varepsilon,0+) \le \|\mu - \mu_*\|_L - \|\mu_\varepsilon - \mu_*\|_L \le \|\mu - \mu_*\|_L.$$

Since $\|2\mu_\varepsilon - \mu - \mu_*\| = \|(\mu_\varepsilon - \mu^*) + (\mu_\varepsilon - \mu)\|$, it follows from the triangle inequality that the left side of the inequality is greater than or equal to $-\|\mu_\varepsilon - \mu\|$. From $\|\mu_\varepsilon - \mu\|_L = \|(\mu_\varepsilon - \mu_*) + (\mu_* - \mu)\|_L$, the triangle inequality and $\|\mu_\varepsilon - \mu_*\|_L < \varepsilon$, we have that

$$-\|\mu_\varepsilon - \mu\|_L \ge -\varepsilon - \|\mu_* - \mu\|_L.$$

Hence

$$-\varepsilon - \|\mu - \mu_*\|_L \le \gamma'(\varepsilon,0+) \le \|\mu - \mu_*\|_L. \tag{7.6.12}$$

The right-hand derivative $\rho_\varepsilon'(0+)$ is the sum of $A'(0)$, $B'(0)$, and $\varepsilon\gamma'(\varepsilon,0+)$. Hence

$$A'(0) + B'(0) + \varepsilon\gamma_\varepsilon'(0+) \ge 0. \tag{7.6.13}$$

Since

$$\begin{aligned}\widehat{f}(t,\psi(t),\mu_t - \mu_{\varepsilon t}) &\equiv \int_{\Omega(t)} \widehat{f}(t,\psi(t),z)d\mu_t - \int_{\Omega(t)} \widehat{f}(t,\psi(t),z)d\mu_{\varepsilon t}\\ &\equiv \widehat{f}(t,\psi(t),\mu_t) - \widehat{f}(t,\psi(t),\mu_{\varepsilon t}),\end{aligned}$$

we can transpose those terms in (7.6.13) involving μ_t to the right-hand side of the inequality and then divide through by $M(\varepsilon) > 0$ and use (7.6.9) to get

$$\begin{aligned}&\int_0^1 \Big[-M(\varepsilon)^{-1} f^0(t,\psi_\varepsilon(t),\mu_{\varepsilon t}) + \langle \lambda(\varepsilon,t), f(t,\psi_\varepsilon(t),\mu_{\varepsilon t})\rangle \\ &\quad - 2M(\varepsilon)^{-1}\langle \psi_\varepsilon'(t) - \psi_*'(t), f(t,\psi_\varepsilon(t),\mu_{\varepsilon t})\rangle\Big] dt + \varepsilon\gamma'(\varepsilon,0+)M(\varepsilon)^{-1}\\ &\ge \int_0^1 [-M(\varepsilon)^{-1} f^0(t,\psi_\varepsilon(t),\mu_t) + \langle \lambda(\varepsilon,t), f(t,\psi_\varepsilon(t),\mu_t)\rangle \\ &\quad - 2M(\varepsilon)^{-1}\langle \psi_\varepsilon'(t) - \psi_*'(t), f(t,\psi_\varepsilon(t),\mu_t)\rangle] dt.\end{aligned} \tag{7.6.14}$$

Equations (7.6.10), (7.6.11), and the inequality (7.6.14) constitute the ε-maximum principle, which a solution $(\psi_\varepsilon,\mu_\varepsilon)$ of the ε-problem satisfies.

7.7 The Maximum Principle; Compact Constraints

In this section we complete the proof of Theorem 6.3.5 by letting $\varepsilon \to 0$.

Let $L = \liminf_{\varepsilon\to 0} M(\varepsilon)$, where $M(\varepsilon)$ is given by (7.6.8). Then $1 \le L \le +\infty$. If L is finite, there exists a sequence $\varepsilon_n \to 0$ and a real number $0 < \lambda^0 \le 1$ such that

$$\lim_{n\to\infty} 1/M(\varepsilon_n) = \lambda^0.$$

If $L = +\infty$, then for every sequence $\varepsilon_n \to 0$, $\lim_{n\to\infty} M(\varepsilon_n) = +\infty$. Thus, we may always select a sequence $\varepsilon_n \to 0$ such that

$$\lim_{n\to\infty} (1/M(\varepsilon_n)) = \lambda^0 \qquad 0 \le \lambda^0 \le 1, \tag{7.7.1}$$

where if possible we select a sequence $\{\varepsilon_n\}$ such that $\lambda^0 > 0$.

It follows from (7.6.8) and (7.6.9) that $|\lambda(\varepsilon, 0)| \le 1$ for all $0 < \varepsilon < \varepsilon_1$. Hence there exists a subsequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$,

$$\lambda(0) = \lim_{n\to\infty} \lambda(\varepsilon_n, 0) \tag{7.7.2}$$

exists, and $|\lambda(0)| \le 1$.

We rewrite the differential equation (7.6.10) for $\lambda(\varepsilon, t)$ as

$$\lambda'(\varepsilon, t) = -f_x(\varepsilon, t)^T \lambda(\varepsilon, t) + N(\varepsilon, t), \tag{7.7.3}$$

where

$$N(\varepsilon, t) = M(\varepsilon)^{-1}[f_x^0(\varepsilon, t) + 2f_x(\varepsilon, t)^T(\psi'_\varepsilon(t) - \psi'_*(t))]. \tag{7.7.4}$$

Let $\Lambda(\varepsilon, t)$ denote the fundamental matrix of solutions with $\Lambda(\varepsilon, 0) = I$ for the homogeneous part of (7.7.3), namely the fundamental matrix of solutions of

$$q' = -f_x(\varepsilon, t)^T q. \tag{7.7.5}$$

Then by the variation of parameters formula

$$\lambda(\varepsilon, t) = \Lambda(\varepsilon, t)[\lambda(\varepsilon, 0) + \int_0^t \Lambda(\varepsilon, s)^{-1} N(\varepsilon, s) ds]. \tag{7.7.6}$$

By Lemma 6.6.2, Eq. (7.7.6) can be written as

$$\lambda(\varepsilon, t) = \Lambda(\varepsilon, t)[\lambda(\varepsilon, 0) + \int_0^t P(\varepsilon, s) N(\varepsilon, s) ds], \tag{7.7.7}$$

where $P(\varepsilon, t)$ is the fundamental matrix with $P(\varepsilon, 0) = I$ for the system $p' = f_x(\varepsilon, t)p$, which is adjoint to (7.7.5).

Lemma 7.7.1. *There exists a constant C such that for $0 \le t \le 1$ and $0 < \varepsilon \le \varepsilon_1$,*

$$|\lambda(\varepsilon, t)| \le C. \tag{7.7.8}$$

Moreover, the integrals

$$\int_E \lambda'(\varepsilon, t)dt \quad 0 < \varepsilon \le \varepsilon_1 \quad E \subset [0,1]$$

are equi-absolutely continuous.

Proof. The lemma will follow from a sequence of bounds on the terms in the right-hand sides of (7.7.3) and (7.7.7). □

By Lemma 7.5.2, $(\psi_\varepsilon, \mu_\varepsilon)$ is in $\mathcal{D}_\varepsilon$ for $0 < \varepsilon \le \varepsilon_1$. Hence by (7.5.2) and (7.5.3), all points $(t, \psi_\varepsilon(t))$ with $0 \le t \le 1$ and $0 < \varepsilon \le \varepsilon_1$ are contained in a compact set $[0,1] \times \mathcal{X} \subset \mathcal{I}_0 \times \mathcal{X}_0$. Hence by Assumption 6.3.1 and Remark 6.3.3 there exists an L_2 function M such that for $0 \le t \le 1$ and $0 < \varepsilon \le \varepsilon_1$,

$$|\widehat{f}_x(\varepsilon, t)| \equiv |\widehat{f}_x(t, \psi_\varepsilon(t), \mu_{\varepsilon t})| \le M(t) \qquad \text{a.e.,} \tag{7.7.9}$$

where $\widehat{f} = (f^0, f^1, \ldots, f^n)$.

From (7.7.9) we get that for $0 \le t \le 1$ and $0 < \varepsilon \le \varepsilon_0$

$$|\langle p, f_x(\varepsilon, t)p\rangle| \le \|p\|^2 M(t) \le M(t)(|p|^2 + 1) \quad \text{a.e.}$$

Hence by Lemma 4.3.14 there exists a constant $A > 0$ such that for $0 \le t \le 1$ and $0 < \varepsilon \le \varepsilon_1$,

$$|P(\varepsilon, t)| \le A. \tag{7.7.10}$$

Similarly, for $0 \le t \le 1$ and $0 < \varepsilon \le \varepsilon_1$

$$|\Lambda(\varepsilon, t)| \le A. \tag{7.7.11}$$

From (7.6.8) we get that $0 < M(\varepsilon)^{-1} \le 1$. From this, and from (7.7.4), (7.7.9), and (7.7.10), we get that

$$\left| \int_0^t P(\varepsilon, t)N(\varepsilon, s)ds \right| \le A \int_0^t M(s)ds + 2A \int_0^t M(s)|\psi_\varepsilon'(s) - \psi_*'(s)|ds.$$

From the Cauchy-Schwarz inequality and from the fact that $\psi_\varepsilon \in \mathcal{D}_\varepsilon$ we get that the second integral on the right is less than or equal to

$$\left(\int_0^1 M^2(s)ds \right)^{1/2} \|\psi_\varepsilon' - \psi_*'\| < (\varepsilon/2) \left(\int_0^1 M^2(s)ds \right)^{1/2}.$$

Thus,

$$\left| \int_0^t P(\varepsilon, s)N(\varepsilon, s)ds \right| \le A \int_0^1 M(s)ds + A\varepsilon \left(\int_0^1 M^2(s)ds \right)^{1/2}.$$

Hence there exists a constant $B > 0$ such that for all $0 \leq t \leq 1$ and $0 < \varepsilon \leq \varepsilon_0$

$$\left| \int_0^t P(\varepsilon, s)N(\varepsilon, s)ds \right| \leq B. \tag{7.7.12}$$

By (7.7.2) the sequence $\{\lambda(\varepsilon_n, 0)\}$ is bounded. From this and from (7.7.7), (7.7.11), and (7.7.12) the inequality (7.7.8) now follows.

We now show that the integrals $\int_E |\lambda'(\varepsilon, t)|dt$ are equi-absolutely continuous. From (7.7.3), (7.7.4), (7.7.8), (7.7.9) and the inequality $0 \leq M(\varepsilon)^{-1} \leq 1$, we get the existence of a function M in $L_2[0,1]$ such that for any measurable set $E \subset [0,1]$

$$\int_E |\lambda'(\varepsilon, t)|dt \leq (C+1)\int_E M(t)dt + 2\int_E M(t)|\psi_\varepsilon'(t) - \psi_*'(t)|dt.$$

From the Cauchy-Schwarz inequality and $\psi_\varepsilon \in \mathcal{D}_\varepsilon$, we get that the second integral on the right is less than or equal to

$$\left(\int_E M^2(t)dt\right)^{1/2} \|\psi_\varepsilon' - \psi_*'\| < (\varepsilon/2)\left(\int_E M^2(t)dt\right)^{1/2}.$$

Thus,

$$\int_E |\lambda'(\varepsilon t)|dt \leq (C+1)\int_E M(t)dt + \varepsilon\left(\int_E M^2(t)dt\right)^{1/2},$$

from which the equi-absolute continuity follows.

Let $\{\varepsilon_n\}$ be the sequence in (7.7.2) and let $\{\lambda(\varepsilon_n, \cdot)\}$ be the corresponding sequence of functions from the set $\{\lambda(\varepsilon, \cdot) \colon 0 < \varepsilon \leq \varepsilon_1\}$. It follows from Lemma 7.7.1 and Ascoli's theorem that there exists a subsequence $\{\lambda_n\} \equiv \{\lambda(\varepsilon_n, \cdot)\}$ that converges uniformly on $[0,1]$ to a function λ. The equi-absolute continuity of the integrals $\int_E |\lambda'(\varepsilon, t)|dt$ implies that the functions $\{\lambda_n\}$ are equi-absolutely continuous. Hence, by Lemma 5.3.3, the function λ is absolutely continuous. Corresponding to the sequence $\{\lambda_n\}$ there exist subsequences $\{\psi_n\} \equiv \{\psi_{\varepsilon_n}\}$ and $\{\mu_n\} \equiv \{\mu_{\varepsilon_n}\}$. We assert that $\psi_n \to \psi_*$ uniformly and that μ_n converges weakly to μ_*. We have

$$\begin{aligned} |\psi_n(t) - \psi_*(t)| &\leq |\psi_n(0) - \psi_*(0)| + \int_0^1 |\psi_n'(t) - \psi_*'(t)|dt \\ &\leq |\psi_n(0) - \psi_*(0)| + \|\psi_n' - \psi_*'\|. \end{aligned}$$

Since $(\psi_n, \mu_n) \in \mathcal{D}_{\varepsilon_n}$ and $\varepsilon_n \to 0$, it follows from (7.5.2) that ψ_n converges uniformly to ψ_* on $[0,1]$. It also follows from (7.5.2) that $\|\mu_n - \mu_*\|_L \to 0$ as $n \to \infty$. Hence by Corollary 7.3.4, $\mu_n \to \mu_*$ weakly.

Since $\psi_n \to \psi_*$ uniformly and $\mu_n \to \mu_*$ weakly, we get from Lemma 4.3.3 that for any measurable set $\Delta \subset [0,1]$

$$\lim_{n\to\infty} \int_\Delta \widehat{f}(t, \psi_n(t), \mu_{nt})dt = \int_\Delta \widehat{f}(t, \psi_*(t), \mu_{*t})dt.$$

Since Δ is arbitrary we get that

$$\lim_{n\to\infty} \widehat{f}(t,\psi_n(t),\mu_{nt}) = \widehat{f}(t,\psi_*(t),\mu_{*t}) \quad \text{a.e.} \tag{7.7.13}$$

A similar argument applied to $\widehat{f}_x$ gives

$$\lim_{n\to\infty} \widehat{f}_x(t,\psi_n(t),\mu_{nt}) = \widehat{f}_x(t,\psi_*(t),\mu_{*t}) \quad \text{a.e.} \tag{7.7.14}$$

By (vi) of Assumption 6.3.1, the convergence in (7.7.13) and (7.7.14) is dominated by an L_2 function M.

Define:

$$\begin{aligned}
\lambda_n(t) &= \lambda(\varepsilon_n,t)\\
f^0_{nx}(t) &\equiv f^0_x(\varepsilon_n,t) = f^0_x(t,\psi_n(t),\mu_{nt})\\
f_{nx}(t) &\equiv f_x(\varepsilon_n,t) = f_x(t,\psi_n(t),\mu_{nt}),
\end{aligned}$$

where f_x is the matrix with $i-j$ entry $(\partial f^i/\partial x^j)$. Then from (7.7.3) and (7.7.4) we have that

$$\lambda_n'(t) = M(\varepsilon_n)^{-1}f^0_{nx}(t) - f_{nx}(t)^T\lambda_n(t) + 2M(\varepsilon_n)^{-1}f_{nx}(t)(\psi_n'(t)-\psi_*'(t)),$$

where the superscript T denotes transpose. Hence

$$\begin{aligned}
\lambda_n(t) = \lambda_n(0) &+ \int_0^t [M(\varepsilon_n)^{-1}f^0_{nx}(s) - f_{nx}(s)^T\lambda_n(s)]ds \\
&+ 2M(\varepsilon_n)^{-1}\int_0^t f_{nx}(s)(\psi_n'(s)-\psi_*'(s))ds.
\end{aligned}\tag{7.7.15}$$

We now let $n\to\infty$ in (7.7.15). We first consider the rightmost term in (7.7.15). From (7.6.8) we have that $0 < M(\varepsilon)^{-1} \le 1$. From (7.7.9) we have that $|f_{nx}(t)| \le M(t)$, where M is in $L_2[0,1]$. Hence for all t in $[0,1]$

$$\begin{aligned}
\left|2M(\varepsilon)^{-1}\int_0^t f_{nx}(s)(\psi_n'(s)-\psi_*'(s))ds\right| &\le 2\int_0^t M(t)(|\psi_n'(s)-\psi_*'(s)|)ds \\
&\le 2\left(\int_0^1 M^2(s)ds\right)^{1/2}\|\psi_n'-\psi_*'\| < \varepsilon_n\left(\int_0^1 M^2(t)dt\right)^{1/2},
\end{aligned}\tag{7.7.16}$$

where the last inequality follows from (7.5.2). From (7.7.16) we get that the rightmost terms in (7.7.15) tend to zero uniformly on $[0,1]$ as $n\to\infty$.

Recall that $\lambda_n \to \lambda$ uniformly, that λ is absolutely continuous, that $|\lambda_n(t)| \le C$, that (7.7.1) holds, and that (7.7.13) and (7.7.14) hold, with $|f^0_n(t)| \le M(t)$ and $|f_{nx}(t)| \le M(t)$. Therefore, letting $n\to\infty$ in (7.7.15) gives the existence of an absolutely continuous function λ and a constant $0 \le \lambda^0 \le 1$ such that for $0 \le t \le 1$

$$\lambda(t) = \lambda(0) + \int_0^t [\lambda^0\overline{f}^0_x(s) - \overline{f}_x(s)^T\lambda(s)]ds, \tag{7.7.17}$$

where

$$\overline{f}^0_x(t) = f^0_x(t, \psi_*(t), \mu_{*t}) \quad \overline{f}_x = f_x(t, \psi_*(t), \mu_{*t}).$$

Differentiating both sides of (7.7.17) gives

$$\lambda'(t) = \lambda^0 \overline{f}^0_x(t) - \overline{f}_x(t)^T \lambda(t). \tag{7.7.18}$$

If we adjoin the equation $\lambda^{0'} = 0$ to the system (7.7.18) we see that (λ^0, λ) is the solution of a system of linear homogeneous differential equations. Hence $(\lambda^0, \lambda(t))$ never vanishes or is identically equal to zero. We shall show that $(\lambda^0, \lambda(0)) \neq 0$, and therefore $(\lambda^0, \lambda(t))$ never vanishes. If $\lambda^0 > 0$, then there is nothing to prove. By (7.7.1), if $\lambda^0 = 0$, then $M(\varepsilon_n) \to \infty$, where $M(\varepsilon_n)$ is given by (7.6.8). From (7.6.8) we have that $|\bar{\lambda}(\varepsilon_n, 0)| \to +\infty$. From (7.6.9) we have that

$$|\lambda_n(0)| = |\lambda(\varepsilon_n, 0)| = |\bar{\lambda}(\varepsilon_n, 0)|[1 + |\bar{\lambda}(\varepsilon_n, 0)]^{-1} = [\bar{\lambda}(\varepsilon_n, 0)^{-1} + 1]^{-1}.$$

Letting $n \to \infty$ gives $|\lambda(0)| = 1$. Thus, $(\lambda^0, \lambda(0)) \neq 0$ and therefore $(\lambda^0, \lambda(t))$ never vanishes.

Let $\{\varepsilon_n\}$ denote the subsequence such that $\lambda_n = \lambda(\varepsilon_n)$ converges uniformly to λ. We shall let $\varepsilon_n \to 0$ in the transversality condition (7.6.11). Since $\psi_n \to \psi_*$ uniformly and g is continuously differentiable, we have that $g_{x_i}(e(\psi_n)) \to g_{x_i}(e(\psi_*))$ for $i = 0, 1$. Also, $(\psi_n(0) - \psi_*(0)) \to 0$ and $M(\varepsilon_n)^{-1} \to \lambda^0$. Hence, as $\varepsilon_n \to 0$, the left-hand side of (6.11) tends to

$$\langle \lambda^0 g_{x_0}(e(\psi_*)) - \lambda(0), dx_0 \rangle + \langle \lambda^0 g_{x_1}(e(\psi_*)) + \lambda(1),\ dx_1 \rangle = 0. \tag{7.7.19}$$

We now let $\varepsilon_n \to 0$ in (7.6.14). We first consider the third term on the left. From $0 < M(\varepsilon)^{-1} \leq 1$, from (v) of Assumption 6.3.1, and from (7.5.2) we have that

$$\begin{aligned}
&\left| \int_0^1 2M(\varepsilon_n)^{-1} \langle \psi_n'(t) - \psi_*'(t), f(t, \psi_n(t), \mu_{nt}) \rangle dt \right| \\
&\qquad \leq 2 \int_0^1 |\langle \psi_n'(t) - \psi_*'(t), f(t, \psi_n(t), \mu_{nt}) \rangle| dt \\
&\qquad \leq 2 \int_0^1 |\psi_n'(t) - \psi_*'(t)| M(t) dt \\
&\qquad \leq 2\|M\| \|\psi_n' - \psi_*'\| < \varepsilon_n \|M\|,
\end{aligned}$$

where $\|\ \|$ denotes the L_2 norm. A similar estimate holds for the third term on the right. Therefore, these terms tend to zero as $\varepsilon_n \to 0$.

From (7.6.12) and $0 < M(\varepsilon_n)^{-1} \leq 1$, it follows that $\varepsilon_n \gamma'(\varepsilon_n, 0+) M(\varepsilon_n)^{-1} \to 0$ as $\varepsilon_n \to 0$.

To find the limit as $\varepsilon_n \to 0$ of the first two terms on the left in (7.6.14), let

$$f^0_\varepsilon(t) = f^0(t, \psi_\varepsilon(t), \mu_{\varepsilon t}) \qquad f_\varepsilon(t) = f(t, \psi_\varepsilon(t), \mu_{\varepsilon t})$$

and write the sum of these terms as

$$\int_0^1 (-M(\varepsilon_n)^{-1} + \lambda^0) f^0_{\varepsilon_n}(t)dt + \int_0^1 (\lambda(\varepsilon_n, t) - \lambda(t)) f_{\varepsilon_n}(t)dt \qquad (7.7.20)$$
$$+ \int_0^1 [-\lambda^0 f^0_{\varepsilon_n}(t) + \langle \lambda(t), f_{\varepsilon_n}(t)\rangle]dt.$$

The sum of the first two terms in (7.7.20) is in absolute value less than

$$| - M(\varepsilon_n)^{-1} + \lambda^0| \int_0^1 M(t)dt + \int_0^1 |\lambda(\varepsilon_n, t) - \lambda(t)| M(t)dt.$$

Since $M(\varepsilon_n)^{-1} \to \lambda^0$ and $\lambda(\varepsilon_n, t) \to \lambda(t)$ uniformly as $\varepsilon_n \to 0$, the sum of the first two terms in (7.7.20) tends to zero as $\varepsilon_n \to 0$. Since $\psi_n \to \psi_*$ uniformly and $\mu_{\varepsilon n} \to \mu_*$ weakly, it follows from Lemma 4.3.3 that as $\varepsilon_n \to 0$, the integrand in the last term in (7.7.20) converges to $-\lambda^0 \overline{f}^0(t) + \langle \lambda(t), \overline{f}(t)\rangle$, where $\overline{f}^0(t) = f^0(t_1\psi_*(t), \mu_{*t})$ and $\overline{f}(t) = f(t, \psi_*(t), \mu_*(t))$. Moreover, the convergence is dominated by $(1+A)M(t)$, where A is a bound for the sequence $\{|\lambda(\varepsilon, 0)|\}$. Thus, the left side of (7.6.14) converges to

$$\int_0^1 [-\lambda^0 \overline{f}^0(t) + \langle \lambda(t), \overline{f}(t)\rangle]dt.$$

A similar argument shows that the right side of (7.6.14) converges to

$$\int_0^1 [-\lambda^0 f^0(t, \psi_*(t), \mu_t) + \langle \lambda(t), f(t, \psi_*(t), \mu_t)\rangle]dt.$$

In summary, we get that as $\varepsilon_n \to 0$, the inequality (7.6.14) tends to

$$\int_0^1 [-\lambda^0 \overline{f}^0(t) + \langle \lambda(t), \overline{f}(t)\rangle dt \qquad (7.7.21)$$
$$\geq \int_0^1 [-\lambda^0 f^0(t, \psi_*(t), \mu_t) + \langle \lambda(t), f(t, \psi_*(t), \mu_t)\rangle]dt$$

for all relaxed controls μ with $\mu_t \in \Omega(t)$.

If in (7.7.18), (7.7.19), and (7.7.20) we set $\widetilde{\lambda}^0 = -\lambda^0$, and then relabel $\widetilde{\lambda}^0$ as λ^0 we obtain the conclusion of Theorem 6.3.5.

If (ψ_*, μ_*) is an admissible relaxed optimal pair, then there exists a constant $-1 \leq \lambda^0 \leq 0$ and an absolutely continuous function λ such that for all $t \in [0, 1]$, $(\lambda^0, \lambda(t)) \neq 0$ and

$$\lambda'(t) = -\lambda^0 \overline{f}^0_x(t) - \overline{f}_x(t)^T \lambda(t), \qquad (7.7.22)$$

where $\overline{f}^0_x(t)$ and $\overline{f}_x(t)$ are defined in (7.7.17). At the end point $e(\psi_*)$, $(\lambda^0, \lambda(t))$ satisfies

$$\langle -\lambda^0 g_{x_0}(e(\psi_*)) - \lambda(0), dx_0\rangle + \langle -\lambda^0 g_{x_1}(e(\psi_*)) + \lambda(1),\ dx_1\rangle = 0 \qquad (7.7.23)$$

for all tangent vectors (dx_0, dx_1) *to* $\mathcal{B}$ *at* $e(\psi_*)$. *If* $e(\psi_*)$ *is a boundary point of* $\mathcal{B}$, *then (7.7.23) holds with* $=$ *replaced by* ≥ 0, *for all tangent vectors* (dx_0, dx_1) *whose projections point into* $\mathcal{B}$. *For all relaxed controls* μ *with* $\mu_t \in \Omega(t)$,

$$\begin{aligned}
&\int_0^1 [\lambda^0 \overline{f}^0(t) + \langle \lambda(t), \overline{f}(t) \rangle] dt \\
&\geq \int_0^1 [\lambda^0 f^0(t, \psi_*(t), \mu_t) + \langle \lambda(t), f(t, \psi_*(t), \mu_t \rangle] dt.
\end{aligned}$$

We leave the statement of the above result in terms of the function H_r to the reader.

7.8 Proof of Theorem 6.3.9

Since in the proof we will again compare the optimal pair with other admissible pairs, we denote the optimal pair by (ψ^*, μ^*), where

$$\mu_t^* = \sum_{i=1}^{n+2} p^{*i}(t) \delta_{u_i^*(t)}.$$

If all the functions u_i^*, $i = 1, \ldots, n+2$ are bounded, then their values $u_i^*(t)$ are all contained in a compact set Z in $\mathbb{R}^m$. Hence, (ψ^*, μ^*) will be optimal for the problem with constraint condition $\mu_t \in \Omega'(t)$, where $\Omega'(t) = \Omega(t) \cap Z$. Since all the sets $\Omega'(t)$ are contained in the compact set Z, Theorem 6.3.5 is applicable to this problem. We therefore assume that not all of the u_i^* are bounded.

Since each u_i^* is finite a.e. there exists a positive integer k_0 such that for $k > k_0$,

$$G_k = \{t \colon |u_i^*(t)| \leq k, \quad i = 1, \ldots, n+2\}$$

is non-empty and measurable. For $k > k_0$ and $i = 1, \ldots, n+2$, set

$$E_{ki} = \{t \colon |u_i^*(t)| > k\},$$

and set $E_k = \cup_{i=1}^{n+2} E_{ki}$. For each k the sets E_{ki} are measurable and all are not empty, so therefore E_k is non-empty and measurable. If $I = [0, 1]$, then

$$I = G_k \cup E_k \qquad \text{meas } E_k \to 0 \text{ as } k \to \infty.$$

For each $k > K$ define a mapping Ω_k from I to subsets of $\mathbb{R}^m$ by the formula

$$\Omega_k(t) = (\text{cl } \ \Omega(t)) \cap \text{cl } \ B(0, k),$$

where cl denotes closure and $B(0,k)$ is the ball in $\mathbb{R}^m$ of radius k, centered at the origin. All the sets $\Omega_k(t)$ are compact and are contained in the compact set cl $B(0,k)$. By hypothesis, all the mappings $\Omega_k : t \to \Omega_k(t)$ are u.s.c.i. Also,

$$\Omega_k(t) \subseteq \Omega_{k+1}(t) \quad \text{and} \quad \Omega(t) = \bigcup_{k=1}^{\infty} \Omega_k(t).$$

For each positive integer $k > k_0$ we define Problem k to be:

$$\begin{aligned}
\text{Minimize:} \quad & g(e(\psi)) + \int_0^1 [\mathcal{X}_{E_k}(t) f^0(t,x,\mu_t^*) + \mathcal{X}_{G_k}(t) f^0(t,x,\mu_t)]\, dt \\
\text{Subject to:} \quad & \frac{dx}{dt} = \mathcal{X}_{E_k}(t) f(t,x,\mu_t^*) + \mathcal{X}_{G_k}(t) f(t,x,\mu_t) \\
& \mu_t \in \Omega_k(t) \quad e(\psi) \in \mathcal{B},
\end{aligned}$$

where $\mathcal{X}_{E_k}$ is the characteristic function of E_k and $\mathcal{X}_{G_k}$ is the characteristic function of G_k.

For each $k > K$ define a relaxed control μ_k as follows:

$$\mu_{kt} = \begin{cases} \mu_t^* & \text{if } t \in G_k \\ & \text{an arbitrary discrete measure control, } \mu_t \in \Omega_k(t) \\ & \text{if } t \in E_k \end{cases}$$

By Lemma 3.4.5 $\mu_t \in \Omega_k(t)$ exists. For $t \in E_k$, since μ_{kt} is a probability measure,

$$\int_{\Omega_k(t)} \widehat{f}(t,x,\mu_t^*) d\mu_{kt} = \widehat{f}(t,x,\mu_t^*),$$

where $\widehat{f} = (f^0, f)$. The trajectory ψ_k corresponding to μ_k satisfies

$$\begin{aligned}
\psi_k'(t) &= \mathcal{X}_{E_k}(t) f(t,\psi_k(t),\mu_t^*) + \mathcal{X}_{G_k}(t) f(t,\psi_k(t),\mu_t^*) \\
&= f(t,\psi_k(t),\mu_t^*).
\end{aligned}$$

The optimal trajectory ψ^* also satisfies this differential equation. It follows from (6.3.13) in the hypothesis of Theorem 6.3.9 and from standard uniqueness theorems for differential equations that if we set $\psi_k(0) = \psi^*(0)$, then $\psi_k(t) = \psi^*(t)$. From this it further follows that

$$\int_0^t f^0(s,\psi_k(s),\mu_{ks}) ds = \int_0^t f^0(s,\psi^*(s),\mu_s^*) ds$$

and that $e(\psi_k) = e(\psi^*) \in \mathcal{B}$. Hence

$$J(\psi_k,\mu_k) = J(\psi^*,\mu_k) = J(\psi^*,\mu^*). \tag{7.8.1}$$

We assert that (ψ^*,μ_k) is optimal for Problem k. To prove this assertion we

first note that Problem k satisfies the hypotheses of Theorem 4.3.5, and thus an optimal pair $(\overline{\psi}, \overline{\mu})$ exists. This pair is admissible for the original problem. Thus, if (ψ^*, μ_k) were not optimal for Problem k we would have that

$$J(\overline{\psi}, \overline{\mu}) < J(\psi_k, \mu_k) = J(\psi^*, \mu^*),$$

the equality coming from (7.8.1). This, however, would contradict the optimality of (ψ^*, μ^*), which proves the assertion.

Since (ψ^*, μ_k) is optimal for Problem k and, as can be readily checked, the data of Problem k satisfy the hypotheses of Theorem 6.3.5, the pair (ψ_*, μ_k) satisfies the necessary conditions of Theorem 6.3.5. Thus, there exists a constant $\lambda_k^0 \leq 0$ and an absolutely continuous function $\lambda_k = (\lambda_k^1, \ldots, \lambda_k^n)$ on $[0, 1]$ such that $(\lambda_k^0, \lambda_k(t)) \neq 0$ for all $t \in [0, 1]$, and such that:
(i)

$$\begin{aligned} \lambda_k'(t) &= -\lambda_k^0[\mathcal{X}_{E_k}(t) f_x^0(t, \psi^*(t), \mu_t^*) + \mathcal{X}_{G_k}(t) f_x^0(t, \psi^*(t), \mu_{kt})] \\ &\quad - [\mathcal{X}_{E_k}(t) f_x^T(t, \psi^*(t), \mu_t^*) + \mathcal{X}_{G_k}(t) f_x^T(t, \psi^*(t), \mu_{kt})]\lambda_k(t) \\ &= -\lambda_k^0 f_x^0(t, \psi^*(t), \mu_t^*) - f_x^T(t, \psi^*(t), \mu^*(t))\lambda_k^T(t), \end{aligned} \tag{7.8.2}$$

where f_x is the matrix with entry in row i column j $(\partial f^i/\partial x^j)$ and the superscript T indicates transpose,
(ii)

$$\langle -\lambda_k^0 g_{x_0}(e(\psi^*)) - \lambda_k(0), dx_0\rangle + \langle -\lambda_k^0 g_x(e(\psi^*)) + \lambda_k(1), dx_1\rangle = 0 \tag{7.8.3}$$

for all tangent vectors (dx_0, dx_1) to $\mathcal{B}$ at $e(\psi^*)$, and
(iii)

$$\begin{aligned} &\int_0^1 [\lambda_k^0 f^0(t, \psi^*(t), \mu_t^*) + \langle \lambda_k(t), f(t, \psi^*(t), \mu_t^*)\rangle]\, dt \\ &\geq \int_0^1 \{\mathcal{X}_{E_k}(t)[\lambda^0 f^0(t, \psi^*(t), \mu_t^*) + \langle \lambda_k(t), f(t, \psi^*(t), \mu_t^*)\rangle] \\ &\quad + \mathcal{X}_{G_k}(t)[\lambda_k^0 f^0(t, \psi^*(t), \mu_t) + \langle \lambda_k(t), f(t, \psi^*(t), \mu_t)\rangle]\} dt \end{aligned} \tag{7.8.4}$$

for all $\mu_t \in \Omega_k(t)$.

All (λ_k^0, λ_k) satisfy the differential equation (7.8.2), but may be different for different k because of different initial conditions $(\lambda_k^0, \lambda_k(0))$. By Remark 6.3.8, we may assume that for each k, $|(\lambda_k^0, \lambda_k(0))| = 1$. Hence there exists a subsequence $\{(\lambda_k^0, \lambda_k(0))\}$ and a vector $(\lambda^0, \lambda(0))$ such that $(\lambda_k^0, \lambda_k(0))$ converges to $(\lambda^0, \lambda(0))$. Since $|(\lambda_k^0, \lambda_k(0))| = 1$, we have $|(\lambda^0, \lambda(0))| = 1$.

Let $\Lambda(t)$ denote the fundamental matrix of solutions of

$$q' = -f_x^T(t, \psi^*(t), \mu_t^*)q \tag{7.8.5}$$

with $\Lambda(0) = I$. Let $P(t)$ denote the fundamental matrix of solutions of $p' = f_x(t, \psi^*(t), \mu_t^*)p$, the system adjoint to (7.8.5), with $P(0) = I$. Then

by arguments used to establish (7.7.10) and (7.7.11) we get that there exists a positive constant C such that $|\Lambda(t)| \leq C$ and $|P(t)| \leq C$. By the variation of parameters formula and (7.8.2)

$$\lambda_k(t) = \Lambda(t)[\lambda_k(0) - \int_0^t \lambda_k^0 P(s) f_x^0(s, \psi^*(s), \mu_s^*) ds].$$

Let

$$\lambda(t) = \Lambda(t)[\lambda(0) - \int_0^t \lambda^0 P(s) f_x^0(s, \psi^*(s), \mu_s^*) ds].$$

Then

$$\lambda'(t) = -\lambda^0 f_x^0(t, \psi^*(t), \mu_t^*) - f_x^T(t, \psi^*(t), \mu_t^*)\lambda(t) \tag{7.8.6}$$

and

$$\begin{aligned}|\lambda_k(t) - \lambda(t)| &\leq |\Lambda(t)|[|\lambda_k(0) - \lambda(0)| \\ &\quad + \int_0^t |P(s)||\lambda_k^0 - \lambda^0| |f_x^0(s, \psi^*(s), \mu_s^*)| ds] \\ &\leq C[|\lambda_k(0) - \lambda(0)| + |\lambda_k^0 - \lambda^0| \int_0^1 |f_x^0(t, \psi^*(t), \mu_t^*)| dt].\end{aligned}$$

By (6.3.1), the last integral on the right is bounded. Since $\lambda_k^0 \to \lambda^0$ and $\lambda_k(0) \to \lambda(0)$, it follows that $\lambda_k \to \lambda$ uniformly.

Also, $|(\lambda^0, \lambda(0))| = 1$. By adjoining $\lambda^{0'} = 0$ to (7.8.6) we get that (λ^0, λ) is a solution to a system of linear homogeneous differential equations. Therefore, (λ^0, λ) either never vanishes or is identically zero. Since $(\lambda^0, \lambda(0)) \neq 0$, we get that $(\lambda^0, \lambda(t))$ never vanishes.

If we let $k \to \infty$ in (7.8.3) we get that

$$\langle -\lambda^0 g_{x_0}(e(\psi^*)) - \lambda(0), dx_0 \rangle + \langle -\lambda^0 g_{x_1}(e(\psi^*)) + \lambda(1), dx_1 \rangle = 0 \tag{7.8.7}$$

for all tangent vectors (dx_0, dx_1) to $\mathcal{B}$ at $e(\psi^*)$.

The inequality (7.8.4) can be written as

$$\begin{aligned}&\int_0^1 \mathcal{X}_{G_k}(t)[\lambda^0 f^0(t, \psi^*(t), \mu_t^*) + \langle \lambda_k(t), f(t, \psi^*(t), \mu_t^*) \rangle] \, dt \\ &\geq \int_0^1 \mathcal{X}_{G_k}(t)[\lambda^0 f^0(t, \psi^*(t), \mu_t) + \langle \lambda_k(t), f(t, \psi^*(t), \mu_t) \rangle] \, dt\end{aligned} \tag{7.8.8}$$

for all $\mu_t \in \Omega_k(t)$. Let μ be an arbitrary discrete measure control with $\mu_t \in \Omega(t)$ and

$$\mu_t = \sum_{i=1}^{n+2} p^i(t) \delta_{u_i(t)}.$$

Then

$$\widehat{f}(t, \psi^*(t), \mu_t) = \sum_{i=1}^{n+2} p^i(t) \widehat{f}(t, \psi^*(t), u_i(t)).$$

By Lemma 3.4.5, for each positive integer $k > K$ there exists a measurable function $\widetilde{u}$ such that $\widetilde{u}_k(t) \in \Omega_k(t)$ a.e. in $[0,1]$. For each $k > K$ let

$$v_{ki}(t) = \begin{cases} u_i(t) & \text{if } |u_i(t)| \leqq k \\ \widetilde{u}_k(t) & \text{if } |u_i(t)| > k \end{cases}$$

Let ν_k be the discrete measure control given by

$$\nu_{kt} = \sum_{i=1}^{n+2} p^i(t)\delta v_{ki}(t)$$

Then (7.8.8) holds with μ_t replaced by ν_{kt}. Since the controls u_i are finite a.e., for a.e. t in $[0,1]$ there exists a positive integer $k_0(t)$ such that $t \in G_k$ and $|u_i(t)| \leqq k$ for all $k > k_0(t)$. Hence for a.e. fixed t and $k > k_0(t)$, $\mathcal{X}_{G_k}(t)\widehat{f}(t,\psi^*(t),\nu_{kt}) = \widehat{f}(t,\psi^*(t),\mu_t)$. In other words,

$$\lim_{k\to\infty} \mathcal{X}_{G_k}(t)\widehat{f}(t,\psi^*(t),\mu_{kt}) = \widehat{f}(t,\psi^*(t),\mu_t)$$

for a.e. t in $[0,1]$. Also, $(\lambda^0,\lambda_k(t))$ converges uniformly to $(\lambda^0,\lambda(t))$. Thus, the integrand on the right-hand side of (7.8.8) with μ_t replaced by ν_{kt} converges a.e. to

$$\lambda^0 f^0(t,\psi^*(t),\mu_t) + \langle\lambda(t), f(t,\psi^*(t),\mu_t)\rangle.$$

Since for all z in the union of the ranges of $u_1,\ldots,u_{n+2}$,

$$|\widehat{f}(t,\psi^*(t),z)| \leq M(t),$$

where M is in $L_1[0,1]$, the convergence is dominated. Also, $\lim \mathcal{X}_{G_k}(t) = 1$ a.e. Using this result in the integral on the left in (7.8.8) and the preceding result gives

$$\begin{aligned}\int_0^1 [\lambda^0 f^0(t,\psi^*(t),\mu_t^*) + \langle\lambda(t), f(t,\psi^*(t),\mu_t^*)\rangle]\,dt \\ \geq \int_0^1 [\lambda^0 f^0(t,\psi^*(t),\mu_t) + \langle\lambda(t), f(t,\psi^*(t)),\mu_t)\rangle]dt\end{aligned} \tag{7.8.9}$$

for all μ with $\mu_t \in \Omega(t)$. Relations (7.8.6), (7.8.7), (7.8.9), and $(\lambda^0,\lambda(t)) \neq 0$ for all t establish Theorem 6.3.9.

7.9 Proof of Theorem 6.3.12

Conclusions (i) and (iii) of Theorem 6.3.12 follow immediately from Theorems 6.3.5 and 6.3.9. The proof of (ii) requires the notion of *point of density* of a measurable set and the notion of *approximate continuity* of a measurable function.

Definition 7.9.1. Let E be a measurable set on the line, let t_0 be a point in E, let $h > 0$, and let $I(h)$ denote the interval $[t_0 - h, t_0 + h]$. The point t_0 is a *point of density* of E if

$$\lim_{h \to 0} \text{ meas } (E \cap I(h))/2h = 1.$$

If meas $E > 0$, then *almost all points of E are points of density of E*. See [74, pp. 260–261].

Definition 7.9.2. Let f be a measurable function defined on a closed interval $[a, b]$. If there exists a measurable subset of E of $[a, b]$ having a point t_0 as a point of density such that f is continuous at t_0 with respect to E, then f is said to be *approximately continuous* at t_0.

A measurable function f defined on a closed interval $[a, b]$ is *approximately continuous at almost all points of* $[a, b]$. See [74, p. 262].

We now take up the proof of (ii). Let $\mathcal{Z}_1$ denote the set of points of $\mathcal{C}$ with rational coordinates. Let $\mathcal{Z}_2$ denote the set of points of $\mathcal{C}$ that are isolated points or limit points belonging to $\mathcal{C}$ of isolated points of $\mathcal{C}$. Then $\mathcal{Z}_2$ is denumerable (see [74, Theorem 2, p. 50]), and therefore so is the set $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2$.

If (6.3.15) were not true, then there would exist a set $E_k \subset P_k$ of positive measure such that for $t \in E_k$

$$H(t, \psi(t), u_k(t), \widehat{\lambda}(t)) < M(t, \psi(t), \widehat{\lambda}(t)). \tag{7.9.1}$$

For each z_i in $\mathcal{Z}$, the function

$$t \to H(t, \psi(t), u_k(t), \widehat{\lambda}(t)) - H(t, \psi(t), z_i, \widehat{\lambda}(t)) \tag{7.9.2}$$

is approximately continuous at all $t \in [0, 1]$ except for those in a set $T_i \subset [0, 1]$ of measure zero. Let $T = [0, 1] - \bigcup T_i$. Then *all* of the functions defined in (7.9.2) are approximately continuous at all points of T.

Let $\tau \in E_k \cap T$. Then from (7.9.1) we get that there exists a z in $\mathcal{Z}$, and therefore in $\mathcal{C}$, such that

$$H(\tau, \psi(\tau), u_k(\tau), \widehat{\lambda}(t)) - H(\tau, \psi(\tau), z, \widehat{\lambda}(t)) < 0.$$

From the approximate continuity at $t = \tau$ of the function defined in (7.9.2) with z_i replaced by z, we get that there exists a measurable set $E \subset E_k \cap T$ of positive measure such that for $t \in E$

$$H(t, \psi(t), u_k(t), \widehat{\lambda}(t)) < H(t, \psi(t), z, \widehat{\lambda}(t)). \tag{7.9.3}$$

Let ν be the discrete measure control given by $\nu_t = \mu_t$ if $t \notin E$ and

$$\nu_t = p^k(t)\delta_z + \sum_{i \neq k} p^i(t)\delta_{u_i(t)}$$

if $t \in E$. Then

$$\int_0^1 \{H_r(t, \psi(t), \mu_t, \widehat{\lambda}(t)) - H_r(t, \psi(t), \nu_t, \widehat{\lambda}(t))\}dt$$
$$= \int_E p^k(t)\{H(t, \psi(t), u_k(t), \widehat{\lambda}(t)) - H(t, \psi(t), z, \widehat{\lambda}(t))\}dt < 0.$$

This contradicts (6.3.7), which establishes (ii).

Remark 7.9.3. If we assume that $\widehat{f}$ is continuous on $\mathcal{I}_0 \times \mathcal{X}_0 \times \mathcal{U}_0$, the proof of (ii) is somewhat simpler. We again note that if (6.3.15) were not true, then there would exist a set $E_k \subset P_k$ of positive measure such that (7.9.1) holds. The function h defined by $h(t) = H(t, \psi(t), u_k(t), \widehat{\lambda}(t))$ is measurable. Hence by Lusin's theorem there exists a closed set E_0 of positive measure such that $E_0 \subset E_k$ and h is continuous on E_0. Let τ be a point of density of E_0. Then $H(\tau, \psi(\tau), u_k(\tau), \widehat{\lambda}(\tau)) < M(\tau, \psi(\tau), \widehat{\lambda}(\tau))$, so there exists a z in $\mathcal{C}$ such that

$$H(\tau, \psi(\tau), u_k(\tau), \widehat{\lambda}(\tau)) < H(\tau, \psi(\tau), z, \widehat{\lambda}(t)).$$

Since h is continuous on E_0, the function $t \to H(t, \psi(t), z, \widehat{\lambda}(t))$ is continuous in E_0 and since τ is a point of density of E_0, there exists a measurable set $E \subset E_0$ of positive measure such that (7.9.3) holds for $t \in E$.

7.10 Proof of Theorem 6.3.17 and Corollary 6.3.19

The only conclusions of the theorem that require proof are (6.3.21) and (6.3.22). We noted in (6.3.11) that if μ is a discrete measure control, then

$$H_r(t, x, \mu_t, q^0, q) = \langle \widehat{q}, \widehat{f}(t, x, \mu_t)\rangle = \sum_{k=1}^{n+2} p^k(t)\left(\sum_{j=0}^{n+1} q^j f^j(t, x, u_k(t)\right)$$
$$= \sum_{k=1}^{n+2} p^k(t) H(t, x, u_k(t), q^0, q).$$

The functions $(p^1, \ldots, p^{n+2}, u_1, \ldots, u_{n+2})$ are all measurable. Hence there exists a measurable set $E \subseteq [0, 1]$ of full measure such that all of these functions are approximately continuous on E. Thus, if ω is any of the p^i or u_i, and t' is a point of E, then

$$\lim_{\substack{t \to t' \\ t \in E}} \omega(t) = \omega(t').$$

We shall describe this succinctly by saying that μ is approximately continuous on E and that

$$\lim_{\substack{t \to t' \\ t \in E}} = \mu_{t'}.$$

We are assuming that all of the $u_i, i = 1, \ldots, n+2$ are bounded; that is there exists a constant K such that $|u_i(t)| \leq K$ for all $t \in [0,1]$ and all u_i, $i = 1, \ldots, n+2$. The p^i are by definition bounded. We shall summarize this by saying that μ is bounded. Now let (ψ, μ) be an optimal relaxed pair and (λ^0, λ) a corresponding set of multipliers. Then, since $\widehat{f}$ is assumed to be continuous on $\mathcal{I}_0 \times \mathcal{X}_0 \times \mathcal{U}_0$, for $t' \in E$

$$\lim_{\substack{t \to t' \\ t \in E}} H(t, \psi(t), \mu_t, \lambda^0, \lambda(t)) = H(t', \psi(t'), \mu_{t'}, \lambda^0, \lambda(t')).$$

We now establish (6.3.21). Let

$$h(t) = H_r(t, \psi(t), \mu_t, \widehat{\lambda}(t)). \tag{7.10.1}$$

Let T_1 denote the set of points t in $[0,1]$ at which (6.3.18) holds. Then meas $T_1 =$ meas $[0,1]$. Let t and t' be in T_1 with $t > t'$ and let

$$\Delta t = t - t' \quad \Delta\psi = \psi(t) - \psi(t') \quad \Delta\widehat{\lambda} = \widehat{\lambda}(t) - \widehat{\lambda}(t').$$

Let

$$P(s) = P(s, t', t) = (t' + s\Delta t,\ \psi(t') + s\Delta\psi, \mu_t, \widehat{\lambda}(t') + s\Delta\widehat{\lambda}).$$

Then, by (7.10.1) and (6.3.18),

$$\begin{aligned} h(t) - h(t') &\leq H_r(t, \psi(t), \mu_t, \widehat{\lambda}(t)) - H_r(t', \psi(t'), \mu_t, \widehat{\lambda}(t')) \\ &= H_r(P(1)) - H_r(P(0)). \end{aligned} \tag{7.10.2}$$

The function $s \to H_r(P(s))$ is continuously differentiable on $[0,1]$. Hence, by the Mean Value Theorem, there exists a real number θ in $(0,1)$ such that

$$H_r(P(1)) - H_r(P(0)) = dH_r/ds]_{s=\theta}. \tag{7.10.3}$$

We have

$$dH_r/ds]_{s=\theta} = H_{rt}(P(\theta))\Delta t + \langle H_{rx}(P(\theta)), \Delta\psi\rangle + \langle H_{rq}(P(\theta)), \Delta\widehat{\lambda}\rangle. \tag{7.10.4}$$

Since ψ and λ are continuous on $[0,1]$ and λ^0 is a constant, all of these functions are bounded on $[0,1]$. Hence so are the $\Delta\psi$ and $\Delta\widehat{\lambda}$ for all points t', t in T_1. By assumption μ is bounded on $[0,1]$. Hence there exists a closed ball B of finite radius such that for all t', t in T_1 and all $0 \leq s \leq 1$, the points $P(s)$ are in B. It then follows from the continuity of f, $\widehat{f}_t$, and $\widehat{f}_x$ on $\mathcal{I}_0 \times \mathcal{X}_0 \times \mathcal{U}_0$ that there exists a constant $K_1 > 0$ such that for all t, t' in T_1 and all $0 \leq s \leq 1$,

$$|H_t(P(s;t',t))| \leq K_1 \quad |H_x(P(s;t',t))| \leq K_1 \quad |H_q(P(s;t',t))| \leq K_1. \tag{7.10.5}$$

From (6.3.14) we have that

$$\Delta\psi = \int_{t'}^{t} H_q(s, \psi(s), \mu_s; \widehat{\lambda}(s))ds \quad \Delta\lambda = -\int_{t'}^{t} H_x(s, \psi(0), \mu_s, \widehat{\lambda}(0))ds.$$

From these relations, the boundedness of ψ, μ, and $\widehat{\lambda}$ on $[0,1]$, and the continuity of H_x and H_q, there exists a constant $K_2 > 0$ such that for all t', t in T_1

$$|\Delta\psi| \leq K_2\Delta t \qquad |\Delta\lambda| \leq K_2\Delta t.$$

From this and from (7.10.5), (7.10.4), (7.10.3), and (7.10.2), we get that there exists a constant $K > 0$ such that for all t', t in T_1,

$$h(t) - h(t') \leq K(t - t'). \tag{7.10.6}$$

Also,

$$h(t) - h(t') \geq H_r(t, \psi(t), \mu_{t'}, \lambda(t)) - H_r(t', \psi(t'), \mu_{t'}, \widehat{\lambda}(t')).$$

By arguments similar to those used to obtain (7.10.6) we get that $h(t)-h(t') \geq -K(t-t')$. Combining this inequality with (7.10.6) gives

$$|h(t) - h(t')| \leq K|t - t'|$$

for all t', t in T_1.

Thus, the function h is Lipschitz continuous on a dense set in $[0,1]$. It is an easy exercise in elementary analysis to show that h can be extended to a function $\widetilde{h}$ that is Lipschitz on $[0,1]$ with the same Lipschitz constant as h. Since $\widetilde{h}$ is Lipschitz continuous, it is absolutely continuous. If we now write $\widetilde{h}$ as h, we have (6.3.21). Since h is absolutely continuous, it is differentiable almost everywhere and

$$h(t) = c + \int_0^t h'(s)ds.$$

We now calculate h'. Recall that T_1 denotes the set of points t in $[0,1]$ at which (6.3.18) holds. Let $T_2 = E$, the set of points at which μ is approximately continuous. Let T_3 denote the set of points at which h is differentiable, let T_4 denote the set of points at which ψ is differentiable, and let T_5 denote the set of points at which λ is differentiable. Let T_6 denote the set of points at which (6.3.18) holds. Let T denote the intersection of the sets $T_i, i = 1, \ldots, 6$. The set has full measure. Let t be a point of T. Since $h'(t)$ exists we have that

$$h'(t) = \lim_{t_k \to t} \frac{h(t_k) - h(t)}{t_k - t},$$

where $\{t_k\}$ is a sequence of points in T. Let

$$\Delta t_k = t_k - t \qquad \Delta\psi_k = \psi(t_k) - \psi(t) \qquad \Delta\widehat{\lambda}_k = \widehat{\lambda}(t_k) - \widehat{\lambda}(t),$$

and let

$$\widetilde{P}(s;\Delta t_k) = (t + s\Delta t_k, \psi(t) + s\Delta\psi_k, \mu_{t_k}, \lambda(t) + s\Delta\lambda_k) \qquad 0 \le s \le 1.$$

As in (7.10.2) we have

$$h(t_k) - h(t) \le H_r(\widetilde{P}(1;\Delta t_k)) - H_r(\widetilde{P}(0;\Delta t_k)).$$

The function $s \to H_r(\widetilde{P}(s;\Delta t_k))$ is continuously differentiable on $[0,1]$. Hence, by the Mean Value Theorem, there exists a real number θ in $(0,1)$ such that

$$H_r(\widetilde{P}(1;\Delta t_k)) - H_r(\widetilde{P}(0;\Delta t_k)) = dH_r/ds|_{s=\theta}.$$

Hence

$$\frac{h(t_k) - h(t)}{t_k - t} \le H_{rt}(\widetilde{P}(\theta;\Delta t_k)) + \left\langle H_{rx}(\widetilde{P}(\theta;\Delta t_k)),\ \frac{\Delta\psi_k}{\Delta t_k}\right\rangle + \left\langle H_{rq}(\widetilde{P}(\theta;\Delta t_k)), \frac{\Delta\lambda_k}{\Delta t_k}\right\rangle.$$

If we now let $t_k \to t$, we get that

$$h'(t) \le H_{rt}(\pi(t)) + \langle H_{rx}(\pi(t)),\ \psi'(t)\rangle + \langle H_{rq}(\pi(t)), \lambda'(t)\rangle, \tag{7.10.7}$$

where $\Pi(t) = (t, \psi(t), \mu_t, \widehat{\lambda}(t))$, as in Definition 6.3.11 at t. From (6.3.14) we have that

$$\lambda'(t) = -H_{rx}(\pi(t)) \qquad \psi'(t) = H_{rq}(\pi(t)).$$

Substituting these into (7.10.7) gives

$$h'(t) \le H_{rt}(\pi(t)). \tag{7.10.8}$$

We also have that

$$h(t_k) - h(t) \ge H(t_k, \psi(t_k), \mu_t, \widehat{\lambda}(t_k)) - H(t, \psi(t), \mu_t, \widehat{\lambda}(t)).$$

An argument similar to the one in the preceding paragraphs gives $h'(t) \ge H_{rt}(\pi(t))$. Combining this with (7.10.8) gives

$$h'(t) = H_{rt}(\pi(t)), \tag{7.10.9}$$

which is (6.3.22).

We conclude this section with a proof of Corollary 6.3.19. It follows from Definition 6.3.18 that since μ is piecewise continuous on $[0,1]$ there exist points $0 = \tau_0 < \tau_1 < \cdots < \tau_k = 1$ in $[0,1]$ such that each of the functions p^i, u_i is continuous on the open subintervals $(\tau_j, \tau_{j+1}), j = 0, \ldots, k-1$ and has one-sided limits at the points $\tau_j, j = 0, \ldots, k$. We summarize the last statement by saying that $\mu_{(\tau_j+0)}$ and $\mu_{(\tau_j-0)}$ exist for $j = 1, \ldots, k-1$ and that $\mu_{(\tau_0+0)}$ and $\mu_{(\tau_k-0)}$ exist.

By the theorem, the mapping

$$t \to H_r(t, \psi(t), \mu_t, \widehat{\lambda}(t)) \tag{7.10.10}$$

is almost everywhere equal to an absolutely continuous function. Since μ is continuous on $(\tau_j, \tau_{j+1}), j = 0, \ldots, k-1$, the function (7.10.10) is absolutely continuous on (τ_j, τ_{j+1}). Since $\mathcal{C}$ is closed, $\mu_{\tau_j - 0}$ and $\mu_{\tau_j + 0}$ are in $\mathcal{C}$ for each $j = 1, \ldots, k-1$. From the continuity of μ on each (τ_j, τ_{j+1}) and from (6.3.18) we get that for each $\tau_j, j = 1, \ldots, k-1$

$$H_r(\tau_j, \psi(\tau_j), \mu_{(\tau_j+0)}, \widehat{\lambda}(\tau_j)) \leq H_r(\tau_j, \psi(\tau_j), \mu_{(\tau_j-0)}, \widehat{\lambda}(\tau_j))$$
$$H_r(\tau_j, \psi(\tau_j), \mu_{(\tau_j+0)}, \widehat{\lambda}(\tau_j)) \geq H_r(\tau_j, \psi(\tau_j), \mu_{(\tau_j-0)}, \widehat{\lambda}(\tau_j)).$$

Hence the mapping (7.10.10) is continuous on $(0, 1)$. We make it continuous on $[0, 1]$ by taking the value of the mapping (7.10.10) at $t = 0$ to be $H_r(0, \psi(0), \mu_{(0+0)}, \widehat{\lambda}(0))$ and at $t = 1$ to be $H_r(1, \psi(t_1), \mu_{(1-0)}, \widehat{\lambda}(1))$.

From (6.3.22) we have that

$$H_r(t, \psi(t), \mu_t, \widehat{\lambda}(t)) = \int_0^t H_{rt}(s, \psi(s), \mu_s, \widehat{\lambda}(s))ds + C, \tag{7.10.11}$$

with the relation now holding everywhere. From the continuity of $\widehat{f}_t$ on $\mathcal{I}_0 \times \mathcal{X}_0 \times \mathcal{U}_0$, it follows that the integrand is continuous on $[0, 1]$. Hence

$$\frac{dH_r}{dt} = H_{rt}(t, \psi(t), \mu_t, \widehat{\lambda}(t))$$

at all t in $[0, 1]$.

7.11 Proof of Theorem 6.3.22

In Section 2.4 we transformed the ordinary problem with possibly variable initial and terminal times into a problem with fixed initial time $t_0 = 0$ and fixed terminal time $t_1 = 1$. Henceforth we shall call the times (t_0, t_1), *end times*. For the relaxed problem we again make the change of variable

$$t = t_0 + s(t_1 - t_0) \qquad 0 \leq s \leq 1, \tag{7.11.1}$$

make time t state variable, make s the new independent variable, and introduce a new state variable w to transform the problem with possibly variable end times into a problem with fixed end times $(0, 1)$. The state equations for the transformed problem are

$$\frac{dt}{ds} = w \qquad \frac{dw}{ds} = 0 \qquad \frac{dx}{ds} = f(t, x, \widetilde{\mu}_s)w \tag{7.11.2}$$

where $\widetilde{\mu}$ is a relaxed control defined on $[0,1]$. The integrand in the transformed problem is $f^0(t,x,\widetilde{\mu}_s)w$ and the terminal set $\widetilde{\mathcal{B}}$ is given by

$$\begin{aligned}\widetilde{\mathcal{B}} = \{(s_0,t_0,x_0,w_0,s_1,t_1,x_1,w_1)\colon s_0=0,\ s_1=1 \\ (t_0,x_0,t_0,x_1)\in\mathcal{B} \quad w_0=w_1=t_1-t_0\}.\end{aligned} \tag{7.11.3}$$

As in Section 2.4, it is readily checked that if $(\widetilde{\psi},\widetilde{\mu}) = (\tau,\xi,\omega,\widetilde{\mu})$ denotes a relaxed admissible pair for the fixed end time problem and (ψ,μ) a relaxed admissible pair for the variable end time problem, then there is a one-one correspondence between the pairs $(\tau,\xi,\omega,\widetilde{\mu})$ and (ψ,μ) defined by

$$t=\tau(s) \quad \psi(t)=\xi(s) \quad \mu_t=\widetilde{\mu}_s \quad \omega(s)=t_1-t_0, \tag{7.11.4}$$

where s and t are related by the one to one mapping (7.11.1). Moreover, if the pairs $(\tau,\xi,\omega,\widetilde{\mu})$ and (ψ,μ) are in correspondence, then

$$J(\psi,\mu)=\widetilde{J}(\tau,\xi,\omega,\widetilde{\mu}), \tag{7.11.5}$$

where

$$\widetilde{J}(\tau,\xi,\omega,\widetilde{\mu}) = g(e(\tau,\xi,\omega)) + \int_0^1 f^0(\tau(s),\xi(s),\widetilde{\mu}_s)\omega(s)ds.$$

In Theorems 6.3.5 through 6.3.17, $\widehat{f}$ is assumed to be measurable in t and the end times are assumed to be fixed at $(0,1)$. The state equations of the transformed problem are given by (7.11.2) and the payoff by (7.11.5). Thus, if $\widehat{f}$ is assumed to be measurable in t, the function $\widehat{\widetilde{f}}$ in the transformed problem is not of class $C^{(1)}$ in the state variables, as required in (iii) of Assumption 6.3.1. In Theorem 6.3.22 we assume the $\widehat{f}$ is $C^{(1)}$ in (t,x), so the function $\widehat{\widetilde{f}}$ of the transformed problem does satisfy (iii) of Assumption 6.3.1. The other hypotheses of Theorem 6.3.22 are such that the transformed problem satisfies hypotheses of Theorems 6.3.5 through 6.3.17. Hence the necessary conditions hold for a solution of the transformed problem. In particular the transversality condition holds for the transformed problem. Translating this back into the variables of the original problem will give the conclusion of Theorem 6.3.5. We now proceed to carry out the proof just outlined.

Let (ψ,μ) be an optimal relaxed admissible pair for the variable end time problem. Let $(\tau,\xi,\omega,\widetilde{\mu})$ be the admissible pair for the fixed end time problem that corresponds to (ψ,μ) via (7.11.4) and (7.11.1). Then by virtue of (7.11.5) the admissible pair $(\tau,\xi,\omega,\widetilde{\mu})$ is optimal for the fixed end time problem and satisfies the conclusions of Theorems 6.3.5 to 6.3.17.

Set

$$H(t,x,z,p^0,p) = p^0 f^0(t,x,z) + \langle p, f(t,x,z)\rangle,$$

set

$$\widetilde{H}(t,x,w,z,p^0,p,a,b) = p^0 f^0(t,x,z)w + \langle p, wf(t,x,z)\rangle + aw + b_0,$$

and as in (6.3.5) set

$$\widetilde{H}_r(t,x,w,\widetilde{\mu}_s,p^0,p,a,b) = \int_{\Omega(t)} \widetilde{H}(t,x,w,z,p^0,p,a,b)d\widetilde{\mu}_s,$$

where $0 \le s \le 1$. Then there exists a constant $\widetilde{\lambda}^0 \le 0$ and absolutely continuous functions $(\widetilde{\lambda},\alpha,\beta)$ defined on $[0,1]$ such that if

$$\widetilde{H}_r(s) = \widetilde{H}_r(\tau(s),\xi(s),\omega(s),\widetilde{\mu}_s,\lambda^0,\widetilde{\lambda}(s),\alpha(s),\beta(s))$$

then

$$\frac{d\widetilde{\lambda}}{ds} = -\widetilde{H}_{rx}(s) \quad \frac{d\alpha}{ds} = -\widetilde{H}_{rt}(s) \quad \frac{d\beta}{ds} = -\widetilde{H}_{rw}(s). \tag{7.11.6}$$

From the first equation in (7.11.6) we get that

$$\frac{d\widetilde{\lambda}}{ds} = \omega(s)[-\widetilde{\lambda}_0 f^0_{rx}(s) - f^T_{rx}(s)\widetilde{\lambda}], \tag{7.11.7}$$

where $\widehat{f}_{rx}(s) = \widehat{f}_x(\tau(s),\xi(s),\widetilde{\mu}_s)$ and the superscript T denotes transpose. Using the first equation in (7.11.2), the equations in (7.11.4), and setting $\lambda^0 = \widetilde{\lambda}^0$ and $\lambda(t) = \widetilde{\lambda}(s)$, where s and t are related by (7.11.1) we can write (7.11.7) as

$$\begin{aligned}\frac{d\lambda}{dt} &= -\lambda^0 f^0_x(t,\psi(t),\mu_t) - f^T_x(t,\psi(t),\mu(t))\lambda(t)\\ &= -H_{rx}(t,\psi(t),\mu_t,\lambda^0,\lambda(t)),\end{aligned}$$

where H_r is given by (6.3.5). Thus, the first equation in (7.11.6) gives us no new information.

For the fixed end time problem, the condition (6.3.7) takes the form

$$\int_0^1 \widetilde{H}_r(s)ds \ge \int_0^1 \widetilde{H}_r(\tau(s),\xi(s),\omega(s),\widetilde{\mu}_s,\widetilde{\lambda}^0,\widetilde{\lambda}(s),\alpha(s),\beta(s))ds. \tag{7.11.8}$$

Using the first equation in (7.11.2), the relation $(\widetilde{\lambda}^0,\widetilde{\lambda}(s)) = (\lambda^0,\lambda(t))$ and the fact that the terms involving α and β are independent of $\widetilde{\mu}$, we transform (7.11.8) into (6.3.7), so we again get no new information.

In summary, we have shown that the necessary conditions, other than the transversality condition, for the problem with variable end times are the same as those for the fixed end time problem. We next take up the transversality condition.

The end points of (τ,ξ,ω) are given by

$$\tau(0) = t_0 \quad \tau(1) = t_1 \quad \xi(0) = x_0 \quad \xi(1) = x_1 \quad \omega(0) = \omega(1) = t_1 - t_0, \tag{7.11.9}$$

where $(t_0, x_0, t_1, x_1) \in \mathcal{B}$. From Theorems 6.3.5 to 6.3.17 we get that the transversality condition for the fixed end time problem is

$$-\widetilde{\lambda}_0 dg + \langle \widetilde{\lambda}(1), dx_1 \rangle + \alpha(1)dt_1 + \beta(1)dw_1 - \langle \widetilde{\lambda}(0), dx_0 \rangle \\ - \alpha(0)dt_0 - \beta(0)dw_0 = 0 \tag{7.11.10}$$

for all tangent vectors $(dt_0, dx_0, dw_0, dt_1, dx_1, dw_1)$ to $\widetilde{\mathcal{B}}$ at the end point of (τ, ξ, ω), and where

$$dg = g_{t_0} dt_0 + \langle g_{x_0}, dx_0 \rangle + g_{t_1} dt_1 + \langle g_{x_1}, dx_1 \rangle$$

with the partials of g evaluated at the end point of (τ, ξ, ω). If we use (7.11.1) and set $(\lambda^0, \lambda(t)) = (\widetilde{\lambda}^0, \widetilde{\lambda}(s))$, we can write (7.11.10) as

$$-\lambda^0 dg + \langle \lambda(t_1), dx_1 \rangle + \alpha(t_1)dt_1 + \beta(t_1)dw_1 - \langle \lambda(t_0), dx_0 \rangle \\ - \alpha(t_0)dt_0 - \beta(t_0)dw_0 = 0. \tag{7.11.11}$$

Henceforth we shall always set $(\lambda^0, \lambda(t)) = (\widetilde{\lambda}^0, \widetilde{\lambda}(t))$.

From the second equation in (7.11.6) we get that

$$\frac{d\alpha}{ds} = -\omega(s)[\lambda^0 f_t^0(\tau(s), \xi(s), \widetilde{\mu}_s) + \langle \lambda(s), f_t(\tau(s), \xi(s), \widetilde{\mu}_s].$$

Using the first equation in (7.11.2) and (7.11.1) gives

$$\begin{aligned}\frac{d\alpha}{dt} &= -[\lambda^0 f_t^0(t, \psi(t), \mu_t) + \langle \lambda(t), f_t(t, \psi(t), \mu_t) \rangle] \\ &= -H_{rt}(t, \psi(t), \lambda^0, \lambda(t)) \quad t_0 \le t \le t_1.\end{aligned}$$

Hence

$$\alpha(t) - \alpha(t_0) = -\int_{t_0}^{t} H_{rt}(s, \psi(s), \mu_s, \lambda^0, \lambda(s)) ds.$$

The hypotheses of Theorem 6.3.22 imply those of Theorem 6.3.17, so from (6.3.22) we get that

$$\alpha(t) - \alpha(t_0) = -\overline{H}_r(t) + \overline{H}_r(t_0), \tag{7.11.12}$$

where $\overline{H}_r(t) = H_r(t, \psi(t), \mu_t, \lambda^0, \lambda(t))$.

From the last equation in (7.11.6) we get that

$$\frac{d\beta}{ds} = -\lambda^0 f^0(\tau(s), \xi(s), \widetilde{\mu}_s) - \langle \lambda(s), f(\tau(s), \xi(s), \widetilde{\mu}_s) \rangle - \alpha(s)$$

and so

$$\frac{d\beta}{dt}\frac{dt}{ds} = -\lambda^0 f^0(t, \psi(t), \mu_t) - \langle \lambda(t), f(t, \psi(t), \mu_t) \rangle - \alpha(t).$$

Using the first two equations in (7.11.2) and the relation $w_1 = (t_1 - t_0)$ in (7.11.3) gives

$$\frac{d\beta}{dt} = -(t_1 - t_0)^{-1}[\overline{H}_r(t) + \alpha(t)].$$

From this and (7.11.13) we get that

$$\frac{d\beta}{dt} = -(t_1 - t_0)^{-1}[\overline{H}_r(t_0) + \alpha(t_0)],$$

and so

$$\beta(t) = -\frac{(t - t_0)}{(t_1 - t_0)}\,[\overline{H}_r(t_0) + \alpha(t_0)] + \beta(t_0). \tag{7.11.13}$$

From (7.11.12) we get that

$$\alpha(t_1)dt_1 = [-\overline{H}_r(t_1) + \alpha(t_0) + \overline{H}_r(t_0)]dt_1. \tag{7.11.14}$$

From (7.11.13) and $dw_1 = (dt_1 - dt_0)$ we get that

$$\beta(t_1)dw_1 = [-\overline{H}_r(t_0) - \alpha(t_0)]dt_1 + [\overline{H}_r(t_0) + \alpha(t_0)]dt_0 + \beta(t_0)dw_1. \tag{7.11.15}$$

Substituting (7.11.15) and (7.11.14) into (7.11.11) and recalling that $w_1 = w_2$ we obtain

$$-\lambda^0 dg - \overline{H}_r(t_1)dt_1 + \langle\lambda(1), dx_1\rangle + \overline{H}_r(t_0)dt_0 - \langle\lambda(0), dx_0\rangle = 0$$

for all tangent vectors (dt_0, dx_0, dt_1, dx_1) to $\mathcal{B}$ at the end point $e(\psi)$ and where the partials of g are evaluated at $e(\psi)$.

Chapter 8

Examples

8.1 Introduction

In this chapter we illustrate the use of results presented in the preceding chapters to determine optimal controls and optimal trajectories.

8.2 The Rocket Car

A car, which we take to be a point mass, is propelled by rocket thrusts along a linear track endowed with coordinates. Units are assumed to be normalized so that the equation of motion is $\ddot{x} = u$, where u is the thrust force constrained to satisfy $-1 \leq u \leq 1$. Initially the car is at a point x_0 with velocity y_0. The problem is to determine a thrust program u that brings the car to rest at the origin in minimum time.

If we consider the state of the system to be x, the position of the car, and y, the velocity of the car, then we can write the equations governing the state of the system by

$$\begin{aligned} \dot{x} &= y \qquad x(0) = x_0 \\ \dot{y} &= u \qquad y(0) = y. \end{aligned} \tag{8.2.1}$$

The problem is to determine a control function u such that

$$|u(t)| \leq 1 \tag{8.2.2}$$

that minimizes

$$J = t_f = \int_0^{t_f} dt, \tag{8.2.3}$$

where t_f is the time at which the state (x, y) reaches the origin $(0, 0)$. We have cast the problem as the control problem: Minimize (8.2.3) subject to the state equations (8.2.1), control constraint (8.2.2), initial condition $\mathcal{T}_0 = (0, x_0, y_0)$, and terminal condition $\mathcal{T}_1 = (t_f, 0, 0)$.

As an exercise, we ask the reader to show that for each initial point $(0, x_0, y_0)$, the set of admissible trajectories is not empty. In this example we have that

$$\begin{aligned}|\langle x, f(t,x,z)\rangle| &= |xy + yz| \le |xy| + |yz| \\ &\le (|x|^2 + |y|^2)/2 + (|y|^2/2) + 1/2 \le (|x|^2 + |y|^2 + 1).\end{aligned}$$

It then follows from Lemma 4.3.14 that for a fixed initial point (x_0, y_0), the set of admissible trajectories lie in a compact set. It is readily verified that all the other hypothesis of Theorem 4.4.2 are satisfied. Thus, there exists an ordinary admissible pair that is the solution of both the relaxed and ordinary problems. Having shown the existence of a solution the next step is to determine the extremal trajectories and if there is more than one, determine which is the solution.

Since the problem is a time optimal problem that is linear in the state and with terminal state the origin, a preferable procedure is the following. The state equation can be written as

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix} u = A\begin{pmatrix}x\\ y\end{pmatrix} + Bu.$$

The matrix B is an $n \times 1$ column vector and the vectors $B,\ AB$ are linearly independent. Hence by Corollary 6.7.16 the system is strongly normal. The other hypotheses of Theorem 6.8.2 are clearly satisfied. Hence an extremal trajectory is optimal, so we only need to determine extremal trajectories. From Theorem 6.8.1 we get that the optimal control only takes on the values $+1$ and -1.

The data of the problem satisfy the hypothesis of Theorem 6.3.27. Therefore an extremal trajectory and corresponding control satisfy the conclusion of Theorem 6.3.27.

The function H defined by (6.3.4) becomes

$$H = q^0 + q^1 y + q^2 z.$$

The second set of equations (6.3.28) are

$$\begin{aligned}\frac{d\lambda^1}{dt} &= -H_x = 0 \\ \frac{d\lambda^2}{dt} &= -H_y = -\lambda^1.\end{aligned} \tag{8.2.4}$$

Hence

$$\lambda^1(t) = c, \qquad \lambda^2(t) = -c_1 t + c_2. \tag{8.2.5}$$

The terminal set $\mathcal{B} = \{(0, x_0, y_0, t_f, 0, 0) : t_f \text{ free}\}$. Therefore, the transversality condition (6.3.31) becomes

$$\lambda^0 + \lambda^1(t_f)y(t_f) + \lambda^2(t_f)u(t_f) = 0. \tag{8.2.6}$$

If $c_1^2 + c_2^2 = 0$, then $\lambda^1(t) = \lambda^2(t) = 0$ for all t. Then by (8.2.6) $\lambda^0 = 0$, which contradicts the conclusion $(\lambda^0, \lambda^1(t), \lambda^2(t) \neq (0, 0, 0)$. Hence $c_1^2+c_2^2 \neq 0$.

From (8.2.5) we get that

$$H = \lambda^0 + c_1 y + (c_2 - c_1 t)z, \tag{8.2.7}$$

and from (8.2.6) we get that

$$\lambda^0 + (c_2 - c_1 t_f)u(t_f) = 0.$$

At time t the value $u(t)$ of the optimal control maximizes (8.2.7) over the interval $|z| \leq 1$. Hence

$$u(t) = \text{ signum } (c_2 - c_1 t).$$

Thus, if $c_1 = 0$, then $c_2 \neq 0$ and $u(t) = \text{signum } c_2$ for all t. Let $c_2 > 0$. Then $u(t) = 1$ for all t. We shall "back out of the target along the extremal trajectory." That is, we shall reverse time and determine all initial states (x_0, y_0) that can be reached using $u(t) = 1$ along the entire trajectory. Thus, setting $t = -\tau$ gives

$$\frac{dx}{d\tau} = -y \qquad \frac{dy}{d\tau} = -u \qquad x(0) = y(0) = 0,$$

and so with $u(t) = 1$

$$y(\tau) = -\tau \qquad x(\tau) = \tau^2/2 \qquad \tau \geq 0. \tag{8.2.8}$$

Equations (8.2.8) are the parametric equations of the parabolic segment OA (See Fig. 8.1) whose Cartesian equation is

$$y = -\sqrt{2x} \qquad x \geq 0. \tag{8.2.9}$$

Thus, we may take any point (x_0, y_0) on OA as an initial point. The optimal control will be $u(t) = 1$ for $0 \leq t \leq t_f$ and the optimal trajectory will be the portion of OA with $0 \leq x \leq x_0$.

Let $W(x_0, y_0)$ denote the terminal time of an extremal, and hence optimal trajectory with initial point (x_0, y_0) on OA. It follows from (8.2.8) that

$$W(t_0, x_0) = -y_0. \tag{8.2.10}$$

Recall that $y_0 < 0$ for $(x_0, y_0) \in OA$.

A similar analysis for $c_1 = 0$ and $c_2 < 0$ gives the existence of the parabolic segment OB whose Cartesian equation is

$$y = \sqrt{-2x}, \quad x \leq 0 \tag{8.2.11}$$

such that for any initial point (x_0, y_0) on OB, the optimal trajectory with

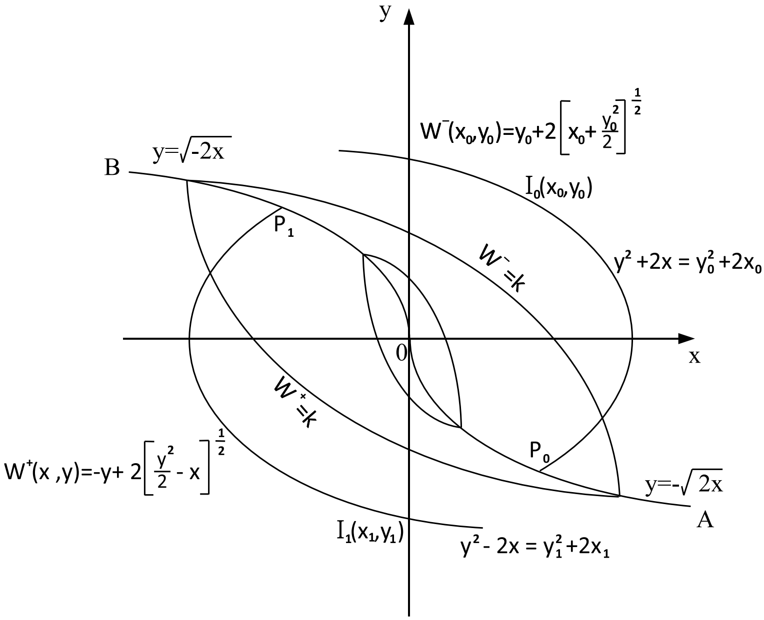

FIGURE 8.1

(x_0, y_0) is the portion of OB given by $x_0 \leq x \leq 0$ and the optimal control is $u(t) = -1$. Also,

$$W(x_0, y_0) = y_0. \tag{8.2.12}$$

If $c_2 = 0$ and $c_1 > 0$ we again get the curve OA.

If $c_2 = 0$ and $c_1 < 0$ we again get OB.

We now consider the case in which $c_1c_2 \neq 0$. We again reverse time by setting $t = -\tau$ and "back out from target." Then $u(\tau) =$ signum $(c_2 + c_1\tau)$ maximizes (8.2.7) over $|z| \leq 1$. Thus if $c_1 > 0$ and $c_2 > 0$, we again get OA. If $c_1 < 0$ and $c_2 < 0$ we again get OB.

We now consider the case $c_1 < 0$ and $c_2 > 0$. Then $u(\tau) = 1$ on some interval $[0, \tau_s]$, where

$$\tau_s = -c_2/c_1.$$

From (8.2.8) we get that at $\tau = \tau_s$

$$y_s \equiv y(\tau_s) = -\tau_s \qquad x_s \equiv x(\tau_s) = \tau_s^2/2. \tag{8.2.13}$$

For $\tau \geq \tau_s$, the optimal control is $u(\tau) =$ signum $(c_2 + c_1\tau) = -1$. Hence for $\tau \geq \tau$ optimal trajectory is the curve defined by

$$\frac{dy}{d\tau} = -u(\tau) = 1 \qquad \frac{dx}{d\tau} = -y \tag{8.2.14}$$

with initial conditions given by (8.2.13). It then follows that the optimal trajectory for $\tau \geq \tau_s$ is given parametrically by

$$y(\tau) = \tau - 2\tau_s \qquad x(\tau) = -(\tau - 2\tau_s)^2/2 + \tau_s^2. \tag{8.2.15}$$

To summarize, in the case $c_1 < 0,\ c_2 > 0$ the optimal trajectory traversed "backwards in time" is the segment of OA corresponding to the time interval $[0, \tau_s]$, followed by the curve defined by (8.2.15) on $[\tau_s, \infty)$. The time τ_s is called *switching time.* The parabolic segment defined by (8.2.15) lies on the parabola whose Cartesian equation is

$$y^2 + 2x = 4x. \tag{8.2.16}$$

Let $\Sigma = OA \cup OB$. The curve Σ is called a *switching curve.* It follows from (8.2.9) and (8.2.11) that if we denote the region above the curve Σ by $\mathcal{R}^-$, then

$$\mathcal{R}^- \equiv \{(x, y) : y > -(\text{signum } x)\sqrt{2|x|}\}.$$

We assert that for each point (x_0, y_0) in $\mathcal{R}^-$ there exists a solution to the problem defined by Eqs. (8.2.1) to (8.2.3). The optimal trajectory consists of a segment of the parabola given by (8.2.16), followed by a segment of OA. The optimal control is $u(t) = -1$ for the segment given by (8.2.16) and $u(t) = 1$ for the segment in OA.

We shall prove this assertion by showing that given (x_0, y_0) in $\mathcal{R}^-$ and reversing time by setting $\tau = -t$, there is a trajectory consisting of a segment of OA with initial point the origin corresponding to an interval $[0, \tau_s]$, followed by the extremal defined parametrically by (8.2.15) that reaches (x_0, y_0) at some time $\tau_0 > \tau_s$. Thus, we need to show that there exists a $\tau_0 > \tau_s$ such that

$$y_0 = (\tau_0 - 2\tau_s) \qquad x_0 = -(\tau_0 - 2\tau_s)^2/2 + \tau_s^2.$$

Hence the point (x_0, y_0) must lie on a parabola whose equation is (8.2.16). This parabola is a translate to the right of the parabola on which OB lies. It is geometrically obvious that the parabola (8.2.16) intersects OA at some point (x_s, y_s). This proves the assertion.

The point (x_0, y_0) is on the parabola (8.2.16) and therefore (x_0, y_0) satisfies (8.2.16). Combining this just with the fact that x_s satisfies (8.2.9) gives:

$$x_s = (y_0^2 + 2x_0)/4 \qquad y_s = -(y_0^2/2 + x_0)^{1/2}. \tag{8.2.17}$$

We now determine the time τ_0 required to traverse an optimal trajectory from (x_0, y_0) in $\mathcal{R}^-$ to the origin. That is, we determine the value function $W(x_0, y_0)$ in $\mathcal{R}^-$. We first determine the time required to go from y_0 to y_s. From (8.2.14) we get that

$$y_0 - y_s = y(\tau_0) - y(s_0) = \tau_0 - \tau_s.$$

From this and from (8.2.17) we get that

$$\tau_0 = y_0 + (y_0^2/2 + x_0)^{1/2} + \tau_s.$$

From (8.2.13) we get that τ_s, the time required to go from (x_s, y_s) to $(0,0)$, is given by $\tau_s = -y_s$. It then follows from (8.2.17) that

$$W(t_0, x_0) = y_0 + 2(y_0^2/2 + x_0)^{1/2} \qquad (x_0, y_0) \in \mathcal{R}^-. \tag{8.2.18}$$

For points (x_0, y_0) on OA, the right-hand side of (8.2.18) equals $-y_0$. It then follows from (8.2.10) that (8.2.18) *is valid for all* (x_0, y_0) *in* $\mathcal{R}^- \cup OA$.

Let $\mathcal{R}^+$ denote *the region below* Σ. Then

$$\mathcal{R}^+ = \{(x,y) : y < -(\text{signum } x)(2|x|)^{1/2}, -\infty < x < \infty\}.$$

If $(x_0, y_0) \in \mathcal{R}^+$, then the optimal control is $u(t) = 1$ on an interval $[0, t_s]$ and then $u(t) = -1$. The optimal trajectory is a segment of the parabola $y^2 - 2x = y_0^2 - 2x_0$. Corresponding to $u(t) = 1$, until the parabola intersects OB. The motion proceeds with $u(t) = -1$ along OB to the origin. For (x_0, y_0) in $\mathcal{R}^+ \cup OB$

$$W(x_0, y_0) = -y^0 + 2(y_0^2/2 - x_0)^{1/2}. \tag{8.2.19}$$

We leave the arguments justifying the statements in this paragraph to the reader.

Our analysis has also provided an *optimal synthesis* $U(x,y)$. At each point in $\mathbb{R}$, $U(x,y)$ is the value of the optimal control at (x,y) for an optimal trajectory starting at the point (x,y). We have that $U(x,y) = -1$ for (x,y) in $\mathcal{R}^- \cup OB$ and $U(x,y) = 1$ for (x,y) in $\mathcal{R}^+ \cup OA$.

Since (8.2.18) holds on $\mathcal{R}^- \cup OA$ and (8.2.19) holds on $\mathcal{R}^+ \cup OB$, it follows that W is continuous on all of $\mathbb{R}^2$. It further follows from (8.2.18) and (8.2.19) that W is continuously differentiable on $\mathcal{R}^- \cup \mathcal{R}^+$ and is Lipschitz continuous on compact subsets of $\mathcal{R}^- \cup \mathcal{R}^+$.

Exercise 8.2.1. (i) Let $(\xi, \eta) \in OA$.

(a) Find $\lim W_x(x,y)$ and $\lim W_y(x,y)$ as $(x,y) \to (\xi, \eta)$, where $(x,y) \in \mathcal{R}^-$.

(b) Find $\lim W_x(x,y)$ and $\lim W_y(x,y)$ as $(x,y) \to (\xi, \eta)$, where $(x,y) \in \mathcal{R}^+$.

(ii) Let $(\xi, \eta) \in OB$.

(a) Find $\lim W_x(x,y)$ and $\lim W_y(x,y)$ as $(x,y) \to (\xi, \eta)$, where $(x,y) \in \mathcal{R}^-$.

(b) Find $\lim W_x(x,y)$ as $(x,y) \to (\xi, \eta)$, where $(x,y) \in \mathcal{R}^+$.

(iii) Determine and sketch the curves of constant W.

Exercise 8.2.2. Determine the optimal controls and optimal trajectories for the time optimal problem with state equations (8.2.1), control constraint (8.2.2) and terminal set $y = 0$.

Exercise 8.2.3. Find the optimal controls and optimal trajectories for the time optimal problem with state equations (8.2.1), control constraints (8.2.2), and terminal set $x^2 + y^2 = \varepsilon^2$. What happens as $\varepsilon \to 0$?

8.3 A Non-Linear Quadratic Example

Consider the system

$$\frac{dx}{dt} = -xu, \tag{8.3.1}$$

where x and u are scalars. Let the end conditions be $t_0 = 0$, $t_1 = 1$, $x_0 = 1$, and x_1 free. Let the constraint condition be $0 \le u(t) \le 1$ and let the payoff be

$$J(\phi, u) = \phi(1)^2/2 + \frac{1}{2}\int_0^1 u^2(t)dt. \tag{8.3.2}$$

Show that an optimal control and trajectory exist and find them.

In the notation used in this text, in this problem $g(x_1) = x_1^2/2$, $f = z^2/2$, $\Omega(t) = [0,1]$ and $\mathcal{B} = \{(t_0, x_0, t_1, x_1) : t_0 = 0,\ x_0 = 1,\ t_1 = 1,\ x_1 = \sigma,\ -\infty < \sigma < \infty\}$. For each (t,x), the set $Q^+(t,x)$ is the set of points (y^0, y^1) such that $y^0 \ge \frac{1}{2}(z)^2$, and $y^1 = -xz$, where $0 \le z \le 1$. It follows from (8.3.1) and $\Omega(t) = [0,1]$ that all admissible trajectories are contained in the compact set bounded by $x = 1$, $x(t) \ge e^{-t}$, $0 \le t \le 1$, and $t = 1$. The other hypotheses of Theorem 4.4.2 hold, so an ordinary solution that is also a solution of relaxed problem exists.

The admissible control $u(t) \equiv 0$ results in the trajectory $\phi(t) \equiv 1$ and $J(\phi, u) = 1/2$. The admissible control $u(t) \equiv 1$ results in the trajectory $\phi(t) = e^{-t}$ and $J(\phi, u) = [e^{-2} + 1]/2 > 1/2$. Thus, the end point of the optimal trajectory lies in the interval $(e^{-1}, 1]$. We first assume that the end point of the optimal trajectory ϕ is in the open interval $(e^{-1}, 1)$, and will use Theorem 6.3.27 to determine ϕ and the optimal control u.

The function H is given by

$$H = q^0 z^2/2 - qxz. \tag{8.3.3}$$

The differential equations (6.3.28) become

$$\begin{aligned} \phi'(t) &= H_\lambda = -\phi(t)u(t) \\ \lambda'(t) &= -H_x = \lambda(t)u(t). \end{aligned} \tag{8.3.4}$$

From (8.3.4) we get that

$$(\lambda\phi)' = \lambda'\phi + \lambda\phi' = \lambda\phi u + \lambda(-\phi u) = 0.$$

Hence

$$\lambda(t)\phi(t) = \text{ const.} = \lambda(1)\phi(1). \tag{8.3.5}$$

The differential equations in (8.3.4) are adjoint to each other. Thus, (8.3.5) is a special case of a general result established in Lemma 6.6.2.

Since we are assuming that $\phi(1) \in (e^{-1}, 1)$, we may take the set $\mathcal{B}$ to be given parametrically by

$$t_0 = 0 \quad x_0 = 1 \quad t_1 = 1 \quad x_1 = \sigma, \quad \sigma \in (e^{-1}, 1).$$

The unit tangent vector to $\mathcal{B}$ at the point $(0, 1, 1, \phi(1))$ is $(0, 0, 0, 1)$. The transversality condition (6.3.31) becomes

$$0 = -\lambda^0 g_{x_1}(\phi(1)) + \lambda(1) = -\lambda^0 \phi(1) + \lambda(1).$$

Hence $\lambda(1) \neq 0$, for if $\lambda(1) = 0$, the $\lambda^0 = 0$, which contradicts $(\lambda^0, \lambda(t)) \neq (0, 0)$ for all $t \in [0, 1]$. We may therefore take $\lambda^0 = -1$. The transversality condition becomes $\phi(1) + \lambda(1) = 0$, and so $\lambda(1) = -\phi(1)$. From this and from (8.3.4) we get that for all $t \in [0, 1]$

$$\lambda(t)\phi(t) = -[\phi(1)]^2 = -s^2, \tag{8.3.6}$$

where s is the value of the parameter σ such that $\phi(1) = s$.

Setting $q^0 = \lambda^0 = -1$, $q = \lambda(t)$ and $x = \varphi(t)$ in (8.3.3) and then using (8.3.6), we get from (6.3.29) that the value $u(t)$ of the optimal control at time t maximizes

$$F(z) = -z^2/2 + s^2 z \qquad 0 \leq z \leq 1.$$

From $F'(z) = -z + s^2$ and $F''(z) = -1$ we get that F is concave on $[0, 1]$ and is maximized at $u(t) = s^2$. The trajectory corresponding to the control $u(t) \equiv s^2$ is $\phi(t) = \exp(-s^2 t)$. Thus, the optimal pair (ϕ, u) is

$$\phi(t) = e^{-s^2 t} \qquad u(t) \equiv s^2, \tag{8.3.7}$$

and by (8.3.2)

$$J(\phi, u) = (e^{-2s^2} + s^4)/2. \tag{8.3.8}$$

From (8.3.7) we have $\phi(1) = \exp(-s^2)$. By the definition of s, $\phi(1) = s$. Hence s must satisfy the equation

$$\sigma = e^{-\sigma^2}. \tag{8.3.9}$$

It is readily seen that this equation has a unique solution s, lying in the interval $(0, 1)$. Moreover, it is also readily verified that

$$.6500 < s < .6551. \tag{8.3.10}$$

Using (8.3.9) and (8.3.10) we get that

$$J(\phi, u) = (s^2 + s^4)/2 < 0.3067. \tag{8.3.11}$$

The preceding analysis was based on the assumption that $\phi(1) \in (e^{-1}, 1)$. We already showed that the control $u(t) \equiv 1$, which yields a trajectory with end point e^{-1}, cannot be optimal. To conclude that (ϕ, u) is indeed optimal we must show that $u(t) \equiv 0$, which yields a trajectory with end point one, cannot be optimal. We already showed that the payoff using the control $u(t) \equiv 0$ equals $1/2$. Comparing this with (8.3.11) shows that (ϕ, u) is indeed optimal.

Exercise 8.3.1. Investigate the problem with terminal time $T > 1$.

8.4 A Linear Problem with Non-Convex Constraints

A boat moves with velocity of constant magnitude one relative to a stream of constant speed s. It is required to transfer the boat to a point (ξ, η) in minimum time. The equations of motion of the boat are

$$\frac{dx}{dt} = s + u_1 \qquad \frac{dy}{dt} = u_2 \qquad (u_1)^2 + (u_2)^2 = 1. \tag{8.4.1}$$

(i) Show that whenever the set of admissible pairs is not empty there exists an ordinary admissible pair (ϕ, u) that is a solution of the ordinary and relaxed problems.

(ii) Determine the unique optimal pair in this case.

(iii) Find the minimum transfer time as a function of (ξ, η), that is, find the value function.

(iv) For each of the cases $s < 1$, $s = 1$, $s > 1$ find the set S of points (ξ, η) for which the problem has a solution.

(v) For $s > 1$ show that for points on the boundary of S, $\lambda^0 = 0$ along the optimal trajectory.

The sets $Q^+(x, y)$ in this problem are not convex. The problem is linear, however, and all of the other hypotheses of Theorem 4.7.8 are fulfilled, so if there exists a trajectory that reaches (ξ, η), then there exists an ordinary optimal pair that is also a solution of the relaxed problem.

Let (ξ, η) be a point that can be reached from the origin. Let (ϕ, u) denote the optimal pair. The function H is given by

$$H = q^0 + q_1(s + z_1) + q_2 z_2.$$

The end set $\mathcal{B}$ is given by

$$\mathcal{B} = \{(t_0, x_0, y_0, t_1, x_1, y_1) : t_0 = x_0 = y_0 = 0,\ t_1 \text{ free}, x_1 = \xi, y_1 = \eta\}.$$

The unit tangent vector to $\mathcal{B}$ is $(0, 0, 0, 1, 0, 0)$. The transversality condition gives

$$\lambda^0 + \lambda_1(t_1)(s + u_1(t)) + \lambda_2(t_1)u_2(t_1) = 0.$$

From this we get that $|\lambda_1(t_1)| + |\lambda_2(t_1)| \neq 0$, for otherwise $\lambda^0 = 0$, which cannot be since $(\lambda^0, \lambda_1(t), \lambda_2(t)) \neq (0, 0, 0)$ for all t.

Let $\lambda(t_1) = c_1$ and $\lambda(t_2) = c_2$. The equations for the multipliers are

$$\frac{d\lambda_1}{dt} = -H_x = 0 \qquad \frac{d\lambda_2}{dt} = -H_y = 0.$$

Hence for all t

$$\lambda_1(t) = c_1 \qquad \lambda_2(t) = c_2 \tag{8.4.2}$$

for some constants c_1 and c_2 with $|c_1| + |c_2| \neq 0$. We may therefore write H as

$$\lambda^0 + c_1(s + z_1) + c_2 z_2. \tag{8.4.3}$$

Hence the optimal control at time t is a unit vector $(u_1(t), u_2(t))$ that maximizes

$$\lambda_1(t)z_1 + \lambda_2(t)z_2 = c_1 z_1 + c_2 z_2 = \langle (c_1, c_2), (z_1, z_2) \rangle.$$

Thus, $(u_1(t), u_2(t))$ is a unit vector in the direction of (c_1, c_2), and so

$$u_1(t) = c_1/(c_1^2 + c_2^2)^{1/2} \qquad u_2(t) = c_2/(c_1^2 + c_2^2)^{1/2}. \tag{8.4.4}$$

Let $\alpha \equiv u_1(t)$ and $\beta \equiv u_2(t)$. It then follows from (8.4.1) and the initial condition $x(0) = y(0) = 0$ that the optimal trajectory is given by

$$x(t) = (s + \alpha)t \qquad y(t) = \beta t, \tag{8.4.5}$$

which is a line from the origin to (ξ, η).

Let $\tau = \tau(\xi, \eta)$ be the time required to reach (ξ, η) from the origin. Then from (8.4.5) we get

$$\begin{aligned} \xi^2 &= s^2\tau^2 + \alpha^2\tau^2 + 2s\alpha\tau^2 \\ \eta^2 &= \beta^2\tau^2. \end{aligned}$$

From this and from $s\alpha\tau = s\xi - s^2\tau$, which we get from (8.4.5), we get that

$$(1 - s^2)\tau^2 + 2s\xi\tau - (\xi^2 + \eta^2) = 0. \tag{8.4.6}$$

Therefore,

$$\tau = \frac{[-s\xi \pm (\xi^2 + (1 - s^2)\eta^2)^{1/2}]}{(1 - s^2)}. \tag{8.4.7}$$

If $s < 1$, since $\tau > 0$ we must take the plus sign in (8.4.7) and get that

$$\tau = \frac{[-s\xi + (\xi^2 + (1 - s^2)\eta^2)^{1/2}]}{(1 - s^2)}. \tag{8.4.8}$$

Since $(\xi^2 + (1 - s^2)\eta^2)^{1/2}$ is greater than $s\xi$ for $s < 1$ and all ξ, it follows that $\tau > 0$. Thus, all points (ξ, η) can be reached. The boat moves faster than the stream, and even points upstream $(\xi < 0)$ can be reached.

If $s = 1$, then (8.4.6) becomes $2\xi\tau - (\xi^2 + \eta^2) = 0$. Hence, since $\xi \neq 0$

$$\tau = \frac{(\xi^2 + \eta^2)}{2\xi}.$$

Thus, all points with $\xi > 0$, that is, all points downstream, can be reached.

If $s > 1$, we rewrite (8.4.8) as

$$\tau = \frac{[s\xi \pm (\xi^2 - (s^2-1)\eta^2)^{1/2}]}{(s^2-1)}. \tag{8.4.9}$$

From (8.4.5), since $s > 1$, $|\alpha| \leq 1$, and $\tau > 0$, we get that $\xi > 0$. For τ to be real we require that $\xi^2 - (s^2-1)\eta^2 \geq 0$, or equivalently

$$-(s^2-1)^{-1/2} \leq \frac{\eta}{\xi} \leq (s^2-1)^{-1/2}. \tag{8.4.10}$$

Thus if, $s > 1$ any points (ξ, η) that lie in the region subtended by the angle determined by the line segments from the origin with slope $-(s^2-1)^{-1/2}$ and $(s^2-1)^{1/2}$ can be reached.

We assert that in (8.4.9) we take the minus sign. We have

$$(\xi^2 - (s^2-1)\eta^2)^{1/2} = \left(\xi^2\left(1 - \frac{(s^2-1)\eta^2}{\xi^2}\right)\right)^{1/2} \leq (\xi^2)^{1/2} < s\xi,$$

where the next to the last inequality follows from (8.4.10). Therefore since we want the smallest positive root in (8.4.9), we take the minus sign.

It follows from (8.4.2), $z_1 = u_1(t)$, $z_2 = u_2(t)$, and (8.4.3) that H evaluated along an optimal trajectory is given by

$$H = \lambda^0 + (c_1^2 + c_2^2)^{1/2} + c_1 s.$$

From the transversality condition we concluded that H at the end time τ was zero. Therefore, again using (8.4.4), we get

$$-\frac{\lambda_0}{(c_1^2 + c_2)^{1/2}} = 1 + u_1(t)s = 1 + \alpha s.$$

From (8.4.5) we get that $\alpha = (\xi - s\tau)/\tau$. From (8.4.9) and (8.4.10) we get that along a boundary line of the reachable set, $\tau = s\xi/(s^2-1)$. Substituting this into the expression for α gives $1 + \alpha s = 0$. Hence $\lambda^0 = 0$.

8.5 A Relaxed Problem

In Example 3.1.1 we used the following problem to motivate and introduce the concept of relaxed controls.

Problem 8.5.1. Minimize $\int_0^t 1dt$ subject to the state equations

$$\frac{dx}{dt} = y^2 - u^2 \qquad \frac{dy}{dt} = u,$$

control constraints $\Omega(t) = \{z : |z| \leq 1\}$, initial set $\mathcal{T}_0 = \{(t_0, x_0, y_0) = (0, 1, 0)\}$, and terminal set $\mathcal{T}_1 = \{(t_1, x_1, y_1) : x_1^2 + y_1^2 = a^2, t_1 \text{ free}\}$, where $0 < a < 1$. We showed that this problem has no solution, but the corresponding relaxed problem, whose state equations are

$$\frac{dx}{dt} = y^2 - \int_{-1}^{1} z^2 d\mu_t \qquad \frac{dy}{dt} = \int_{-1}^{1} z d\mu_t, \tag{8.5.1}$$

has a solution. Moreover, the solution is the discrete measure control

$$\mu_t = \sum_{i=1}^{2} p^i(t)\delta_{u_i(t)},$$

where $p^1(t) = p^2(t) = 1/2$ and $u_1(t) = 1$, $u_2(t) = -1$. Here we shall use an existence theorem and Theorem 6.3.22 to determine the solution.

In Example 3.1.1 we constructed a minimizing sequence, all of whose trajectories lie in a compact set. Hence, by Corollary 4.3.13, the relaxed problem has a solution, (ψ, μ). We next use Theorem 6.3.22 to determine this solution. Although the direct method used in Example 3.1.1 is simpler, it is instructive to see how Theorem 6.3.17 is used.

The functions H and H_r in the problem at hand are

$$\begin{aligned} H &= q^0 + q_1(y^2 - z^2) + q_2 z \\ H_r &= q^0 + q_1 y^2 - q_1 \int_{-1}^{1} z^2 d\mu_t + q_2 \int_{-1}^{1} z d\mu_t \end{aligned} \tag{8.5.2}$$

Equations (6.3.6) become

$$\frac{d\lambda_1}{dt} = 0 \qquad \frac{d\lambda_2}{dt} = -2\lambda_1 y.$$

Hence

$$\lambda_1(t) = c \qquad \frac{d\lambda_2}{dt} = -2cy \qquad 0 \leq t \leq t_f, \tag{8.5.3}$$

where t_f denotes the time at which the optimal trajectory hits $\mathcal{T}_1$.

Let (t_f, x_f, y_f) be the point at which the optimal trajectory first hits $\mathcal{T}_1$. Let $-\pi/2 \leq \alpha \leq \pi/2$ be the angle such that $x_f = a\cos\alpha$ and $y_f = a\sin\alpha$. Then in a neighborhood $\mathcal{N}$ (t_f, x_f, y_f) the terminal manifold can be given parametrically by

$$x_1 = a\cos(\theta + \alpha) \qquad y_1 = a\sin(\theta + \alpha) \qquad t_1 = \tau$$

for θ and τ each in an interval $(-\delta, \delta)$. The general tangent vector to $\mathcal{T}_1$ at points in the neighborhood $\mathcal{N}$ of (t_f, x_f, y_f) is $(d\tau, -a\sin(\theta + \alpha)d\theta, a\cos(\theta + \alpha)d\theta)$, which at the point (t_f, x_f, y_f) is

$$(d\tau, (-a\sin\alpha)d\theta,\ a\cos\alpha)d\theta).$$

The transversality condition (6.3.24) together with (8.5.1), (8.5.2), and (8.5.4) give

$$\lambda^0 + c[y_f^2 - \int_{-1}^{1} z^2 d\mu_t] + \lambda_2(t_f) \int_{-1}^{1} z d\mu_{t_f} = 0 \tag{8.5.4}$$
$$a[-c\sin\alpha + \lambda_2(t_f)\cos\alpha] = 0.$$

We next show that $x_f \neq 0$ by assuming the contrary and reaching a contradiction. If $x_f = 0$, then $\alpha = \pm\pi/2$ and the second equation in (8.5.4) gives $\lambda_1(t) = c = 0$. It follows from (8.5.4), (6.3.21), and (6.3.22) that for $0 \leq t \leq t_f$

$$\lambda^0 + \lambda_2(t) \int_{-1}^{1} z d\mu_t = 0.$$

Hence $\lambda_2(t) \neq 0$ for all $0 \leq t \leq t_f$, for otherwise $\lambda^0 = 0$ and there would exist a $t' \in [0, t_f]$ such that $(\lambda^0, \lambda_1(t'), \lambda_2(t') = (0, 0, 0)$, which cannot be.

The optimal control μ is a discrete measure control that is not an ordinary control. Thus,

$$\mu_t = \sum_{i=1}^{2} p^i(t)\delta_{u_i(t)} \qquad 0 < p^i < 1 \qquad i = 1, 2 \tag{8.5.5}$$

for t in a set $\mathcal{P}$ of positive measure contained in $[0, t_f]$.

Hence by (6.3.15), for t in $\mathcal{P}$, $\max\{H(t, \psi(t), z, \lambda(t)) : |z| \leq 1\}$ occurs at two distinct points. Since $c = 0$, from (8.5.2) we get that

$$H(t, \psi(t), z, \lambda(t)) = \lambda^0 + \lambda_2(t)z,$$

whose maximum on the interval $|z| \leq 1$ occurs at a unique point, since $\lambda_2(t) \neq 0$ for all t. Thus, $x_f \neq 0$.

The preceding argument also shows that $c \neq 0$. Once we have established that $x_f \neq 0$, a simpler argument exists. Since $x_f \neq 0$, $\cos\alpha \neq 0$. From the second equation in (8.5.4) we get that if $c = 0$, then $\lambda_2(t_f) = 0$. From the first equation in (8.5.4) we get that $\lambda^0 = 0$, and so $(\lambda_1^0, \lambda_1(t_f), \lambda_2(t_f)) = (0, 0, 0)$, which cannot be. Therefore, $c \neq 0$.

We now determine the optimal relaxed controls. From (8.5.2), (8.5.5), and (6.3.15) we get that

$$\max\{Q(z) \equiv -cz^2 + \lambda_2(t) : |z| \leq 1\}$$

must occur at two distinct points of the interval $-1 \leq z \leq 1$. The function Q is a quadratic whose graph passes through the origin. It is easy to see that if $c > 0$, then it is not possible for Q to have two maxima in the interval $[-1, 1]$. It is also easy to see that if $c < 0$, for Q to have two maxima in the interval $[-1, 1]$, we must have $\lambda_2(t) = 0$. The maxima will then occur at $z = 1$ and $z = -1$. Thus,

$$u_1(t) = 1 \text{ and } u_2(t) = -1. \tag{8.5.6}$$

From $\lambda_2(t) \equiv 0$ and (8.5.3) we get, since $c \neq 0$, that $y(t) \equiv 0$. From (8.5.1), (8.5.5), and (8.5.6) we get that $0 = p^1(t) - p^2(t)$. Also, $p^1(t) + p^2(t) = 1$. Hence $p^1(t) = p^2(t) = 1/2$, and the optimal relaxed control is

$$\mu_1 = \frac{1}{2}\,\delta_1 + \frac{1}{2}\,\delta_{-1}.$$

From (8.5.1) and $(t_0, x_0, y_0) = (0, 1, 0)$ we get that the optimal trajectory is

$$x(t) = 1 - t \qquad y(t) = 0$$

and the optimal time is $t_f = 1 - a$.

8.6 The Brachistochrone Problem

In Section 1.6 we formulated the brachistochrone problem, first as a simple problem in the calculus of variations and then as two versions of a control problem. The classical existence theorem, Theorem 5.4.18, that would ensure that the calculus of variations version of the brachistochrone problem has a solution requires that the integral in (1.6.3) be a convex function of y' and that $[1 + (y')^2]^{1/2}/|y'| \to \infty$ as $|y'| \to \infty$. A straightforward calculation shows that $d^2(1 + (y')^2)/d^2y' > 0$, so the integral is convex. The growth condition fails, since $[1 + (y')^2]^{1/2}/|y'| \to 1$ as $|y'| \to \infty$. In the absence of an existence theorem, the problem is solved as follows. The Euler equation is solved to determine the extremals for the problem, which are found to be cycloids. It is then shown that there is a unique cycloid $\mathcal{C}$ passing through P_0 and P_1. A field, in the sense of the calculus of variations, containing $\mathcal{C}$ is constructed, and an argument using the properties of such fields is used to show that $\mathcal{C}$ minimizes. See [15], [27], [87].

The formulation of the brachistochrone problem as in (1.6.4) and (1.6.5) also suffers from the inapplicability of the existence theorem, Theorem 5.4.16, to guarantee even the existence of a relaxed optimal control. The theorem requires $|u|$ to be of slower growth than $[(1 + u^2)/(y - \alpha)]^{1/2}$, which is not the case.

We shall show that the problem formulated in (1.6.7) subject to (1.6.6) has a solution in relaxed controls. The maximum principle, Theorem 6.3.12, will then be used to show that the relaxed optimal control is an ordinary control and that the optimal trajectory is a cycloid.

We change the notation in (1.6.6) and let ξ, η be the state variables and z the control variable. We then have the following:

Problem 8.6.1. Minimize $\int_{t_0}^{t_f} dt$ subject to:

$$\frac{d\xi}{dt} = [2g(\eta - \alpha)]^{1/2} \cos z, \quad \xi(t_0) = x_0 \tag{8.6.1}$$

$$\frac{d\eta}{dt} = [2g(\eta - \alpha)]^{1/2} \sin z, \quad \eta(t_0) = y_0,$$

$\pi \geq z \geq -\pi$ and the terminal conditions $\xi_1 = x_1$, $\eta_1 = y_1$, t_f free, where $x_1 > x_0$, $y_1 > \alpha$ and $\alpha = y_0 - v_0^2/2g$. Thus, $\Omega(t) = [-\pi, \pi]$ for all t and $\tau_1 = \{(t_f, x_0, y_1) : t_f \text{ free}\}$. We further assume that $v_0 \neq 0$; that is, the particle has an initial velocity. Without loss of generality we may take the origin of coordinates to be at the initial point and take the initial time to be zero. Thus, $(t_0, x_0, y_0) = (0, 0, 0)$. The constant α then becomes $\alpha = -v_0^2/2g$, and $y - \alpha = y + v_0^2/2g$.

We now show that the hypotheses of Theorem 4.3.5 are satisfied for Problem 8.6.1, and therefore there exists an optimal relaxed pair $((x^*, y^*); \mu^*)$ that minimizes the transit time from (x_0, y_0) to (x_1, y_1) over all admissible pairs. Let f denote the right-hand side of (8.6.1) and let $\widehat{f} = (f^0, f) = (1, f)$. Then, $\widehat{f}$ is continuous on $\mathbb{R} \times \mathbb{R}^2$. The set $\mathcal{B} = \{(0, 0, 0, t_f, x_1, y_1) : t_f \text{ free}\}$ is closed. The mapping Ω is a constant map, and so is u.s.c.i. It remains to show that we can restrict our attention to a compact interval $\mathcal{I}$ of the time variable, that the set of relaxed pairs $((x, y), \mu)$ is not empty, and that the graphs of these trajectories are contained in a compact subset of $\mathcal{I} \times \mathcal{R}^2$.

If we take $z(t) = \theta$, a constant, then the motion determined by (8.6.1) satisfies $d\eta/d\xi = \tan\theta$, and thus is motion along the line $\eta = \xi \tan\theta$. For the motion to hit the point (x_1, y_1), then we must have

$$\tan\theta = y_1/x_1. \tag{8.6.2}$$

Let (x, y) denote the trajectory obtained from (8.6.1) with θ determined by (8.6.2). For (x, y) to be admissible we must show that there exists a time t_1 such that $x(t_1) = x_1$ and $y(t_1) = y_1$. We now show that this is so.

From the second equation in (8.6.1) we get that

$$dy/dt = [2g(y + v_0^2/2g)]^{1/2} \sin\theta.$$

A straightforward calculation gives

$$[2g(y(t) + v_0^2/2g)]^{1/2} = gt\sin\theta + v_0.$$

To find $x(t)$, the corresponding value of x at time t, we substitute this equation into the first equation in (8.6.1) and

$$dx = [gt\sin\theta\cos\theta + v_0\cos\theta]dt.$$

Hence

$$x(t) = gt^2\sin\theta\cos\theta/2 + v_0\cos\theta t.$$

To find a value of t such that $x(t) = x_1$, in the preceding equation set $x(t) = x_1$ and use (8.6.2) to get $\cos\theta = x_1/d$ and $\sin\theta = y_1/d$, where $d = [x_1^2 + y_1^2]^{1/2} \neq 0$. We get that

$$(gy_1/2d^2)t_1^2 + \frac{v_0}{d}\, t_1 - 1 = 0.$$

Hence

$$t_1 = \{-v_0/d + [(v_0/d)^2 + 4(gy_1/2d^2)]^{1/2}\}/(gy_1/d^2).$$

We have just shown that the set of ordinary admissible pairs is not empty and hence that the set of relaxed admissible pairs is not empty. Since $\inf\{t_f : t_f$ terminal time of an admissible trajectory$\}$ is less than or equal to t_1, it follows that we can restrict our attention to admissible pairs defined on compact intervals $[0, t_f] \subseteq [0, t_1]$. If we set $\mathcal{I} = [0, t_1]$, we need only consider $\widehat{f}$ on $\mathcal{I} \times \mathbb{R}^2$.

It is easy to show that there exists a constant K such that

$$|\langle(\xi, \eta), f(\tau, \xi, \eta)\rangle| \leq K(|\xi|^2 + |\eta|^2 + 1)$$

for all t in $\mathcal{I}$, and all (ξ, η) in $\mathbb{R}^2$ and all $-\pi \leq z \leq \pi$. Hence by Corollary 4.3.15 all admissible relaxed trajectories defined on intervals contained in $\mathcal{I}$ lie in a compact set in $\mathcal{I} \times \mathbb{R}^2$. Thus, all the hypotheses of Theorem 4.3.5 are satisfied in Problem 8.6.1.

Hence we shall only consider optimal relaxed pairs and simply write $((x, y), \mu)$. Such a pair satisfies the Maximum Principle, Theorem 6.3.12, which we now use to determine the unique optimal pair.

The functions H and H_r, defined in (6.3.4) and (6.3.5), are in our problem given by

$$H = q^0 + [2g\eta + v_0^2]^{1/2}[q_1 \cos z + q_2 \sin z]$$
$$H_r = q^0 + [2g\eta + v_0^2]^{1/2}\left[q_1 \int_{-\pi}^{\pi} \cos z d\widetilde{\mu}_t + q_2 \int_{-\pi}^{\pi} \sin z d\widetilde{\mu}_t\right],$$

where $\widetilde{\mu}$ is a discrete measure control on $[-\pi, \pi]$. Thus,

$$H_{r\xi} = 0, \;\; H_{r\eta} = g[2g\eta + v_0^2]^{-1/2}\left[q_1 \int_{-\pi}^{\pi} \cos z d\widetilde{\mu}_t + q_2 \int_{-\pi}^{\pi} \sin z d\widetilde{\mu}_t\right].$$

Let $((x, y), \mu)$ be a relaxed optimal pair defined on the interval $[0, t_1]$. By Theorem 6.3.12, there exists a constant $\lambda^0 \geq 0$ and absolutely continuous functions λ_1 and λ_2 defined on $[0, t_1]$ such that $(\lambda^0, \lambda_1(t), \lambda_2(t)) \neq (0, 0, 0)$ for all t in $[0, t_1]$ and such that

$$\begin{aligned} \lambda_1'(t) = 0, \quad \lambda_2'(t) = & - g[2gy(t) + v_0^2]^{-\frac{1}{2}} \\ & \times \left[\lambda_1(t) \int_{-\pi}^{\pi} \cos z d\mu_t + \lambda_2(t) \int_{-\pi}^{\pi} \sin z d\mu_t\right] \end{aligned} \tag{8.6.3}$$

for almost all t in $[0, t_1]$. Hence

$$\lambda_1(t) = c, \tag{8.6.4}$$

for some constant c. The transversality condition (6.3.24) gives

$$\lambda^0 + [2gy(t_1) + v_0^2]^{1/2}\left[\lambda_1(t_1) \int_{-\pi}^{\pi} \cos z d\mu_{t_1} + \lambda_2(t_1) \int_{-\pi}^{\pi} \sin z d\mu_{t_1}\right] = 0.$$

The problem is autonomous, so by (6.3.21) and (6.3.22)

$$\lambda^0 + [2gy(t) + v_0^2]^{1/2} \left[\lambda_1(t) \int_{-\pi}^{\pi} \cos z d\mu_t + \lambda_2(t) \int_{-\pi}^{\pi} \sin z d\mu_t\right] = 0 \quad (8.6.5)$$

for all t in $[0, t_1]$. Hence, for all t in $[0, t_1]$,

$$\lambda_1^2(t) + \lambda_2^2(t) \neq 0. \quad (8.6.6)$$

Otherwise, there would exist a t' in $[0, t_1]$ such that $\lambda_1(t') = \lambda_2(t') = 0$. This in turn would imply that $\lambda^0 = 0$, and so $(\lambda^0, \lambda_1(t'), \lambda_2(t')) = (0, 0, 0)$ which cannot be.

We next use (6.3.15) to determine the discrete measure control μ. We have, setting $\widehat{\lambda}(t) = (\lambda^0, \lambda_1(t), \lambda_2(t))$,

$$\begin{aligned} M(t, x(t), y(t), \widehat{\lambda}(t)) &= \sup[H(t, x(t), y(t), z, \widehat{\lambda}(t)) : |z| \leq \pi] \\ &= \lambda^0 + [2gy(t) + v_0^2]^{1/2}[\sup\{\lambda_1(t) \cos z + \lambda_2(t) \sin z : |z| \leq \pi\}]. \end{aligned}$$

The term involving the sup can be written, using (8.6.4) as

$$\sup\{\langle(\cos z, \sin z),\ (c, \lambda_2(t))\rangle : |z| \leq \pi\}.$$

By (8.6.6), $(c, \lambda_2(t)) \neq (0, 0)$. Hence, the supremum is attained when $z = \theta(t)$, where $\theta(t)$ is such that $(\cos\theta, \sin\theta)$ is the unit vector in the direction of $(c, \lambda_2(t))$.

Thus,

$$\cos\theta(t) = c/(c^2 + \lambda_2^2(t))^{1/2}, \quad \sin\theta(t) = \lambda_2(t)/(c^2 + \lambda_2^2(t))^{1/2}. \quad (8.6.7)$$

We have shown that at each t in $[0, t_1]$ the supremum is achieved at a unique value $\theta(t)$ in $[-\pi, \pi]$. Hence for each t in $[0, t_1]$ the discrete measure control is such that μ_t is concentrated at $\theta(t)$. Thus, the relaxed optimal control μ is an ordinary control θ. The relaxed optimal trajectory is an ordinary trajectory.

We assert that $c > 0$. If $c = 0$, then by (8.6.7), $\cos\theta(t) = 0$ for all t in $[0, t_1]$. Hence $\theta(t) = \pm\pi/2$ and $\sin\theta(t) = \pm 1$ for all t in $[0, t_1]$. This would imply that the optimal trajectory lies on a vertical line. If $c \leq 0$, then $\cos\theta(t) < 0$, for all t in $[0, t_1]$, and so $dx/dt < 0$ for all t. Since $x_1 > 0$, this is impossible. Hence

$$\cos\theta(t) > 0 \quad \text{and} \quad -\pi/2 < \theta(t) < \pi/2 \text{ a.e.} \quad (8.6.8)$$

It follows from (8.6.8) and the first equation in (8.6.1) that $dx/dt > 0$. Hence the optimal trajectory can be given by $y = y(x)$. This is not assumed *a priori* here, in contrast to the calculus of variations treatments, where a minimizing curve that has the form $y = f(x)$ is sought.

Since c and $\cos\theta(t)$ are greater than zero, we get from (8.6.7) that

$$\lambda_2(t) = c \tan\theta. \quad (8.6.9)$$

If we substitute (8.6.4) and (8.6.9) into (8.6.3) and (8.6.5) and use the fact that μ_t is concentrated at $\theta(t)$ we get that

$$\lambda_2'(t) = -gc[2gy(t) + v_0^2]^{-1/2} \sec\theta(t), \tag{8.6.10}$$

and

$$\lambda^0 + c[2gy(t) + v_0^2]^{1/2} \sec\theta(t) = 0. \tag{8.6.11}$$

From (8.6.11) we conclude that $\lambda^0 \neq 0$. Hence we may take $\lambda^0 = -1$, and (8.6.11) becomes

$$-1 + c[2gy(t) + v_0^2]^{1/2} \sec\theta(t) = 0.$$

Thus,

$$[2gy(t) + v_0^2]^{1/2} = \cos\theta(t)/c, \tag{8.6.12}$$

and so

$$y(t) + v_0^2/2g = \cos^2\theta(t)/2gc^2.$$

Setting $\theta = u/2$ and using the half angle formula gives

$$y + v_0^2/2g = b(1 + \cos u), \qquad b = (4gc^2)^{-1}. \tag{8.6.13}$$

From (8.6.9) and the fact that $\theta(t) \in (-\pi/2, \pi/2)$ we get that $\theta = \arctan(\lambda_2/c)$. Since λ_2 is absolutely continuous, so is θ and $\theta'(t)$ exists for almost all t in $(0, t_1)$. From (8.6.9) we get that

$$\lambda_2'(t) = [c\sec^2\theta(t)]d\theta/dt.$$

Substituting (8.6.12) into (8.6.10) gives

$$\lambda_2'(t) = -gc^2 \sec^2\theta.$$

Hence

$$d\theta/dt = -gc.$$

Therefore, θ is a strictly decreasing differentiable function of t. Hence t is a strictly decreasing differentiable function of θ and

$$dt/d\theta == (d\theta/dt)^{-1} = -(gc)^{-1}. \tag{8.6.14}$$

From the first equation in (8.6.1) and from (8.6.12) we get that

$$dx/dt = \cos^2\theta(t)/c.$$

From this and from (8.6.14) we get that

$$\begin{aligned} dx/d\theta &= (dx/dt)(dt/d\theta) = -(gc^2)^{-1}\cos^2\theta(t) \\ &= -(2gc^2)^{-1}(1 + \cos 2\theta). \end{aligned}$$

Therefore,

$$x = (-4gc^2)^{-1}(u + \sin u) + a,$$

where a is an arbitrary constant. If we now set $u = -w$ in (8.6.13), we get that

$$
\begin{aligned}
y + v_0^2/2g &= b(1 + \cos w) \\
x - a &= b(w + \sin w)
\end{aligned}
\tag{8.6.15}
$$

where $b = (4gc^2)^{-1} > 0$. Equations (8.6.15) use the parametric equations of an inverted cycloid. They represent the locus of a fixed point on the circumference of a circle of radius b as the circle rolls on the lower side of the line $y = v_0^2/2g$.

8.7 Flight Mechanics

In this section we consider the problem formulated in Section 1.4. We change the notation from that in Section 1.4 and denote the position coordinates by (ξ, η), the velocity components by (π^1, π^2), the mass variable by ν, the angle coordinate by z_1, and the thrust coordinate by z_2. We assume that the motion takes place in the earth's gravitational field and that the only external force is gravity.

The equations governing planar rocket flight become:

$$
\begin{aligned}
\frac{d\xi}{dt} &= \pi^1 & \frac{d\eta}{dt} &= \pi^2 \\
\frac{d\pi^1}{dt} &= -(cz_2/\nu)\cos z_1 & \frac{d\pi^2}{dt} &= -g - (c\beta/\nu)\sin z_1 \\
\frac{d\nu}{dt} &= -z_2
\end{aligned}
\tag{8.7.1}
$$

where

$$
\begin{aligned}
&0 \le B \le z_2 \le A, \quad -\pi \le z_1 \le \pi, \quad \nu > M > 0 \\
&\Omega(t) = \{(z_1, z_2) : -\pi \le z_1 \le \pi,\ 0 < B \le z_2 \le A\}.
\end{aligned}
\tag{8.7.2}
$$

Thus, the constraint sets $\Omega(t)$ are constant and given as in (8.7.2) for given constants A, B, M. We take the initial time $t_0 = 0$ and the other initial values $(\xi_0, \eta_0, \pi_0^1, \pi_0^2, \nu_0)$ also to be fixed.

We first consider the hard landing problem. The terminal position (ξ_1, η_1) is fixed and the terminal time, velocity, and mass, $(t_1, \pi_1^1, \pi_1^2, \nu_1)$, are free with $\pi_1^1 > 0$, $\pi_1^2 > 0$.

Thus, $\mathcal{J}_1$ is given by (ξ_1, η_1) fixed and

$$
t_1 = \sigma^0, \quad \pi_1^1 = \sigma^3, \quad \pi_1^2 = \sigma^4, \quad \nu_1 = \sigma^5; \quad \sigma^i > 0, \quad i = 3, 4, \quad \sigma^5 > M.
\tag{8.7.3}
$$

The problem is to minimize

$$\nu_0 - \nu_1 = -\int_0^{t_1} (d\nu/dt)dt = \int_0^{t_1} z_2 dt$$

We leave it as an exercise for the reader to show that the set of admissible trajectories is not empty. For example, if we assume a constant direction $z_1 \equiv \omega_0$ and a constant thrust of magnitude $z_2 \equiv T$, $B \leq T \leq A$ so that

$$\pi_0^2/\pi_0^1 > (\beta/T)\tan\omega_0,$$

$$(\beta/T)(\tan\omega_0)(\xi_1 - \xi_0) - (\eta_1 - \eta_0) \geq 0,$$
$$(\pi_0^1/g)[\pi_0^2/\pi_0^1 - (\beta/T)\tan\omega_0] < \nu_0,$$

then the resulting trajectory will be admissible. The attainment of these inequalities depends on the initial and target points, the magnitude of the maximum initial velocity, and the allowable thrust limits A and B. The first and last requirements above depend on the state of rocket technology. It is a straightforward calculation to show that the condition $|\langle x, f(t,x,z)\rangle| \leq \Lambda(t)[|x|^2+1]$ of Lemma 4.3.14 holds. Since all trajectories have a fixed initial point, it follows that for any compact interval $\mathcal{I}$, all the trajectories restricted to $\mathcal{I}$ lie in a compact set. The sets $Q^+(t,x) = Q^+(\xi,\eta,\pi^1,\pi^2,\nu)$ in this problem are not convex, so we cannot use Theorem 4.4.2 to obtain a solution to our problem.

The hypotheses of Theorem 4.3.5 are fulfilled, however, so we get that the relaxed problem has an optimal relaxed admissible pair, which we denote by $((x,y,p,q,m);\mu)$. Theorem 6.3.27 is applicable to this solution. The functions H and H_r are given by

$$H = \rho^0 z_2 + \rho_1\pi^1 + \rho_2\pi^2 - \rho_3(cz_2/\nu)\cos z_1 + \rho_4(-g - (cz_2/\nu)\sin z_1) - \rho_5 z_2 \tag{8.7.4}$$

and

$$\begin{aligned} H_r = \rho^0 \int_\Omega z_2 d\mu_t + \rho_1\pi^1 + \rho_2\pi^2 - \rho_3 \int_\Omega (cz_2/\nu)\cos z_1 d\mu_t \\ - \rho_4 g - \rho_4 \int_\Omega (cz_2/\nu)\sin z_1 d\mu_t - \rho_5 \int_\Omega z_2 d\mu_t \end{aligned}$$

where μ_t is a discrete control measure on Ω.

There exist a constant $\lambda^0 \leq 0$ and absolutely continuous functions $\lambda_1, \ldots, \lambda_5$ defined on $[0,t_1]$ such that for t in $[0,t_1]$

$$(\lambda^0, \lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), \lambda_5(t)) \neq 0$$

and

$$\begin{aligned} \frac{d\lambda_1}{dt} &= -H_{r\xi} = 0 \qquad \frac{d\lambda_2}{dt} = H_{r\eta} = 0 \\ \frac{d\lambda_3}{dt} &= -H_{r\pi^1} = -\lambda_1(t) \qquad \frac{d\lambda_4}{dt} = -H_{r\pi^2} = -\lambda_2(t) \end{aligned} \tag{8.7.5}$$

$$\frac{d\lambda_5}{dt} = -H_{r\nu} = \lambda_3(t)\int_\Omega \frac{cz_2}{\nu^2}\cos z_1 d\mu_t + \lambda_4(t)\int_\Omega \frac{cz_2}{\nu^2}\sin z_1 d\mu_t.$$

Hence

$$\begin{aligned} &\lambda_1(t) = a_1 \qquad \lambda_2(t) = a_2 \text{ for all } t \text{ in } [0, t_1] \\ &\lambda_3(t) = -a_1 t + a_3 \qquad \lambda_4(t) = -a_2 t + a_4 \end{aligned} \tag{8.7.6}$$

for constants a_1, a_2, a_3, a_4.

Since the initial point is fixed and the terminal set $\mathcal{J}_\infty$ is given by (8.7.3), the transversality condition (6.3.24) gives that the $n+1$ vector $(-H_r(\pi(t_1)), \lambda(t_1))$ is orthogonal to $\mathcal{J}_\infty$ at $(t_1, x(t_1), y(t_1), p(t_1), q(t_1), m(t_1))$. From (8.7.3) we get that the vectors $(d\sigma^0, d\sigma^3, d\sigma^4, d\sigma^5)$ of the form $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$ are a basis to the tangent space to $\mathcal{J}_1$ at $(t_1, x(t_1), y(t_1), p(t_1), q(t_1), m(t_1))$. Thus,

$$H(\pi(t_1)) = 0 \quad \lambda_3(t_1) = \lambda_4(t_1) = \lambda_5(t_1) = 0. \tag{8.7.7}$$

From (8.7.7) and (8.7.6) we get that

$$\lambda_3(t) = a_1(t_1 - t) \qquad \lambda_4(t) = a_2(t_1 - t). \tag{8.7.8}$$

From (8.7.7), with $\widehat{\rho} = \widehat{\lambda}(t)$, and (8.7.4) we get that

$$\lambda^0 \int_\Omega z_2 d\mu_{t_1} + a_1 p(t_1) + a_2 q(t_1) = 0.$$

This implies that $a_1^2 + a_2^2 \neq 0$. For otherwise, since $p(t_1) > 0$ and $q(t_1) > 0$, we would have $\lambda^0 = 0$, and $(\lambda^0, \lambda_1(t_1), \lambda_2(t_1), \lambda_3(t_1), \lambda_4(t_1), \lambda_5(t_1)) = 0$, which cannot be.

We now use (6.3.15) to show that the optimal discrete measure control is an ordinary control. We let $\psi = (x, y, p, q, m)$ denote the optimal trajectory. Then, as in Theorem 6.3.12, we get

$$M(t, \psi(t), \widehat{\lambda}(t) = \sup\{H(t, \psi(t), z, \widehat{\lambda}(t)) : z = (z_1, z_2) \in \Omega\}$$

From (8.7.4) we see that in calculating the sup we need only calculate

$$\sup\{[\lambda^0 - \lambda_5(t)]z_2 - (\frac{cz_2}{m})[\lambda_3(t)\cos z_1 + \lambda_4(t)\sin z_1] : (z_1, z_2) \in \Omega\}. \tag{8.7.9}$$

Since $c > 0, m > 0$ and $z_2 > 0$, we can first maximize with respect to z_1. The quantity $\lambda_3(t)\cos z_1 + \lambda_4(t)\sin z_1$ can be interpreted as the inner product of $(\lambda_3(t), \lambda_4(t))$ with $(\cos z_1, \sin z_1)$. Hence it is maximized at the unique angle $z_1 = \omega(t)$, where

$$\cos\omega(t) = -\lambda_3(t)/(\lambda_3^2(t) + \lambda_4^2(t))^{1/2},$$
$$\sin\omega(t) = -\lambda_4(t)/(\lambda_3^2(t) + \lambda_4^2(t))^{1/2}.$$

From (8.7.8) we get

$$\cos\omega(t) = -a_1/(a_1^2+a_2^2)^{1/2} \qquad \sin\omega(t) = -a_2/(a_1^2+a_2^2)^{1/2}. \tag{8.7.10}$$

Thus, the optimal direction is constant throughout the flight.

From (8.7.8) and (8.7.10) we get that

$$\lambda_3(t)\cos\omega(t) + \lambda_4(t)\sin\omega(t) = (a_1^2+a_2^2)^{1/2}(t_1 - t) \tag{8.7.11}$$

substituting (8.7.11) into (8.7.9), and using the fact that $\omega(t)$ maximizes (8.7.9) over all z_1 in Ω, we get that to complete the determination of the sup in (8.7.9) we need to determine

$$\sup\{[\lambda^0 - \lambda_5(t) + (c/m(t))(a_1^2+a_2^2)^{1/2}(t_1-t)]z_2 : B \le z_2 \le A\}. \tag{8.7.12}$$

Let $F(t)$ equal the expression in square brackets in (8.7.12). The function F is absolutely continuous and

$$F'(t) = -\lambda_5'(t) - \alpha((cm'/m^2(t))(t_1 - t) + c/m(t)),$$

where $\alpha = (a_1^2 + a_2^2)^{1/2}$. From (8.7.5) with $z_1 = \omega(t)$ and (8.7.11) we get that $\lambda_5'(t) > 0$. Hence $F'(t) < 0$, and so F is strictly decreasing. Therefore, the supremum in (8.7.12) is attained at a unique value $z_2 = v(t)$ in $[B, A]$. Therefore, the relaxed optimal control μ is the ordinary control $(v(t), \omega(t))$.

Since $\lambda^0 \le 0$ and $\lambda_5(t_1) = 0$, we get that $F(t_1) = \lambda^0 \le 0$. If $\lambda^0 < 0$, then $F(t_1) < 0$. If $F(0) > 0$, then the supremum is attained at the unique value $z_2 = v(t)$ on some interval $[0, t_s]$ and $v(t) = B$ on $[t_s, t_1]$. If $F(0) \le 0$, then $z_2 = v(t) = B$ uniquely on $[t_0, t_1]$. If $\lambda^0 = 0$, then $F(t_1) = 0$, and so $F(0) > 0$ for $t \in [0, t_1)$. Therefore, $z_2 = v(t) = A$, uniquely on $[0, t_1]$. To summarize we have shown that the supremum in (8.7.9) is achieved at a unique point $(z_1, z_2) = (\omega(t), v(t))$. Therefore, the relaxed optimal control μ is the ordinary control $(\omega(t), v(t))$.

We now consider the soft landing problem. The state equations are given by (8.7.1) and the constraint set by (8.7.2). The initial conditions are fixed as in the hard landing case. The terminal conditions are now $(\xi_1, \eta_1, \pi_1^1, \pi_2^1)$ fixed, with $\pi_1^1 \ge 0$, $\pi_2^1 \ge 0$, and t_1, m_1 free. Thus,

$$\mathcal{J}_2 = \{(t_1, \nu_1) : t_1 = \sigma^1 > 0, \nu_1 = \sigma^2 > u^1\} \cup \{\xi_1, \eta_1, \pi_1^1, \pi_1^2\}.$$

We again leave the verification of the assumption that the set of admissible relaxed pairs is non-empty to the reader. The argument in the hard landing case that all trajectories defined on a given compact interval lie in a compact set is independent of the terminal conditions, and so is valid here also. Thus, an optimal relaxed pair $((x, y, p, q, m); \mu)$ exists and satisfies Theorem 6.3.27.

The functions H and H_r are as in (8.7.4), and there exists a constant λ^0 and absolutely continuous functions $\lambda_1, \ldots, \lambda_5$ defined on $[0, t_1]$ such that for t in $[0, t_1]$ $(\lambda^0, \lambda_1(t), \ldots, \lambda_5(t)) \ne 0$. The functions $\lambda_i, i = 1, \ldots, 5$ satisfy (8.7.5).

Hence $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are given by (8.7.6). The transversality condition (6.3.24) now gives that the two-dimensional vector $(-H(\pi(t_1)), \lambda_5(t_1))$ is orthogonal to $\mathcal{J}_2$ at the end point of the optimal trajectory. Thus, $-H(\pi(t_1))dt + \lambda_5(t_1)dm = 0$ for all tangent vectors (dt, dm) to $\mathcal{J}_2$ at $(t_1, \psi(t_1))$. Hence

$$H(\pi(t_1)) = 0 \qquad \lambda_5(t_1) = 0. \tag{8.7.13}$$

We assert that $\lambda_3(t)$ and $\lambda_4(t)$ cannot both be identically equal to zero. If it were the case that $\lambda_3(t) \equiv 0$ and $\lambda_4(t) \equiv 0$, then $a_1 = 0$ and $a_2 = 0$. It would then follow from (8.7.6) and (8.7.13) that $\lambda^0 = 0$ and $\lambda_i(t) = 0, i = 1, \ldots, 5$, which cannot be.

We now proceed to calculate (8.7.9) in the present case. Since λ_3 and λ_4 are linear functions of t, they can vanish simultaneously at most at one point. We denote such a point, if it exists, as $t = \tau$. As in the hard landing case, we first maximize with respect to z_1. With the possible exception of a point τ at which $\lambda_3(\tau) = \lambda_4(\tau) = 0$, the maximum with respect to z_1 occurs at a unique angle ω, where

$$\begin{aligned} \cos\omega(t) &= -\lambda_3(t)/(\lambda_3(t)^2 + \lambda_4(t)^2)^{1/2} \\ \sin\omega(t) &= -\lambda_4(t)/(\lambda_3(t)^2 + \lambda_4(t)^2)^{1/2}. \end{aligned} \tag{8.7.14}$$

We consider two cases.

Case 1. λ_3 and λ_4 never vanish simultaneously. Then $(\lambda_3(t))^2 + (\lambda_4(t))^2 \neq 0$ for all t, and thus ω is given by (8.7.14) for all t.

Having maximized (8.7.9) with respect to z_1, to complete the calculation of (8.7.9) we need to calculate

$$\sup\{[\lambda^0 - \lambda_5(t) + (c/m(t))(\lambda_3(t)^2 + \lambda_4(t)^2)^{1/2}]z_2 : B \leq z_2 \leq A\},$$

where we have substituted (8.7.14) into (8.7.9). To determine the supremum, we must again determine the sign of the term in the square brackets.

Case 2. There exists τ such that $\lambda_3(\tau) = \lambda_4(\tau)$. Then,

$$\lambda_3(t) = \alpha(t - \tau) \qquad \lambda_4(t) = \beta(t - \tau)$$

for appropriate α, β. Then, for $t \neq \tau$

$$\cos\omega(t) = -\frac{\alpha(t-\tau)}{\sqrt{\alpha^2 + \beta^2}|t-\tau|} \qquad \sin\omega(t) = -\frac{\beta(t-\tau)}{\sqrt{\alpha^2 + \beta^2}|t-\tau|}.$$

Thus, as t "passes through τ" $\cos\omega(t)$ and $\sin\omega(t)$ both change sign, that is,

$$\begin{aligned} \lim_{t\to\tau^\pm} \quad \cos\omega(t) &= \mp\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \\ \lim_{t\to\tau^\pm} \quad \sin\omega(t) &= \mp\frac{\beta}{\sqrt{\alpha^2 + \beta^2}}. \end{aligned}$$

Thus, $\omega(t)$ jumps by $\pm\pi$ as we pass through τ. This is a thrust reversal.

Now maximize with respect to z_2. Again let $F(t)$ denote the coefficient of z_2

$$F(t) = \lambda^0 - \lambda_5(t) + \frac{c}{m(t)}\ [\lambda_3(t)^2 + \lambda_4(t)^2]^{1/2}$$

Let

$$Q(t) = \lambda_3(t)^2 + \lambda_4(t)^2.$$

Note that $Q(t)$ is quadratic in t.

Case 2.1. $Q(t) \neq 0$ for all t. That is, λ_3 and λ_4 do not have simultaneous zeros, that is, no thrust reversal.

$$\frac{dF}{dt} = -\frac{d\lambda_5}{dt} - \frac{cQ(t)^{1/2}}{m^2(t)}\frac{dm}{dt} + \frac{1}{2}\frac{c}{m(t)}\ Q(t)^{-1/2}\frac{dQ}{dt}.$$

Now,

$$\frac{d\lambda_5}{dt} = \frac{cv(t)}{m^2(t)}\ Q^{1/2}(t), \qquad \frac{dm}{dt} = -v(t).$$

So

$$\frac{dF}{dt} = \frac{1}{2}\frac{c}{m(t)}\ Q(t)^{-1/2}\frac{dQ}{dt}$$

Note that dQ/dt is linear. In fact,

$$\begin{aligned}\frac{1}{2}\frac{dQ}{dt} &= \lambda_3(t)(-a_1) + \lambda_4(t)(-a_2)\\ &= (a_1^2 + a_2^2)t - a_1a_3 - a_2a_4.\end{aligned}$$

Hence dQ/dt has a positive slope. Since $\frac{c}{m(t)}\ Q^{-1/2}(t) > 0$ we have

$$\text{sign}\ \frac{dF}{dt} = \ \text{sign}\ \frac{dQ}{dt}.$$

Case 2.2. Now consider the case in which thrust reversal occurs. Again let

$$Q(t) = \lambda_3(t)^2 + \lambda_4(t)^2.$$

Then,

$$\begin{aligned}Q(t) &= (\alpha^2 + \beta^2)(t-\tau)^2\\ Q(t)^{1/2} &= (\alpha^2 + \beta^2)^{1/2}|t-\tau|.\end{aligned}$$

And we now have

$$\begin{aligned}F(t) &= \lambda^0 - \lambda_5(t) + \frac{c}{m(t)}\ Q(t)^{1/2}\\ &= \lambda^0 - \lambda_5(t) + (\alpha^2 + \beta^2)^{1/2}|t-\tau|\left(\frac{c}{m(t)}\right).\end{aligned}$$

Therefore,

$$\frac{dF}{dt} = -(\alpha^2 + \beta^2)^{1/2} \frac{c}{m(t)}, \quad 0 \le t < \tau$$
$$\frac{dF}{dt} = (\alpha^2 + \beta^2)^{1/2} \frac{c}{m(t)}, \quad \tau < t \le t_1.$$

We are now in a position to completely characterize the control $(\omega(t), v(t))$ as we did in the soft landing case. We note that F is decreasing for $t < \tau$ and increasing for $t > \tau$.

8.8 An Optimal Harvesting Problem

In this section we will apply the maximum principle to the example presented in Section 1.7. In dealing with this example and the one in the next section we employ Corollary 4.5.1 and directly proceed to employ the maximum principle. A population model of McKendric type [40], [68] with crowding effect is given by

$$\begin{aligned} \frac{\partial p(r,t)}{\partial t} + \frac{\partial p(r,t)}{\partial r} &= -\mu(r)p(r,t) - f(N(t))p(r,t) - u(t)p(r,t) \\ p(r,0) &= p_0(r) \\ p(0,t) &= \beta \int_0^\infty k(r)p(r,t)dr \\ N(t) &= \int_0^\infty p(r,t)dr \end{aligned} \tag{8.8.1}$$

We consider the problem of maximizing the harvest

$$J(u) = \int_0^T u(t)N(t)dt, \tag{8.8.2}$$

$$0 \le u(\cdot) \le M$$

Assumption 8.8.1. We need some technical assumptions.

(i) The functions k, $p_0 \ge 0$ are continuous. Further, $k(r) = p_0(r) = 0$ if $r \ge r_m$.

(ii) $\mu_{\max} \ge \mu \ge \epsilon_0 > 0$, $\int_0^\infty \exp\left(-\int_0^r \mu(s)ds\right) ds < \infty$.

(iii) $f \in C^1((0,\infty))$, $f' \ge 0$, $f(0) = 0$, $f \not\equiv 0$.

Using the method of characteristics

$$p(r,t)=\begin{cases} p_0(r-t)\exp\left(-\int_{r-t}^{r}\mu(\rho)d\rho\right) \\ \qquad\qquad\qquad \times\exp\left(-\int_0^t[f(N(s))+u(s)]ds\right), & r>t \\ p(0,t-r)\exp\left(-\int_0^r\mu(\rho)d\rho\right) \\ \qquad\qquad\qquad \times\exp\left(-\int_{t-r}^t[f(N(s))+u(s)]ds\right), & r<t \end{cases} \tag{8.8.3}$$

We have

$$\begin{aligned} p(0,t)=\beta\int_0^t & k(r)p(0,t-r)\exp\left(-\int_0^r\mu(\rho)d\rho\right) \\ & \times\exp\left(-\int_{t-r}^t[f(N(s))+u(s)]ds\right)dr \\ & +\beta\int_t^\infty k(r)p_0(r-t,0)\exp\left(-\int_{r-t}^r\mu(\rho)d\rho\right) \\ & \times\exp\left(-\int_0^t[f(N(s))+u(s)]ds\right)dr \end{aligned} \tag{8.8.4}$$

Let

$$w(t)=p(0,t)\exp\left(\int_0^t[f(N(s))+u(s)]ds\right) \tag{8.8.5}$$

Then,

$$\begin{aligned} w(t)=\beta\int_0^t & k(r)w(t-r)\exp\left(-\int_0^r\mu(\rho)d\rho\right)dr \\ & +\beta\int_t^\infty k(r)p_0(r-t)\exp\left(-\int_{r-t}^r\mu(\rho)d\rho\right)dr, \end{aligned} \tag{8.8.6}$$

$$\begin{aligned} N(t) &= \int_0^\infty p(r,t)dr \\ &= \exp\left(-\int_0^t[f(N(s))+u(s)]ds\right)\left[\int_0^t w(t-r)\exp\left(-\int_0^r\mu(\rho)d\rho\right)dr\right. \\ &\quad \left.+\int_t^\infty p_0(r-t,0)\exp\left(-\int_{r-t}^r\mu(\rho)d\rho\right)dr\right] \end{aligned} \tag{8.8.7}$$

Let

$$\begin{aligned} \eta(t)=\int_0^t & w(t-r)\exp\left(-\int_0^t\mu(\rho)d\rho\right)dr \\ & +\int_t^\infty p_0(r-t)\exp\left(-\int_{r-t}^r\mu(\rho)d\rho\right)dr \end{aligned} \tag{8.8.8}$$

Then

$$N(t) = \eta(t) \exp\left(-\int_0^t [f(N(s)) + u(s)]ds\right)$$

$$N'(t) = \left(\frac{\eta'(t)}{\eta(t)} - [f(N(t)) + u(t)]\right) N(t) \tag{8.8.9}$$

$$N(0) = \int_0^\infty p_0(r)dr \equiv \alpha_0. \tag{8.8.10}$$

Integrating by parts we observe that

$$\int_0^T uNdt = \alpha_0 - N(T) + \int_0^T \left[\frac{\eta'(t)}{\eta(t)} - f(N(t))\right] N(t)dt \tag{8.8.11}$$

Thus, we have to choose u to maximize the harvest given in (8.8.11) under the conditions (8.8.9) and (8.8.10). Suppose $(\bar{N}, \bar{u})$ is optimal. The adjoint variable λ satisfies

$$\begin{aligned} -\frac{d\lambda}{dt} &= \left[\frac{\eta'(t)}{\eta(t)} - f'(N(t))N(t) - f(N(t)) - u(t)\right] \lambda \\ &\quad + \frac{\eta'(t)}{\eta(t)} - f'(N(t)N(t) - f(N(t)) \end{aligned} \tag{8.8.12}$$

$$\lambda(T) = -1$$

The maximum principle reads

$$-\lambda(t)\bar{N}(t)\bar{u}(t) \geq -\lambda(t)\bar{N}(t)v \qquad \forall\, v \in [0, M]$$

Thus,

$$\bar{u}(t) = \begin{cases} M, & \lambda(t) < 0 \\ 0, & \lambda(t) > 0 \end{cases}$$

Letting

$$\Phi(t) = \frac{\eta'(t)}{\eta(t)} - f'(\bar{N}(t))\bar{N}(t) - f(\bar{N}(t)) \tag{8.8.13}$$

we have

$$\lambda(t) = -1 + \int_t^T \bar{u}(s) \exp\left(\int_t^s [\Phi(\rho) - \bar{u}(\rho)]d\rho\right) ds \tag{8.8.14}$$

Note that $\lambda(T) = -1$. Thus, we solve for t_1 such that

$$0 = \lambda(t_1) = -1 + \int_{t_1}^T M \exp\left(\int_{t_1}^s [\Phi(\rho) - M]d\rho\right) ds$$

or

$$\int_{t_1}^T \exp\left(\int_{t_1}^s [\Phi(\rho) - M]d\rho\right) ds = \frac{1}{M} \tag{8.8.15}$$

Thus, for $t \leq t_1$ we set $\bar{u} \equiv 0$ and $\bar{u} \equiv M$ for $t \geq t_1$. Note that we can solve for $\bar{N}(t)$ explicitly from (8.8.9) setting $\bar{u} \equiv 0$ for $t \leq t_1$ and $\bar{u} \equiv M$ for $t \geq t_1$. Thus, Φ is known explicitly using (8.8.13). Thus, (8.8.15) is an equation for t_1.

In conclusion we see that the population should be allowed to build up until time t_1, and then harvest afterward with maximal effort, that is, $\bar{u}(t) \equiv M$. (Recall $0 \leq \bar{u}(t) \leq M$.)

8.9 Rotating Antenna Example

In this section we follow [47] and discuss in a sketchy manner the design of a control policy for the antenna problem presented in Example 1.5.2. This problem is discussed in complete detail in [47].

For the antenna problem in question we recall the control torque T is constrained in magnitude by requiring

$$|T| \leq k, \qquad k > 0.$$

Our initial state at time t_0 is $(\theta_0, \dot{\theta}_0)$ and our desired terminal state at $t_1 > t_0$ is $(\theta_1, 0)$. It is convenient to define the following quantities:

$$x_1 = Ik^{-1}(\theta - \theta_1), \tag{8.9.1}$$

$$x_2 = Ik^{-1}\dot{\theta}, \tag{8.9.2}$$

$$\xi = Ik^{-1}(\theta_0 - \theta_1), \tag{8.9.3}$$

$$\sigma = Ik^{-1}\dot{\theta}_0, \tag{8.9.4}$$

$$u = Tk^{-1}. \tag{8.9.5}$$

The equation of motion under an applied torque T is given by

$$I\ddot{\theta} + \beta\dot{\theta} = T,$$

where β is a damping factor and I is the moment of inertia of the system about the vertical axis. We make the simplifying assumption that friction is negligible and the equation of motion is then

$$I\ddot{\theta} = T \tag{8.9.6}$$

In terms of the x_1, x_2 variables, Eq. (8.9.6) becomes

$$\dot{x}_1 = x_2, \tag{8.9.7}$$

$$\dot{x}_2 = u,$$

and we have $x_1(t_0) = \xi$, $x_2(t_0) = \sigma$. In the plane determined by the coordinates (x_1, x_2) our objective will be achieved by passing to any point in the set

$$\Omega = \left\{ \left(\pm \frac{2\pi}{k}\, n, 0 \right) : n = 0, 1, 2, \ldots \right\},$$

where

$$|u| \leq 1$$

and the performance index

$$J = \int_{t_0}^{t_1} (\lambda_1 + \lambda_2 x_2^2 + \lambda_3 |u|) dt,$$

where $\lambda_1 = \gamma_1$, $\lambda_2 = \gamma_2 k^2 I^{-2}$, and $\lambda_3 = k\gamma_3$. We can, without loss of generality, assume that $\lambda_1 + \lambda_2 + \lambda_3 = 1$, since multiplying J by a positive constant does not affect our problem.

We can apply Theorem 6.3.12 to deal with this problem. Let (ψ_1, ψ_2) be the adjoint variable. Then, consider the Hamiltonian given by

$$H = \lambda^0(\lambda_1 + \lambda_2 x_2^2(t) + \lambda_3 |u(t)|) - \psi_1(t) x_2(t) - \psi_2(t) u(t), \qquad \lambda^0 \geq 0.$$

The adjoint equations are given by

$$\psi_1' = 0$$

$$\psi_2' = 2\lambda^0 \lambda_2 x_2 - \psi_1.$$

If $\lambda^0 = 0$, then

$$H = -\psi_1(t_0) x_2(t) - \psi_2(t) u(t) \qquad \text{a.e.}$$

Thus,

$$u(t) = \operatorname{sign} \psi_2(t), \quad \text{a.e. } t \in [t_0, t_1) \cap \{t : \psi_2(t) \neq 0\}.$$

If $\lambda^0 = 0$,

$$\psi_2(t) = \psi_2(t_0) - \psi_1(t_0)(t - t_0).$$

If $\psi_1(t_0) = 0$ also, $\psi_2(t) \equiv \psi_2(t_0)$ must be nonzero and u is either identically 1 or -1. If $\psi_1(t_0) \neq 0$, ψ_2 can vanish in at most one point, and u has to change sign across this point, that is, from -1 to 1 or 1 to -1. However, this can be eliminated.

Next, consider $\lambda^0 > 0$, in which case λ^0 may be taken to be 1. In this case

$$\psi_2' = 2\lambda_2 x_2 - \psi_1(t_0),$$

$$x_1(t) = \xi_1 + \int_{t_0}^{t} x_2(\tau) d\tau.$$

Thus,

$$\psi_2(t) = \psi_2(t_0) + 2\lambda_2(x_1(t) - \xi) - \psi_1(t_0)(t - t_0),$$

and the Hamiltonian becomes

$$H(t) = \lambda_1 + \lambda_2 x_2^2 + \lambda_3|u| - \psi_1(t_0)x_2(t) - (\psi_2(t_0) \\ + 2\lambda_2(x_1(t) - \xi) - \psi(t_0)(t - t_0))u(t),$$

and $H(t)$ is minimized if

$$\lambda_3|u| - \psi_2(t)u(t)$$

is minimized. Thus, we have

$$u(t) = \begin{cases} 1 & \text{if } \psi_2(t) > \lambda_3, \\ 0 & \text{if } |\psi_2(t)| < \lambda_3, \\ -1 & \text{if } \psi_2(t) < -\lambda_3. \end{cases} \tag{8.9.8}$$

If $\psi_2(t) = \lambda_3$, then $0 \leq u(t) \leq 1$, and if $\psi_2(t) = -\lambda_3$, then $-1 \leq u(t) \leq 0$.

We note that

$$\dot{\psi}_2(t) = 2\lambda_2 x_2 - \psi_1(t_0).$$

Thus,

$$\ddot{\psi}_2(t) = 2\lambda_2 \dot{x}_2(t) = 2\lambda_2 u(t). \tag{8.9.9}$$

Thus,

$$\begin{aligned} &u(t) = 1 \quad \text{and } \ddot{\psi}_2(t) > 0 \quad \text{if } \psi_2(t) > \lambda_3 \\ &u(t) = 0 \quad \text{and } \ddot{\psi}_2(t) = 0 \quad \text{if } |\psi_2(t)| < \lambda_3 \\ &u(t) = -1 \text{ and } \ddot{\psi}_2(t) < 0 \quad \text{if } \psi_2(t) < -\lambda_3 \end{aligned} \tag{8.9.10}$$

If $\psi_2(t) \equiv \lambda_3$ on an interval, then $\dot{\psi}_2 = \ddot{\psi}_2 = 0$ on the same interval. Likewise if $\psi_2(t) \equiv -\lambda_3$ on an interval. Hence, we have

$$\ddot{\psi}(t) = 0 \qquad \text{if } |\psi_2(t)| \leq \lambda_3, \tag{8.9.11}$$

and from (8.9.9) and (8.9.11) it follows that $u \equiv 0$ on any interval on which $|\psi_2(t)| = \lambda_3$. We observe from (8.9.10) and (8.9.11) that ψ_2 is linear on any interval $[a, b]$ on which $|\psi_2(t)| \leq \lambda_3$. Since $\dot{\psi}_2$ is continuous it follows that either ψ_2 is constant on $[a, b]$ or $\dot{\psi}_2$ is constant and nonzero on $[a, b]$. The four generic possibilities are illustrated in Fig. 8.2.

If $\psi_2(t_0) > \lambda_3$ the possible ψ_2-trajectories are as in Fig. 8.3, and the possible sequence of values assumed by u are

(a) $\{+1\}$ (c) $\{+1, 0, +1\}$

(b) $\{+1, 0\}$ (d) $\{+1, 0, -1\}$

If $\psi_2(t_0) < -\lambda_3$ the sequence of values assumed by u are

(a) $\{-1\}$ (c) $\{-1, 0, -1\}$

(b) $\{-1, 0\}$ (d) $\{-1, 0, 1\}$

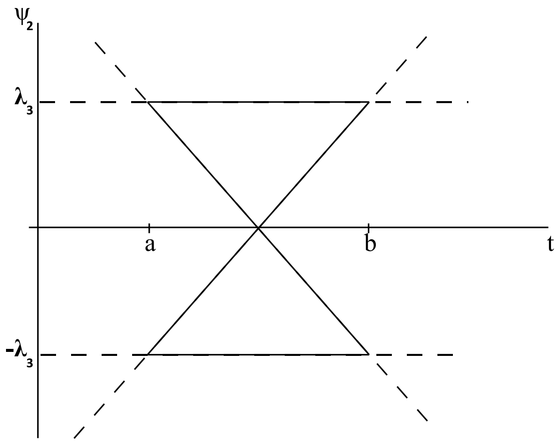

FIGURE 8.2

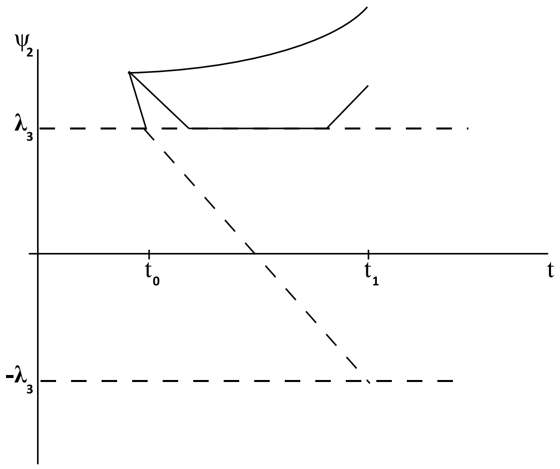

FIGURE 8.3

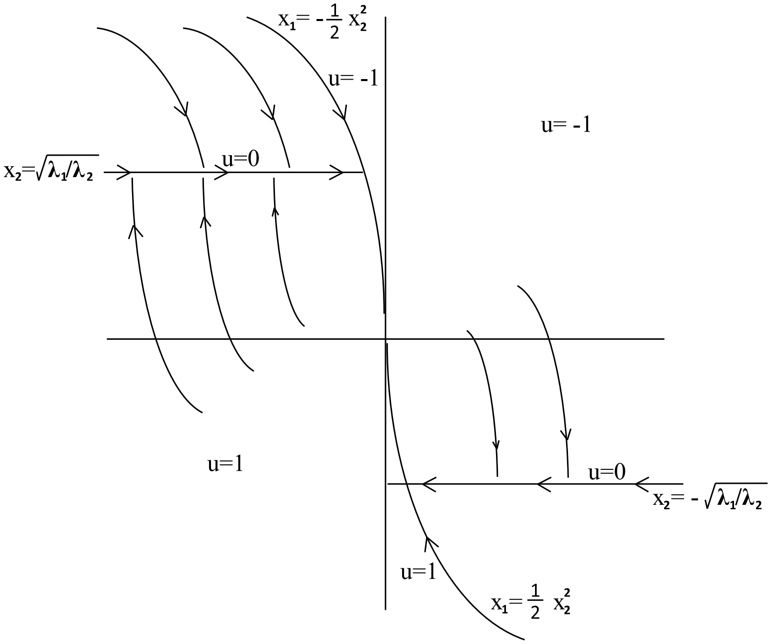

FIGURE 8.4

If $|\psi_2(t_0)| \leq \lambda_3$, the possible sequence of values assumed by u are

(a) $\{0\}$

(b) $\{0, 1\}$

(c) $\{0, -1\}$

If $\lambda_3 = 0$, we immediately see from (8.9.8) that u takes values 1 or -1, except at a single point which we can neglect. That is, the sequence of control values assumed are

(a) $\{-1, 1\}$

(b) $\{1, -1\}$

When $\lambda_3 = 0$ the trajectory pattern and control synthesis could be deduced from Fig. 8.4. Note that as $\lambda_2 \to 0$ the lines $x_2 = \sqrt{\lambda_1/\lambda_2}$ and $x_2 = -\sqrt{\lambda_1/\lambda_2}$ pass to infinity and the trajectory pattern gets simple.

If λ_2 and λ_3 are positive the trajectory pattern and control synthesis are as in Fig. 8.5.

Finally, we analyze the cost calculation relevant to the target $(0, 0)$ as a function of the present state of the system. That is, we shall determine a function C defined on the (x_1, x_2)-plane such that $C(x_1, x_2)$ is the value of J computed for an optimal trajectory starting at (x_1, x_2) and passing to the origin.

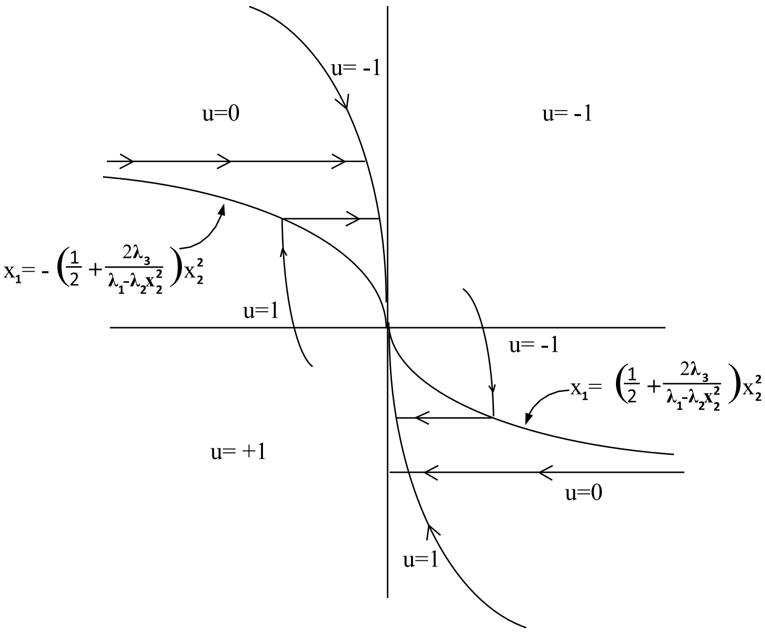

FIGURE 8.5

Let $\eta(0)$ denote the curve in the (x_1, x_2)-plane defined by the formula

$$x_1 = -\frac{1}{2}\,x_2^2, \qquad x_2 \geq 0,$$
$$x_1 = \frac{1}{2}\,x_2^2, \qquad x_2 \leq 0.$$

For (μ, ν) on $\eta(0)$, we have

$$C(\mu,\nu) = \int_0^{t_1} (\lambda_1 + \lambda_2 x_2(\tau)^2 + \lambda_3 |u(\tau)|)d\tau = \lambda_1 + \lambda_3 t_1 + \lambda_2 \int_0^{t_1} (\nu - \mathrm{sign}(\nu)\tau)^2 d\tau,$$

and $t_1 = |\nu|$. Hence,

$$C(\mu,\nu) = (\lambda_1 + \lambda_3)|\nu| + \frac{\lambda_2}{3}\,|\nu|^3.$$

Let S_1 be the region in the (x_1, x_2)-plane to the left of $\eta(0)$, on or above the curve ψ, and on or below the line $x_2 = \sqrt{\lambda_1/\lambda_2}$. Then,

$$\begin{aligned} C(\mu,\nu) &= (\lambda_1 + \lambda_3)\nu + \frac{\lambda_2}{3}\,\nu^3 + \int_0^{\tau_2} (\lambda_1 + \lambda_2 \nu^2) d\tau \\ &= (\lambda_1 + \lambda_3)\nu + \frac{\lambda_2}{3}\,\nu^3 + (\lambda_1 + \lambda_2 \nu^2)\tau_2, \end{aligned}$$

where τ_2 is the time when the trajectory reached $\eta(0)$. τ_2 is easily computed to be given by the formula

$$\tau_2 = \frac{|\mu|}{\nu} - \frac{1}{2}\,\nu.$$

We refer the reader to [47] for cost calculation relevant to all the terminal states $\Omega = \left\{\left(\pm\frac{2\pi}{k}\,n, 0\right) : n = 0, 1, 2, \dots\right\}$.

Chapter 9

Systems Governed by Integrodifferential Systems

9.1 Introduction

In this chapter we treat problems governed by integrodifferential systems [65], [67]. In Section 9.2 we present a version of the problems of interest and discuss existence. In Section 9.3 we specialize to systems linear in the state. In the next section we discuss linear systems and the bang-bang principle. In Section 9.5 we present a second version of the problem of interest. In Section 9.6 we discuss so-called linear plant quadratic cost criterion. We close the chapter by presenting minimum principle for a constrained problem governed by an integral equation.

9.2 Problem Statement

We consider a process governed by the equation

$$\phi(t) = F(t) + \int_0^t L(t, \phi(s), u(s), s)ds \tag{9.2.1}$$

under the conditions

$$W(\phi(t_1)) + \int_0^{t_1} M(\phi(t), u(t), t)dt = 0 \tag{9.2.2}$$

$$u(t) \in \Omega(t, \phi(t)). \tag{9.2.3}$$

It is required that the cost

$$\int_0^{t_1} F^0(\phi(t), u(t), t)dt \tag{9.2.4}$$

be minimum.

Assumption 9.2.1. (i) Let $\mathcal{X}, \mathcal{U}$ be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $\mathcal{I}_0$ an open interval of $\mathbb{R}$.

(ii) The function $L : \mathcal{I}_0 \times \mathcal{X} \times \mathcal{U} \times \mathcal{I}_0 \to \mathbb{R}$ is such that $L(t, x, u, s)$ is continuous in (t, x, u) for each fixed s, and measurable in s for fixed (t, x, u).

(iii) The function $M(x, u, t)$ from $\mathcal{X} \times \mathcal{U} \times \mathcal{I}_0$ into $\mathbb{R}$ is differentiable in x and continuous in (x, u) for fixed t, and measurable in t for fixed (x, u). We assume the same for the function $(x, u, t) \mapsto F^0(x, u, t)$.

(iv) For each compact subset $\Gamma \subset \mathcal{X} \times \mathcal{U}$ $\exists\ \Lambda \in L_2(\mathbb{R})$ such that $|\partial_x L(t, x, u, s)| + |L(t, x, u, s)| \leq \Lambda(s)$ for almost every $s \in \mathcal{I}_0$ and $|M_1(x, u, t)| + |M(x, u, t)| + |F_1^0(x, u, t)| + |F^0(x, u, t)| \leq \Lambda(t)$, for almost every t, where $\partial_x L$, M_1, F_1^0 denote matrices of partial derivatives with respect to x. Here and in what follows $|\cdot|$ denotes the euclidean norm of the vector or matrix in question. For existence we only need Lipschitz condition in the x-variable for L, M, and F^0.

(v) The function F is absolutely continuous with $F(0) \in \mathcal{X}$. We have imposed this condition for the purpose of obtaining necessary conditions. For existence we only need F to be continuous and $F(0) \in \mathcal{X}$.

(vi) The function $W : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. For existence we only need it to be lower semi-continuous.

(vii) The set valued map $\Omega(t, x)$, $(t, x) \in \mathcal{I}_0 \times \mathcal{X}$ is u.s.c.i. on $\mathcal{I} \times \mathcal{X}$, and for each $(t, x) \in \mathcal{I}_0 \times \mathcal{X}$, $\Omega(t, x)$ is compact.

Notation 9.2.2. We denote by $\mathcal{A}_R$ the set of all admissible pairs (ϕ, ν). That is, the set of all pairs (ϕ, ν) that satisfy (9.2.1), (9.2.2), and (9.2.3). Here, ν is a relaxed control.

Theorem 9.2.3. *Assume $\mathcal{A}_R \neq \emptyset$. Then, with Assumption 9.2.1 in force there exists a relaxed pair (ϕ, ν) satisfying Eqs. (9.2.1) to (9.2.3) and giving the cost in (9.2.4) its infimum value.*

Exercise 9.2.4. Prove Theorem 9.2.3.

Following Theorem 5.4.18 we also have the following:

Theorem 9.2.5. *In Assumption 9.2.1 we replace (vii) by the assumption that Ω is u.s.c. on $\mathcal{I}_0 \times \mathcal{X}$ and*

$$\bigcap_{\delta > 0} cl\, \Omega(N_{x\delta}(t_0, x_0)) \subseteq \Omega(t_0, x_0) \qquad \forall\ (t_0, x_0) \in \mathcal{I}_0 \times \mathcal{X},$$

$F^0(t, x, u) \geq \Phi(u)$, where Φ is continuous and $|u|^{-1}\Phi(u) \to \infty$ as $|u| \to \infty$, $u \in \mathcal{U}$. Then, problem (9.2.1)–(9.2.4) *has a relaxed solution.*

9.3 Systems Linear in the State Variable

Consider a plant governed by the system

$$\phi(t) = F(t) + \int_0^t A(t,s)\phi(s)ds + \int_0^t B(t,s)h(s,u(s))ds, \tag{9.3.1}$$

with the restrictions

$$W(\phi(t_1)) + \int_0^{t_1} [M(s)\phi(s) + k(s,u(s))]ds = 0, \tag{9.3.2}$$

and

$$u(t) \in \Omega(t). \tag{9.3.3}$$

The cost to be minimized is given by

$$\int_0^{t_1} [A_0(s)\phi(s) + h_0(s,u(s))]ds. \tag{9.3.4}$$

We assume that Assumption 9.2.1 remains in force. To apply Assumption 9.2.1 to the problem (9.3.1)–(9.3.4) we identify $L(t,\phi(s),u(s),s)$ of (9.2.1) with $A(t,s)\phi(s) + B(t,s)h(s,u(s))$ of (9.3.1). Similarly $M(\phi(t),u(t),t)$ of (9.2.2) is identified with $M(s)\phi(s) + k(s,u(s))$ of (9.3.2). We also assume that $A(t,s)$ and $B(t,s)$, $M(t)$ satisfy (ii), (iii), and (iv) of Assumption 9.2.1. We also remark that $A_0(s) + h_0(s,u)$ in (9.3.4) satisfies the same condition as F^0 that is stated in (iv) of Assumption 9.2.1. We enforce Assumption 9.2.1 in Sections 9.4 and 9.7. In (9.3.1)–(9.3.4), in addition to Assumption 9.2.1 we assume that

$$|A_0(s)| + |h_0(s,u(s))| \leq \Lambda(s)$$

where Λ is as in (iv) of Assumption 9.2.1. Note that, here, the set-valued map $\Omega(t)$ is independent of x.

Theorem 9.3.1. *Assume Assumption 9.2.1 is in force. We also assume that $h(\cdot,\cdot)$, $h_0(\cdot,\cdot)$, and $k(\cdot,\cdot)$ are continuous in both of their arguments. Suppose (ϕ,ν) is a relaxed admissible pair for* (9.3.1)–(9.3.4)*. Then, there exist an ordinary control $\widetilde{u}$ such that $(\phi,\widetilde{u})$ is also admissible giving the same value to the cost.*

Proof. From (9.3.4) we can define

$$\psi(t) = \int_0^t [A_0(s)\phi(s) + h_0(s,u(s))]ds,$$

and replace the cost by $\psi(t_1)$. Thus, it is no loss of generality to assume that the cost is in the form $g(\phi(t_1))$ in the proof, and we do.

Set

$$\zeta(t) = W(\phi(t_1)) + \int_0^t [M(s)\phi(s) + k(s,\nu_s)]ds,$$
$$\widetilde{\psi}(t) = \int_0^t [A_0(s)\phi(s) + h_0(s,\nu_s)]ds.$$

Now, corresponding to the relaxed admissible pair (ϕ, ν) we set

$$\begin{aligned}
\phi(t) - F(t) - \int_0^t A(t,s)\phi(s)ds &= \sum_{i=1}^{n+1} \int_0^t B(t,s)\lambda_i(s)h(s,u_i(s))ds \\
\widetilde{\psi}(t) - \int_0^t A_0(s)\phi(s)ds &= \sum_{i=1}^{n+1} \int_0^t \lambda_i(s)h_0(s,u_i(s))ds \qquad (9.3.5) \\
\zeta(t) - W(\phi(t_1)) - \int_0^t M(s)\phi(s)ds &= \sum_{i=1}^{n+1} \int_0^t \lambda_i(s)k(s,u_i(s))ds
\end{aligned}$$

where $\lambda_i \geq 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$. Our objective is to show that in (9.3.5), an ordinary control $\widetilde{u}$ exists which can be used in place of $\nu_s = \sum_{i=1}^{n+1} \lambda_i(s)\delta_{u_i}(s)$.

Define a mapping T from $L_2([0,t])$ to $\mathbb{R}^{n+2}$ as follows

$$T\rho = \int_0^t \begin{pmatrix} B(t,s) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rho(s)ds, \qquad \rho \in L_2([0,t]). \qquad (9.3.6)$$

The mapping T is continuous from the strong topology of $L_2([0,t])$ to $\mathbb{R}^{n+2}$. Letting $a \in \mathbb{R}^{n+2}$ be the left-hand side of (9.3.6) $T^{-1}(a)$ is closed and convex in $L_2([0,t])$. Let Σ denote the intersection of $T^{-1}(a)$ and $\Theta = \{\xi \mid \xi(s) \in co\,\widetilde{h}(s,\Omega(s)),\ \widetilde{h} = (h, h_0, k)^T$ a.e.$\}$. Let ξ_0 be an extreme point of Σ. Then, ξ_0 has the form $\xi_0(s) = \sum_{i=1}^{n+1} \lambda_i(s)\widetilde{h}(s,u_i(s))$, $u_i(s) \in \Omega(s)$. We can then prove that on no measurable subset E of $[0,t]$ with positive measure can we have $0 < \epsilon \leq \lambda_i(s) \leq 1-\epsilon$ for some i in the set $\{i = 1,\ldots,n+1\}$ and some $\epsilon > 0$. The argument for this has been presented in the proof of Theorem 4.7.7, and the proof is complete. ☐

Corollary 9.3.2. *With Assumption 9.2.1 in force, the problem of minimizing the cost (9.3.4) under the conditions (9.3.1) to (9.3.3) has a solution where the optimal control is ordinary.*

9.4 Linear Systems/The Bang-Bang Principle

For linear systems we modify (9.3.1) to (9.3.4) and consider

$$\phi(t) = F(t) + \int_0^t A(t,s)\phi(s)ds + \int_0^t B(t,s)u(s)ds, \tag{9.4.1}$$

$$W(\phi(t_1)) + \int_0^{t_1} M(s)\phi(s)ds + \int_0^{t_1} k(s)u(s)ds = 0, \tag{9.4.2}$$

$$u(t) \in \Omega(t). \tag{9.4.3}$$

The cost to be minimized is given by

$$\int_0^{t_1} [A_0(s)\phi(s) + h_0(s)u(s)]ds. \tag{9.4.4}$$

The control constraint set is independent of x and compact for each t. We make the additional assumptions

$$|A(t+h,s) - A(t,s)| \leq \Lambda(s)|h|,$$
$$|B(t+h,s) - B(t,s)| \leq \Lambda(s)|h|,$$

where Λ is as in (iv) of Assumption 9.2.1. We also assume that the set of admissible relaxed pairs is not empty. We have the following.

Remark 9.4.1. The problem of minimizing the cost in (9.4.4) under the conditions (9.4.1) to (9.4.3) with $\Omega(t)$ compact for each t has a solution with the control being ordinary. Note that a minimizing sequence of admissible trajectories for this problem constitutes an equi-continuous family.

Exercise 9.4.2. Verify Remark 9.4.1.

In (9.4.3) we replace Ω by $\mathcal{C}$, a compact convex polyhedron in $\mathbb{R}^m$. Then the above problem has an optimal control taking values in the vertices of $\mathcal{C}$; that is, the control is bang-bang.

9.5 Systems Governed by Integrodifferential Systems

Consider now the problem

$$\min \int_0^{t_0} f^0(\phi(t), u(t), t)dt \tag{9.5.1}$$

subject to:

$$\frac{d}{dt}\,\phi^i(t) = f^i(t, \phi(t), u(t)) + \int_0^t g^i(t, s, \phi(s), u(s))ds \tag{9.5.2}$$

$$u(t) \in \Omega(t, \phi(t)) \tag{9.5.3}$$

$$T(\phi(0), \phi(t_1)) = 0. \tag{9.5.4}$$

For assumptions on $f^0, f^1, \ldots, f^n$ refer to Assumption 6.3.1. For assumptions on g^i, $i = 1, \ldots, n$ refer to Section 2.7. T is continuously differentiable.

Remark 9.5.1. Under Assumption 9.2.1 and additional assumptions on the kernel functions, we can rewrite (9.2.1) to (9.2.4) to fit the form of (9.5.1) to (9.5.4). We may introduce a new state variable ψ such that $\psi'(t) = M(\phi(t), u(t), t)$, and consider the constraint $[W(\phi(t_1)) + \psi(t_1)]^2 + [\phi(0) - F(0)]^2 = 0$. Finally we differentiate both sides of Eq. (9.2.1).

Necessary conditions for systems governed by integrodifferential systems are presented in Section 11.5. The above system is a special version of the problem (11.2.1) to (11.2.8).

9.6 Linear Plant Quadratic Cost Criterion

Consider a process where the cost is given by

$$\int_0^{t_1} \langle \phi(t), X(t)\phi(t)\rangle dt + \int_0^{t_1} \langle u(t), R(t)u(t)\rangle dt, \tag{9.6.1}$$

where the matrices X and R are symmetric, X is nonnegative, R is positive definite, and

$$\phi'(t) = A_0(t)\phi(t) + B_0 u(t) + \int_0^t [A_1(t,s)\phi(s) + B_1(t,s)u(s)]ds, \tag{9.6.2}$$

$$u(t) \in \mathbb{R}^m, \tag{9.6.3}$$

$$T(\phi(t_1)) = 0. \tag{9.6.4}$$

The data of the problem obey the same assumptions as in (9.5.1) to (9.5.4).

Suppose (ϕ_0, u_0) is optimal. Then from Theorem 11.5.3 and Remark 11.5.4 we must have

$$\begin{aligned} &-\lambda^0 \langle u_0(t), R(t)u_0(t)\rangle + \Phi(t)\cdot B_0(t)u_0(t) \qquad (9.6.5)\\ &+ \int_t^{t_1} \Phi(s)\cdot B_1(t,s)u_0(s)ds \geq -\lambda^0\langle u(t), R(t)u(t)\rangle + \Phi(t)\cdot B_0(t)u(t) \end{aligned}$$

$$+ \int_t^{t_1} \Phi(s) \cdot B_1(t,s)u(s)ds,$$

where

$$\Phi(t) - \int_t^{t_1} \Phi(s)A_0(s)ds + \int_t^{t_1} \int_s^{t_1} \Phi(\tau)A_1(\tau,s)d\tau\, ds \tag{9.6.6}$$
$$+ 2\lambda_0 \int_t^{t_1} \phi_0(s)X(s)ds = \Phi(t_1^-)$$

Integrating both sides of the inequality from $t = 0$ to $t = t_1$ and switching order of integration we obtain the inequality

$$- \lambda^0 \langle u_0(t), R(t)u_0(t)\rangle + \Phi(t) \left[B_0(t) + \int_0^t B_1(s,t)ds\right] u_0(t) \tag{9.6.7}$$
$$\geq -\lambda^0 \langle u(t), R(t)u(t)\rangle + \Phi(t) \left[B_0(t) + \int_0^t B_1(s,t)ds\right] u(t)$$

From (9.6.7) we can eliminate the possibility $\lambda^0 = 0$. Thus, we may take $\lambda^0 = 1$ in what follows, and

$$u_0^T(t) = \frac{1}{2}\Phi(t) \left[B_0(t) + \int_0^t B_1(s,t)ds\right] R^{-1}(t) \tag{9.6.8}$$

From (9.6.6) we see the dependence of Φ on $\Phi(t_1^-)$, and hence of $u_0(t)$ on $\Phi(t_1^-)$. We have to determine $\Phi(t_1^-)$ consistent with (iii) and (iv) of Theorem 11.5.3. In this case $\Phi(t_1^-) = c\nabla T(\phi_0(t_1))$, c a constant. If we still have undetermined parameters we determine $\phi_0(t)$ using u_0 as given by (9.6.8) and insist on $T(\phi_0(t_1)) = 0$.

Exercise 9.6.1. Under Assumption 9.2.1, and additional regularity assumptions as needed, use Theorem 11.5.3 to obtain a set of necessary conditions of optimality for (9.2.1) to (9.2.4).

9.7 A Minimum Principle

In Exercise 9.6.1 the reader was asked to get necessary optimality conditions for (9.2.1) to (9.2.4) using Theorem 11.5.3 under Assumption 9.2.1 and additional regularity assumptions. In this section we will relax the assumptions on the data and obtain necessary optimality conditions.

We consider the system governed by

$$\phi(t) = F(t) + \int_0^t L(t, \phi(s), u(s), s)ds \tag{9.7.1}$$

$$W(\phi(t_1)) + \int_0^{t_1} M(\phi(t), u(t), t)dt = 0, \tag{9.7.2}$$

$$u(t) \in \Omega(t), \tag{9.7.3}$$

and the cost to be minimized is given by

$$g(\phi(t_1)). \tag{9.7.4}$$

To obtain necessary conditions at any optimal control ν_0 we define the functional

$$F_K(\nu) = g(\phi(t_1)) + \epsilon\|\nu - \nu_0\|_L + K\left[W(\phi(t_1)) + \int_0^{t_1} M(\phi(t), \nu_t, t)dt\right]^2 \tag{9.7.5}$$

under the restriction

$$\phi(t) = F(t) + \int_0^t L(t, \phi(s), \nu_s, s)ds, \tag{9.7.6}$$

and proceed as in Section 7.6.

Assumption 9.7.1. (i) F is continuous.

(ii) W and g are continuously differentiable.

(iii) The functions L and M obey the assumptions in (i) through (iv), in Assumption 9.2.1.

(iv) $|L(t+h, x, u, s) - L(t, x, u, s)| \leq \Lambda(s)|h|$,

$|L_x(t, x, u, s) - L_x(t, x', u, s)| \leq \Lambda(s)|x - x'|$,

where Λ is as in Assumption 9.2.1.

(v) $\Omega(t)$ is contained in a fixed compact set Ω.

Suppose the problem of minimizing $g(\phi(t_1))$ subject to (9.7.1) to (9.7.3) has a solution (ϕ_0, ν_0), where ν_0 is a relaxed control. We assume that given any relaxed control ν (9.7.1) has a unique solution. Without loss of generality we assume that $g(\phi_0(t_1)) = 0$.

Given $0 < \epsilon < 1$ we can verify that there exists $K(\epsilon)$ such that $F_{K(\epsilon)}(\nu) > 0$ for all ν such that $\|\nu - \nu_0\|_L = \epsilon$. Let

$$\mathcal{A}_r = \{\nu : \|\nu - \nu_0\|_L \leq \epsilon\}.$$

Suppose ν_ϵ minimizes $F_{K(\epsilon)}(\nu)$ in $\mathcal{A}_r$. We note that $\|\nu_\epsilon - \nu_0\|_L < \epsilon$. Corresponding to ν_ϵ we have the trajectory ϕ_ϵ such that

$$\phi_\epsilon(t) = F(t) + \int_0^t L(t, \phi_\epsilon(s), \nu_{\epsilon s}, s)ds.$$

We note that $\phi_\epsilon \to \phi_0$ uniformly as $\epsilon \to 0$. For $0 < \theta < 1$ let $\phi_{\epsilon\theta}$ be such that

$$\phi_{\epsilon\theta}(t) = F(t) + \int_0^t L(t, \phi_{\epsilon\theta}(s), \nu_{\epsilon s} + \theta(\nu - \nu_\epsilon)_s, s)ds.$$

We note that $\{\phi_{\epsilon\theta}\}_{\substack{0<\epsilon<\epsilon_0 \\ 0<\theta<1}}$ is a uniformly bounded equicontinuous family. We can also verify that $\phi_{\epsilon\theta} \to \phi_\epsilon$ as $\theta \to 0$ and that $\{(\phi_{\epsilon\theta}(t) - \phi_\epsilon(t))/\theta\}_{0<\theta<1}$ is uniformly bounded and converges to $\delta\phi_\epsilon(t)$ pointwise as $\theta \to 0^+$, where $\delta\phi_\epsilon$ satisfies the equation

$$\delta\phi_\epsilon(t) = \int_0^t L_x(t, \phi_\epsilon(s), \nu_{\epsilon s}, s)\delta\phi_\epsilon(s)ds + \int_0^t L(t, \phi_\epsilon(s), (\nu - \nu_\epsilon)_s, s)ds.$$

We can also verify that $\{\delta\phi_\epsilon\}_{0<\epsilon<\epsilon_0}$ is uniformly bounded and that $\delta\phi_\epsilon \to \delta\phi_0$ uniformly, where $\delta\phi_0$ satisfies the integral equation

$$\delta\phi_0(t) = \int_0^t L_x(t, \phi_0(s), \nu_{0s}, s)\delta\phi_0(s)ds + \int_0^t L(t, \phi_0(s), (\nu - \nu_0)_s, s)ds.$$

We can also verify that

$$\lim_{\theta\to 0^+} [F_{K(\epsilon)}(\nu_\epsilon + \theta(\nu - \nu_\epsilon)) - F_{K(\epsilon)}(\nu_\epsilon)]/\theta \geq 0$$

leads to the inequality

$$\begin{aligned} &g'(\phi_\epsilon(t_1))\delta\phi_\epsilon(t_1) + \epsilon\rho'(0^+) + 2K(\epsilon)c_\epsilon \Bigg\{ W'(\phi_\epsilon(t_1))\delta\phi_\epsilon(t_1) \\ &\quad + \int_0^{t_1} M_x(\phi_\epsilon(t), \nu_{\epsilon t}, t)\delta\phi_\epsilon(t)dt + \int_0^{t_1} M(\phi_\epsilon(t), (\nu - \nu_\epsilon)_t, t)dt \Bigg\} \geq 0 \end{aligned}$$

where

$$c_\epsilon = W(\phi_\epsilon(t_1)) + \int_0^{t_1} M(\phi_\epsilon(t), \nu_{\epsilon t}, t)dt$$

and $\rho'(0^+)$ is the right derivative of $\rho(\theta) = \|\nu_\epsilon + \theta(\nu - \nu_\epsilon) - \nu_0\|_L$.

Dividing the above inequality $M(\epsilon) = 1 + 2K(\epsilon)c_\epsilon$ and letting $\epsilon \to 0^+$ through an appropriate subsequence we obtain

$$\begin{aligned} \lambda^0 g'(\phi_0(t_1))\delta\phi_0(t_1) + \lambda \Bigg\{ &W'(\phi_0(t_1))\delta\phi_0(t_1) + \int_0^{t_1} M_x(\phi_0(t), \nu_{0t}, t)\delta\phi_0(t)dt \\ &+ \int_0^{t_1} M(\phi_0(t), (\nu - \nu_0)_t, t)dt \Bigg\} \geq 0 \end{aligned} \tag{9.7.7}$$

where $\lambda^0 \geq 0$ and $\lambda^0 + |\lambda| = 1$, and $\delta\phi_0(t)$ satisfies the integral equation

$$\delta\phi_0(t) = \int_0^t L_x(t, \phi_0(s), \nu_{0s}, s)\delta\phi_0(s)ds$$

$$+\int_0^t L(t,\phi_0(s),(\nu-\nu_0)_s,s)ds \tag{9.7.8}$$

Let $R(t,s)$ be the resolvent kernel for the integral equation (9.7.8). Let

$$q(t)=\int_0^t L(t,\phi_0(s),(\nu-\nu_0)_s,s)ds. \tag{9.7.9}$$

Then,

$$\delta\phi_0(t)=q(t)+\int_0^t R(t,s)q(s)ds.$$

Now we substitute for $\delta\phi_0$ in (9.7.7) to obtain

$$\begin{aligned}
&\lambda^0 g'(\phi_0(t_1))q(t_1)+\lambda^0\int_0^{t_1} g'(\phi_0(t_1))R(t_1,s)q(s)ds\\
&\quad+\lambda W'(\phi_0(t_1))q(t_1)+\lambda\int_0^{t_1} W'(\phi_0(t_1))R(t_1,s)q(s)ds\\
&\quad+\lambda\int_0^{t_1} M_x(\phi_0(t),\nu_{0t},t)\left[q(t)+\int_0^t R(t,s)q(s)ds\right]dt\\
&\quad+\lambda\int_0^{t_1} M(\phi_0(t),(\nu-\nu_0)_t,t)dt\\
&\geq 0.
\end{aligned}$$

We rewrite this inequality as

$$\begin{aligned}
&\lambda^0 g'(\phi_0(t_1))q(t_1)+\int_0^{t_1}\Big\{\lambda^0 g'(\phi_0(t_1))R(t_1,t)\\
&\quad+\lambda W'(\phi_0(t_1))R(t_1,t)+\lambda M_x(\phi_0(t),\nu_{0t},t)\\
&\quad+\lambda\int_t^{t_1} M_x(\phi_0(s),\nu_{0s},s)R(s,t)\Big\}q(t)dt\\
&\quad+\lambda W'(\phi_0(t_1))q(t_1)+\lambda\int_0^{t_1} M(\phi_0(t),(\nu-\nu_0)_t,t)dt\geq 0.
\end{aligned}$$

We rewrite this last inequality. However, we first set

$$\begin{aligned}
\psi(t)=\ &\lambda^0 g'(\phi_0(t_1))R(t_1,t)\\
&+\Big\{W'(\phi_0(t_1))R(t_1,t)+M_x(\phi_0(t),\nu_{0t},t)\\
&+\int_t^{t_1} M_x(\phi_0(s),\nu_{0s},s)R(s,t)ds\Big\},
\end{aligned}$$

a row vector. Now the last inequality can be rewritten as

$$\lambda^0 g'(\phi_0(t_1))q(t_1) + \lambda W'(\phi_0(t_1))q(t_1) + \lambda \int_0^{t_1} M(\phi_0(t), (\nu - \nu_0)_t, t)dt + \int_0^{t_1} \psi(t)q(t)dt \geq 0. \tag{9.7.10}$$

Recalling the definition of $q(t)$ (9.7.9) we can rerite (9.7.10) as

$$\begin{aligned}&\int_0^{t_1} [\lambda^0 g'(\phi_0(t_1)) + \lambda W'(\phi_0(t_1))]L(t_1, \phi_0(t), \nu_t, t)dt \\ &\quad + \int_0^{t_1} \left[\int_t^{t_1} \psi(s)L(s, \phi_0(t), \nu_t, t)ds\right] dt + \lambda \int_0^{t_1} M(\phi_0(t), \nu_t, t)dt \\ &\geq \int_0^{t_1} \left[\lambda^0 g'(\phi_0(t_1)) + \lambda W'(\phi_0(t_1))\right] L(t_1, \phi_0(t), \nu_{0t}, t)dt \\ &\quad + \int_0^{t_1} \left[\int_t^{t_1} \psi(s)L(s, \phi_0(t), \nu_{0t}, t)ds\right] dt + \lambda \int_0^{t_1} M(\phi_0(t), \nu_{0t}, t)dt.\end{aligned} \tag{9.7.11}$$

We use (9.7.11) to state our theorem next.

Theorem 9.7.2. *Under Assumption* 9.7.1 *let* (ϕ_0, ν_0) *be optimal for the problem of minimizing the cost* (9.7.4) *under the conditions* (9.7.1) *to* (9.7.3). *Then the following conditions are met: There exist multipliers* $\lambda^0 \geq 0$, λ, *and* ψ *such that*

(i) $\lambda^0 + |\lambda| = 1$

(ii) $\psi(t) = \lambda^0 g'(\phi_0(t_1)) + R(t_1, t) + \lambda \Big\{ W'(\phi_0(t_1))R(t_1, t) + M_x(\phi_0(t), \nu_{0t}, t) + \int_t^{t_1} M_x(\phi_0(s), \nu_{0s}, s)R(s, t)ds \Big\}$

(iii) $[\lambda^0 g'(\phi_0(t_1)) + \lambda W'(\phi_0(t_1))]L(t_1, \phi_0(t), \nu_t, t) + \lambda M(\phi_0(t), \nu_t, t) + \int_t^{t_1} \psi(s)L(s, \phi_0(t), \nu_t, t)ds \geq [\lambda^0 g'(\phi_0(t_1)) + \lambda W'(\phi_0(t_1))]L(t_1, \phi_0(t), \nu_{0t}, t) + \lambda M(\phi_0(t), \nu_{0t}, t) + \int_t^{t_1} \psi(s)L(s, \phi_0(t), \nu_{0t}, t)ds$ *a.e.*

Remark 9.7.3. Condition (iii) of Theorem 9.7.2 follows from (9.7.11).

Chapter 10

Hereditary Systems

10.1 Introduction

In this chapter we consider processes where the evolution of the state depends upon the present and past history of the state and control.

In the formulation considered, the right-hand side of the dynamics is split into two parts. The first part contains the history of the state, and the second part contains the history of the state as well as the control. Many familiar delay problems are included in the first part of the formulation.

In Section 10.2 we will state the problem and assumptions, and address existence issues briefly. In the next section we state a corresponding minimum principle. In Section 10.4 we will consider linear systems and systems that are linear in the state variable. These systems are concrete versions of the general problem considered in Chapter 11, and the constituents of the minimum principle, Theorem 11.4.4 can be spelled out in these cases easily. In Section 10.5 we consider an example of a linear plant with quadratic cost criterion. In these examples we note that the relaxed optimal controls can be replaced by ordinary controls.

As is well known, hereditary problems are infinite dimensional problems. Our approach is to essentially employ finite dimensional techniques. This will be seen in Chapter 11. In Section 10.6, we deal with a hereditary problem in an infinite dimensional setting.

We remark that systems of the form given in Section 10.4 can be handled by the method presented in Section 10.6.

10.2 Problem Statement and Assumptions

Consider a process governed by the system

$$\phi'(t) = f(\phi(\cdot), u(\cdot), t) \tag{10.2.1}$$

$$\phi(t) = y(t), \qquad -r \le t \le 0, \quad y \in \mathcal{M} \tag{10.2.2}$$

$$u(t) \in \Omega(t, \phi(t)), \tag{10.2.3}$$

$$T(\phi(0), \phi(t_1)) = 0. \tag{10.2.4}$$

It is desired that the cost

$$\int_0^{t_1} f^0(\phi(\cdot), u(\cdot), t)dt \tag{10.2.5}$$

be infimum.

Assumption 10.2.1. For the definition of $f(\phi(\cdot), u(\cdot), t)$, and $f^0(\phi(\cdot), u(\cdot), t)$ see Section 2.7. We assume that the set-valued map $\Omega(t, x)$, $(t, x) \in I_0^{t_1} \times \mathcal{X}$ is u.s.c.i. on $I_0^{t_1} \times \mathcal{X}$, and for each $(t, x) \in I_0^{t_1} \times \mathcal{X}$, $\Omega(t, x)$ is compact. Finally, we assume T is continuously differentiable, although for existence we only need lower semi-continuity. The set $\mathcal{M}$ is a closed convex subset of $H^1(-r, 0)$.

Theorem 10.2.2. *Assume the set of admissible pairs for* (10.2.1) *to* (10.2.4) *is not empty. We assume that all admissible trajectories are such that* $(\phi(0), \phi(t_1))$ *lie in a fixed compact subset of* $\mathbb{R}^n \times \mathbb{R}^n$*. To guarantee existence we also assume that the set* $\mathcal{M}$ *in* (10.2.2) *is bounded in* $H^1(-r, 0)$*. Then, there exists a relaxed admissible pair giving the cost* (10.2.5) *its infimum value. However, the derivation of necessary conditions at an optimal pair known to exist does not require that the set* $\mathcal{M}$ *be bounded in* $H^1(-r, 0)$.

Exercise 10.2.3. Verify Theorem 10.2.2 regarding existence.

For more on existence refer to Chapter 4 and Chapter 5.

10.3 Minimum Principle

The problem stated in Section 10.2 is a special case of what is presented in Chapter 11, Section 11.2. Thus, from Theorem 11.4.4 we obtain the following minimum principle for it. This problem has been considered in [69].

We suggest to the reader to take a quick look at Theorem 11.4.4 and scan its proof to see where Theorem 10.3.1 came from.

Theorem 10.3.1. *For the problem stated in Section 10.2, under Assumption 10.2.1, we have the following necessary conditions met at any optimal relaxed pair* (ϕ_0, ν_0)*: There exist a function of bounded variation* Φ *on* $I_0^{t_1}$*, scalars* $\lambda^0 \geq 0$*,* β*;* $\Gamma(t, s) = (\Gamma^1(t, s), \ldots, \Gamma^n(t, s))$*, and* $\Gamma^0(t, s)$ *such that* $s \mapsto \Gamma^i(t, s)$ *is of bounded variation, continuous from the right and vanishing for* $s > t$.

(i) $\lambda^0 + |\beta| + |\Phi(t_1^-)| = 1$*;*

(ii) $\Phi(t) - \lambda^0 \int_t^{t_1} \Gamma^0(s,t)ds + \int_t^{t_1} \Phi(s)\Gamma(s,t)ds = \Phi(t_1^-)$

$\Phi(s)\Gamma(s,t) = \Phi_1\Gamma_1(s,t) + \cdots + \Phi_n\Gamma_n(s,t)$

(iii) for any relaxed control ν,

$$\begin{aligned}&\lambda^0 f^0(t,\phi_0(\cdot),\nu_{0t}) - \Phi(t)\cdot f(t,\phi_0(\cdot),\nu_{0t})\\ &\quad \geq \lambda^0 f^0(t,\phi_0(\cdot),\nu_t) - \Phi(t)\cdot f(t,\phi_0(\cdot),\nu_t) \qquad a.e.\end{aligned}$$

(iv) $\Phi(0) = \beta\partial_1 T(\phi_0(0),\phi_0(t_1)) + \int_0^{t_1}\{\lambda^0\Gamma^0(t,0) - \Phi(t)\cdot\Gamma(t,0)\}dt$

$+ \int_0^{t_1}\{-\lambda^0\Gamma^0(t,-r) + \Phi(t)\cdot\Gamma(t,-r)\}dt + B(0),$

$\Phi(t_1^-) = -\beta\partial_2 T(\phi(0),\phi_0(t_1)).$

(v) $B''(t) - B(t) + \int_0^{t_1}[-\lambda^0\Gamma^0(s,t) + \Phi(s)\Gamma(s,t)]ds$

$- \int_0^{t_1}[-\lambda^0\Gamma^0(t,-r) + \Phi(t)\Gamma(t,-r)]dt = 0,$

$B(-r) = 0,\ (B',B) \in \partial I_{\mathcal{M}}(y).$

To see how (iv) is obtained consult Theorem 11.4.4(iii) and (ii). Note that since we do not have state constraint here, we remove quantities involving the multiplier λ in Theorem 11.4.4.

Remark 10.3.2. To see how Γ^0 and Γ in the previous theorem arise, see (11.3.12) to (11.3.14). In the following example and the one in Section 10.5 $\Gamma^i(t,s)$, $i = 0,1,\ldots,n$ are explicitly given in terms of the data of the problem.

Now, consider the process (10.2.1) to (10.2.5) where now

$$\begin{aligned}f^i(\phi(\cdot),u(\cdot),t) = &\int_{-r}^t \phi(s)d_s\eta^i(t,s) + k_i(t,u(t)) \qquad (10.3.1)\\ &+ \int_{-r}^t \langle\phi(s),L^i(t,s)\rangle d_s\omega^i(t,s) + \int_{-r}^t M^i(t,s,u(s))d_s\omega^i(t,s),\\ &i = 0,1\ldots,n.\end{aligned}$$

In this case $\Gamma^i(t,s)$, ith entry of $\Gamma(t,s) = (\Gamma^1(t,s),\ldots,\Gamma^n(t,s))$ in Theorem 10.3.1, is given by

$$\Gamma^i(t,s) = \eta^i(t,s) - \int_s^t L^i(t,\tau)d_\tau\omega^i(t,\tau), \qquad (10.3.2)$$

and (iii) of the same theorem is

$$-\lambda^0\left\{k_0(t,\nu_{0t}) + \int_{-r}^t M^0(t,s,\nu_{0s})d_s\omega^0(t,s)\right\} \qquad (10.3.3)$$

$$+ \Phi(t) \cdot \left(k(t, \nu_{0t}) + \int_{-r}^{t} M(t, s, \nu_{0s}) d_s \omega(t, s) \right) \geq$$

$$- \lambda^0 \left\{ k_0(t, \nu_t) + \int_{-r}^{t} M^0(t, s, \nu_t) d_s \omega^0(t, s) \right\}$$

$$+ \Phi(t) \cdot \left(k(t, \nu_t) + \int_{-r}^{t} M(t, s, \nu_s) d_s \omega(t, s) \right) \qquad \text{a.e. } t \in [0, t_1]$$

10.4 Some Linear Systems

We specialize (10.2.1) to (10.2.5) as follows.

We consider

$$\frac{d}{dt} \phi^i(t) = f^i(\phi(\cdot), u(\cdot), t), \qquad i = 1, \ldots, n, \tag{10.4.1}$$

where

$$f^i(\phi(\cdot), u(\cdot), t) = k^i(t, u(t)) + \int_{-r}^{0} \phi(t+s) d\eta(s) \tag{10.4.2}$$

$$+ \int_{-r}^{t} M^i(t, s, u(s)) ds, \qquad i = 1, 2, \ldots, n$$

and it is required to

$$\text{minimize } g(\phi_0(t_1)). \tag{10.4.3}$$

We assume that the set-valued map $\Omega(t, x)$ is independent of x. We also assume that η is of bounded variation and continuous from the right.

Writing

$$L(t, \phi_t) = \int_{-r}^{0} \phi(t+s) d\eta(s), \tag{10.4.4}$$

we define $\Phi(t, s)$ by the equation [41, page 144–146]

$$\Phi(t, s) = \begin{cases} \int_s^t L(\xi, \Phi_\xi(\cdot, s)) d\xi + I, & \text{a.e. in } s, \ t \geq s, \\ 0, & s - r \leq t < s \end{cases} \tag{10.4.5}$$

and T by the formula

$$\Phi_t(\cdot, s) = T(t, s) X_0 \tag{10.4.6}$$

$$X_0(\theta) = \begin{cases} 0, & -r \leq \theta < 0 \\ I, & \theta = 0 \end{cases}$$

Now

$$\phi(t) = (T(t, 0)y)(0) + \int_0^t \Phi(t, s) \left\{ k(s, u(s)) + \int_{-r}^{s} M(s, \xi, u(\xi)) d\xi \right\} ds \tag{10.4.7}$$

That is,

$$\phi(t) = (T(t,0)y)(0) + \int_0^t \Phi(t,s)k(s,u(s))ds + \int_{-r}^0 \int_0^t \Phi(t,\xi)M(\xi,s,u(s))d\xi\, ds$$
$$+ \int_0^t \int_s^t \Phi(t,\xi)M(\xi,s,u(s))d\xi\, ds \tag{10.4.8}$$

Assumption 10.4.1. In (10.4.8) assume $M(\xi,s,u)$ is of the form $M(\xi)h(s,u)$ where both M and h are continuous.

Suppose (ϕ^*,ν^*) is a relaxed optimal pair for (10.4.1) to (10.4.3) with restrictions (10.2.2) to (10.2.4), where as stated previously $\Omega(t,x)$ is independent of x.

Rewriting (10.4.8) as

$$\phi^*(t) - (T(t,0)y^*)(0) - \int_{-r}^0 \int_0^t \Phi(t,\xi)M(\xi,s,\nu_s^*)d\xi\, ds$$
$$= \int_0^t \left[\Phi(t,s)k(s,\nu_s^*) + \int_s^t \Phi(t,\xi)M(\xi,s,\nu_s^*)d\xi\right] ds$$

We can argue, as in the proof of Theorem 9.3.1, that there exists an ordinary control u_0^* such that

$$k(s,\nu_s^*) + \int_s^t M(\xi,s,\nu_s^*)d\xi = k(x,u_0^*(s)) + \int_s^t M(\xi,s,u_0^*(s))ds. \tag{10.4.9}$$

Similarly, there is an ordinary control u_{-r}^* such that

$$\int_{-r}^0 \int_0^t \Phi(t,\xi)M(\xi,s,\nu_s^*)d\xi\, ds = \int_{-r}^0 \int_0^t \Phi(t,\xi)M(\xi,s,u_{-r}^*(s)d\xi\, ds. \tag{10.4.10}$$

Thus, we have the following:

Theorem 10.4.2. *The problem (10.4.1) to (10.4.3) under Assumption 10.4.1 and the restrictions (10.2.2) to (10.2.4), where $\Omega(t,x)$ is independent of x, has an optimal pair where the control is ordinary.*

Necessary conditions for the above problem can be derived using the infinite dimensional approach of Section 10.6. We leave this to the reader.

10.5 Linear Plant-Quadratic Cost

Consider the process governed by the system

$$\begin{aligned}\frac{d}{dt}\phi^i(t) &= \langle k^i(t), u(t)\rangle + \int_{-r}^{t} \phi(s) d_s\omega^i(t,s) \\ &\quad + \int_{-r}^{t} \langle M^i(t,s), u(s)\rangle ds, \quad i = 1, \ldots, n \end{aligned} \tag{10.5.1}$$

$$\phi(t) = y(t), \qquad -r \leq t \leq 0, \quad y \in M \tag{10.5.2}$$

$$u(t) \in \mathbb{R}^m, \qquad u(t) = \widetilde{u}(t), \quad -r \leq t \leq 0 \tag{10.5.3}$$

$$T(\phi(0), \phi(t_1)) = 0 \tag{10.5.4}$$

It is required to

$$\text{minimize} \left\{ \int_0^{t_1} \langle \phi(t), X(t)\phi(t)\rangle dt + \int_0^{t_1} \langle u(t), R(t)u(t)\rangle dt \right\} \tag{10.5.5}$$

where X is a symmetric positive definite matrix, and R is a symmetric strictly positive definite matrix.

Let (ϕ_0, u_0) be optimal for (10.5.1) to (10.5.5). Then, from (10.3.3)

$$\begin{aligned} &-\lambda^0\langle u_0(t), R(t)u_0(t)\rangle + \langle \Phi(t), k(t)u_0(t) + \int_{-r}^{t} M(t,s)u_0(s)ds\rangle \geq \\ &-\lambda^0\langle u(t), R(t)u(t)\rangle + \langle \Phi(t), k(t)u(t) + \int_{-r}^{t} M(t,s)u(s)ds\rangle \end{aligned} \tag{10.5.6}$$

If $\lambda^0 > 0$ we can take it to be 1 and

$$u_0^T(t) = \frac{1}{2} \left\{ \Phi(t)k(t) + \int_t^{t_1} \Phi(s)M(s,t)ds \right\} R^{-1}(t) \tag{10.5.7}$$

Thus, to resolve this problem one must seek Φ consistent with Theorem 10.3.1. In this example, $\Gamma^i(t,s)$ of Theorem 10.3.1 is $\omega_i(t,s)$.

10.6 Infinite Dimensional Setting

In this section we consider hereditary systems, and a slightly different approach to obtain necessary conditions. This approach was initially followed in [70].

For $0 < a < b$, let $C([a,b],\mathbb{R}^n)$ be the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. Let $r > 0$ and $C = C([-r,0],\mathbb{R}^n)$. For $\phi \in C$, let

$$|\phi| = \max\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

Let $\sigma \in \mathbb{R}$ and $A > 0$. If

$$x \in C([\sigma - r, \sigma + A], \mathbb{R}^n),$$

then for any $t \in [\sigma, \sigma + A]$, let $x_t \in C$ be defined by

$$x_t(\theta) = x(t+\theta), \qquad -r \leq \theta \leq 0$$

For $t_1 > 0$ we consider the problem of minimizing the functional

$$J(u,z) = \int_0^{t_1} f^0(x_t, u(t))dt \tag{10.6.1}$$

subject to the autonomous functional differential equation

$$dx(t)/dt = f(x_t, u(t)), \qquad 0 < t < t_1, \tag{10.6.2}$$

$$x_0 = z,$$

and the conditions

$$P(x_{t_1}, x_0) \leq 0 \tag{10.6.3}$$

$$Q(x_{t_1}, x_0) = 0 \tag{10.6.4}$$

Assumption 10.6.1. In (10.6.1) and (10.6.2) we assume that $f^0 : C \times \mathbb{R}^m \to \mathbb{R}$, and $f : C \times \mathbb{R}^m \to \mathbb{R}$ are continuous and that the Frechet derivatives f_1^0, f_1 are continuous. In (10.6.3) and (10.6.4) the functions P and Q are continuously differentiable from $C \times C$ into $\mathbb{R}$.

Let

$$H^1(-r,0) = \{z : [-r,0] \to \mathbb{R}^n \mid z, z' \in L^2(-r,0)\}.$$

Corresponding to $\zeta = (z,\nu)$, $z \in H^1(-r,0)$, ν a relaxed control, denote by $y[\zeta](\cdot)$ the unique solution of the equation

$$dy/dt = f(y_t, \nu_t), \qquad t_0 < t < t_1 \tag{10.6.5}$$

$$y_0 = z$$

Definition 10.6.2. We say that $\zeta = (z,\nu)$ is admissible if $y[\zeta](\cdot)$ satisfies (10.6.2) to (10.6.4).

Assumption 10.6.3. There exists a bounded continuous function $\ell(u)$ such that the function f appearing in (10.6.2) satisfies

(i) $|f(0,u)| \leq \ell(u)$

(ii) $|f(x,u) - f(\bar{x},u)| \leq \ell(u)|x - \bar{x}|$.

10.6.1 Approximate Optimality Conditions

Assume that $\zeta_0 = (z_0, \nu_0)$ is a relaxed solution for (10.6.1) to (10.6.4). Let

$$\mathcal{U}_{ad} = \{(z, \nu) \mid z \in H^1(-r, 0), \nu \text{ relaxed control}\} \tag{10.6.6}$$

Let

$$F_K(z, \nu) = \int_{t_0}^{t_1} f^0(y_t, \nu_t)dt + \|z' - z_0'\|^2 + |z(0) - z_0(0)|^2 \tag{10.6.7}$$
$$+ \epsilon\|\nu - \nu_0\|_L + K\gamma(P(y_{t_1}, y_{t_0})) + KQ^2(y_{t_1}, y_{t_0}),$$

where γ is a smooth convex function such that $\gamma(t) = 0$, $t \leq 0$, $\gamma(t) > 0$ if $t > 0$, and

$$\|\cdot\| = L_2\text{-norm on } [-r, 0],$$

$$\|\nu - \nu_0\|_L = \text{ess sup}\{|\nu(t) - \nu_0(t)|(\Omega) : 0 \leq t \leq t_1\},$$

Ω a fixed compact subset of $\mathbb{R}^m$, and y_t is obtained from (10.6.2) and the initial condition. As mentioned earlier we will indicate this by writing $y[\zeta](\cdot)$, $\zeta = (z, \nu)$, and $y[\zeta]_t$ will stand for the function $\theta \mapsto y[\zeta](t + \theta)$, $-r \leq \theta \leq 0$.

Assume that

$$\int_0^{t_1} f^0(y[\zeta_0]_t, \nu_{0t})dt = 0.$$

We state, without proof, the following two lemmas.

Lemma 10.6.4. *For any $0 < \epsilon \leq 1$ there exists $K(\epsilon) > 0$ such that $F_{K(\epsilon)}(z, \nu) > 0$ if any of the following inequalities is an equality:*

$$|z(0) - z_0(0)| \leq \epsilon, \qquad \|z' - z_0'\| \leq \epsilon, \qquad \|\nu - \nu_0\|_L \leq \epsilon.$$

Lemma 10.6.5. *Let $K(\epsilon)$ be as in Lemma 10.6.4, $0 < \epsilon \leq 1$. Then, the functional $(z, \nu) \to F_{K(\epsilon)}(z, \nu)$ attains its minimum on $B(\epsilon)$. For $0 < \epsilon \leq 1$, let $(z^\epsilon, \nu^\epsilon) \in B(\epsilon)$ be such that*

$$F_{K(\epsilon)}(z^\epsilon, \nu^\epsilon) = \inf\{F_{K(\epsilon)}(z, \nu) \mid (z, \nu) \in B(\epsilon)\}. \tag{10.6.8}$$

Then,

$$\|z^{\epsilon'} - z_0'\| < \epsilon, \quad |z^\epsilon(0) - z_0(0)| < \epsilon, \quad \|\nu^\epsilon - \nu_0\|_L < \epsilon. \tag{10.6.9}$$

From (10.6.8) and (10.6.9), for $(z, \nu) \in \mathcal{U}_{ad}$ we have

$$\lim_{h \to 0} dF_{K(\epsilon)}(z^\epsilon + hz, \nu^\epsilon)/dh = 0, \tag{10.6.10}$$

$$\lim_{h \to 0^+} dF_{K(\epsilon)}(z^\epsilon, \nu^\epsilon + h(\nu - \nu^\epsilon))/dh \geq 0. \tag{10.6.11}$$

Now, let $T_\epsilon(s, \sigma)$ be the solution operator of the homogeneous equation

$$dW/dt = f_1(y[\zeta_\epsilon]_t, \nu^\epsilon)W_t, \tag{10.6.12}$$

where

$$\zeta_\epsilon = (z^\epsilon, \nu^\epsilon) \tag{10.6.13}$$

Also, we define

$$X_0(\theta) = \begin{cases} 0, & -r \le \theta < 0 \\ I, & \theta = 0 \end{cases} \tag{10.6.14}$$

where I is the identity operator. Then, if we set

$$\delta y[\zeta^\epsilon](t) = \lim_{\epsilon \to 0} \frac{y[\zeta^{\epsilon,h}](t) - y[\zeta^\epsilon](t)}{\epsilon}, \tag{10.6.15}$$

where

$$\zeta^{\epsilon,h} = (z^\epsilon + h(z - z^\epsilon), \nu^\epsilon), \qquad 0 < h \ll 1, \tag{10.6.16}$$

we may write

$$\delta y[\zeta^\epsilon]_t = T_\epsilon(t,0)X_0(z - z^\epsilon). \tag{10.6.17}$$

Thus, from (10.6.10), writing $y^\epsilon = y[\zeta^\epsilon](\cdot)$, we obtain

$$\begin{aligned} &\int_0^{t_1} f_1^0(y_t^\epsilon, \nu_t^\epsilon) T_\epsilon(t,0) X_0(z - z^\epsilon)dt \\ &\quad + K(\epsilon)[\gamma'(P(y_{t_1}^\epsilon, y_{t_0}^\epsilon))P_1(y_{t_1}^\epsilon, y_{t_0}^\epsilon) \\ &\quad + 2Q(y_{t_1}^\epsilon, y_0^\epsilon)Q_1(y_{t_1}^\epsilon, y_0^\epsilon)]T_\epsilon(t_1,0)X_0(z - z^\epsilon) \\ &\quad + K(\epsilon)[\gamma'(P(y_{t_1}^\epsilon, y_0^\epsilon))P_2(y_{t_1}^\epsilon, y_0^\epsilon) + 2Q(y_{t_1}^\epsilon, y_0^\epsilon)Q_2(y_{t_1}^\epsilon, y_0^\epsilon)](z - z^\epsilon) \\ &\quad + \int_{-r}^0 2(z_\epsilon' - z_0')(z - z_\epsilon)dt + 2(z_\epsilon(0) - z_0(0))(z(0) - z_\epsilon(0)) = 0 \end{aligned} \tag{10.6.18}$$

If, now, we set

$$\zeta^{\epsilon,h} = (z^\epsilon, \nu^\epsilon + h(\nu - \nu^\epsilon)), \qquad 0 \le h \le 1, \tag{10.6.19}$$

and compute $\delta y[\zeta^\epsilon]$ as in (10.6.15), then by the variation of constants formula we obtain

$$\delta y[\zeta^\epsilon]_t = \int_0^t T_\epsilon(t,s) X_0 f(y_s^\epsilon, \nu_s - \nu_s^\epsilon)ds \tag{10.6.20}$$

and so from (10.6.11) we obtain

$$\begin{aligned} &\int_0^{t_1} \left\{ f^0(y_t^\epsilon, \nu_t - \nu_t^\epsilon) + \int_t^{t_1} f_1^0(y_s^\epsilon, \nu_s^\epsilon) T_\epsilon(s,t) X_0 f(y_t^\epsilon, \nu_t - \nu_t^\epsilon)ds \right\} dt \\ &\quad + \int_0^{t_1} \Big\{ K(\epsilon)[\gamma'(P(y_{t_1}^\epsilon, y_0^\epsilon))P_1(y_{t_1}^\epsilon, y_0^\epsilon) + 2Q(y_{t_1}^\epsilon, y_{t_0}^\epsilon)Q_1(y_{t_1}^\epsilon, y_0^\epsilon)] \cdot \\ &\quad \cdot T_\epsilon(t_1,s) X_0 f(y_s^\epsilon, \nu_s - \nu_s^\epsilon) \Big\} ds + \epsilon \rho_\epsilon'(0^+) \ge 0, \end{aligned} \tag{10.6.21}$$

where

$$\rho_\epsilon(h) = \|\nu^\epsilon + h(\nu - \nu^\epsilon) - \nu_0\|_L, \qquad 0 \le h \le 1.$$

10.6.2 Optimality Conditions

In this section we obtain optimality conditions.

Assumption 10.6.6. We assume that $|P_1(y_1^0, y_0^0)| + |P_2(y_1^0, y_0^0)| \neq 0$. Let

$$\lambda^0 = \frac{1}{\underline{\lim}_{\epsilon \to 0} M(\epsilon)} \tag{10.6.22}$$

From the assumptions on the data one can immediately prove that $\exists$ $\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \cdots \to 0$, $\lambda^1 \geq 0$, $\ell_1, \ell_2 \in C^*([-r, 0])$ such that

$$\lambda^0 + \lambda^1 + |\ell_1| + |\ell_2| \neq 0 \tag{10.6.23}$$

and

$$\begin{aligned} \int_0^{t_1} \lambda^0 f_1^0(y_t^0, \nu_{0t}) T(t,0) dt + (\lambda^1 P_1(y_{t_1}^0, y_0^0) + \ell_1) T(t_1, 0) \\ + (\lambda^1 P_2(y_{t_1}^0, y_0^0) + \ell_2) I = 0, \end{aligned} \tag{10.6.24}$$

$$\begin{aligned} \int_0^{t_1} \left\{ \lambda^0 f^0(y_t^0, \nu_t - \nu_{0t}) + \int_t^{t_1} f_1^0(y_s^0, \nu_s^0) T(s,t) X_0 f(y_t^0, \nu_t - \nu_{0t}) ds \right\} dt \\ + \int_0^{t_1} (\lambda^1 (P_1(y_{t_1}^0, y_0^0) + \ell_1) T(t_1, s) X_0 f(y_s^0, \nu_s - \nu_{0s}) ds \geq 0, \end{aligned} \tag{10.6.25}$$

where

$$y^0(t) \equiv y^0[\zeta_0](t), \tag{10.6.26}$$

$$\zeta_0 = (z_0, \nu_0), \tag{10.6.27}$$

and $T(s, \sigma)$ be the solution operator of the homogeneous system

$$dw/dt = f_1(y_t^0, \nu_{0t}) w_t. \tag{10.6.28}$$

Define

$$\psi_t = \lambda^0 \int_t^{t_1} f_1^0(y_s^0, \nu_{0s}) T(s,t) ds + (\lambda^1 P_1(y_{t_1}^0, y_0^0) + \ell_1) T(t_1, t). \tag{10.6.29}$$

Now, we have the following theorem.

Theorem 10.6.7. *Suppose (z_0, ν_0) provides an optimal control for (10.6.1) to (10.6.4). With Assumptions 10.6.1 to 10.6.6 in force there exist multipliers $\lambda^0, \lambda^1 \geq 0$, $\ell_i \in C^*([-r, 0])$, $i = 1, 2$ and an adjoint variable ψ such that*

(i) $\lambda^0 + \lambda^1 + |\ell_1| + |\ell_2| = 1$, $\lambda^1 P(y_1^0, y_0^0) = 0$.

(ii) $\psi_t = \lambda^0 \int_t^{t_1} f_1^0(y_s^0, \nu_{0s}) T(s,t) ds + (\lambda^1 P_1(y_{t_1}^0, y_0^0) + \ell_1) T(t_1, t)$

(iii) $\psi_0 = -(\lambda^1 P_2(y_{t_1}^0, y_0^0) + \ell_2) I$

(iv) $\forall$ ν *relaxed*

$$\lambda^0 f^0(y_t^0, \nu_t) + \psi_t X_0 f(y_t^0, \nu_t) \geq \lambda^0 f^0(y_t^0, \nu_{0t}) + \psi_t X_0 f(y_t^0, \nu_{0t}) \qquad a.e.$$

Chapter 11

Bounded State Problems

11.1 Introduction

In this chapter we deal with bounded state problems. We give in detail the derivation of necessary conditions for problems governed by hereditary systems. Then, we specialize to problems governed by other systems. We follow the approaches in [66, 69, 70].

In Sections 11.2 to 11.4 we proceed to present the bounded state problems for problems governed by hereditary systems. In Section 11.5 we cover problems governed by integrodifferential systems, and in Section 11.6 problems governed by ordinary differential systems. Since problems governed by ordinary differential systems are more common we have treated them independently.

11.2 Statement of the Problem

The problem considered is

$$\min \int_0^{t_1} f^0(\phi(\cdot), u(\cdot), t)dt \tag{11.2.1}$$

such that

$$\frac{d}{dt}\,\phi(t) = f(\phi(\cdot), u(\cdot), t), \tag{11.2.2}$$

$$\phi(t) = y(t), \qquad -r \le t \le 0, \quad y \in \mathcal{M}, \tag{11.2.3}$$

$$\mathcal{M} \text{ closed convex subset of } H^1(-r, 0), \tag{11.2.4}$$

$$u(t) \in \Omega(t, \phi(t)), \qquad \text{a.e. } t \in [0, t_1], \tag{11.2.5}$$

$$T(\phi(0), \phi(t_1)) = 0, \tag{11.2.6}$$

$$G(t, \phi(t)) \le 0, \qquad 0 \le t \le t_1, \tag{11.2.7}$$

where

$$f(\phi(\cdot), u(\cdot), t) = \langle f^1(\phi(\cdot), u(\cdot), t), \ldots, f^n(\phi(\cdot), u(\cdot), t)\rangle,$$

$$\widehat{f}(\phi(\cdot),u(\cdot),t)=\langle f^0(\phi(\cdot),u(\cdot),t),f(\phi(\cdot),u(\cdot),t)\rangle,$$

and

$$\begin{aligned}\widehat{f^i}(\phi(\cdot),u(\cdot),t)&=h^i(t,\phi(\cdot),u(\cdot)) \\ &\quad+\int_{-r}^{t}g^i(t,s,\phi(s),u(s))d_s\omega^i(t,s), \qquad 0\le i\le n\end{aligned} \tag{11.2.8}$$

Remark 11.2.1. For the assumptions on the data of the problem we go back to Section 2.7.

Assumption 11.2.2. In (11.2.7) the function G is continuous in both arguments, and twice continuously differentiable in the second argument. In (11.2.6) T is assumed to be continuously differentiable.

11.3 ϵ-Optimality Conditions

In what follows we remind the reader that Remark 11.2.1 is in force.

Remark 11.3.1. We assume that all admissible trajectories are such that $(\phi(0),\phi(t_1))$ lie in a fixed compact subset of $\mathbb{R}^n\times\mathbb{R}^n$. To guarantee existence we also assume that the set $\mathcal{M}$ in (11.2.4) is bounded in $H^1(-r,0)$. Then, we leave it to the reader to verify that optimal pairs exist provided the set of admissible pairs is not empty. However, the derivation of necessary conditions at an optimal pair known to exist does not require that the set $\mathcal{M}$ be bounded in $H^1(-r,0)$.

Let (ϕ_0,ν_0) be optimal for the relaxed version of (11.2.1) to (11.2.7). In what follows, we write y_0 for ϕ_0 on $[-r,0]$. We fix (ϕ_0,ν_0) in the rest of our discussion. Without loss of generality, we assume that

$$J(\phi_0,\nu_0)\equiv\int_0^{t_1}f^0(\phi_0(.),\nu_0,t)dt=0.$$

Let

$$\begin{aligned}\mathcal{A}d=\{(\phi,y,\nu)\mid\ &\phi\in\mathcal{AC}(I_0^{t_1},\mathcal{X}),\phi'\in L_2(I_0^{t_1}),\\ &y\in H^1(-r,0),\phi(0)=y(0),\nu\text{ a relaxed control}\}\end{aligned} \tag{11.3.1}$$

$$\begin{aligned}B(\epsilon)=\{(\phi,y,\nu)\in\mathcal{A}d\mid\ &\|\phi'-\phi_0'\|\le\epsilon,\|\nu-\nu_0\|_L\le\epsilon,|\phi(0)-\phi_0(0)|\le\epsilon,\\ &\|y'-y_0'\|_*\le\epsilon,G(t,\phi(t))\le 0,0\le t\le t_1\},\end{aligned} \tag{11.3.2}$$

where

$$\|\nu - \nu_0\|_L = \text{ess-sup}\{|\nu(t) - \nu_0(t)|(\Omega) : -r \le t \le t_1\}$$
$$\|\cdot\| = L_2\text{-norm on } I_0^{t_1}$$
$$\|\cdot\|_* = L_2\text{-norm on } I_{-r}^0$$

Let

$$I_{\mathcal{M}}(y) = \begin{cases} 0, & \text{if } y \in \mathcal{M} \\ \infty, & \text{if } y \notin \mathcal{M} \end{cases}$$

On $H^1(-r, 0)$, and for $K \ge 0$ a scalar, we define

$$P_K(z) = \inf\{2^{-1}K(\|z' - x'\|_*^2 + \|z - x\|_*^2) + I_{\mathcal{M}}(x) : x \in H^1(-r, 0)\} \quad (11.3.3)$$

Note that $z \mapsto P_K(z)$ is lower semi-continuous, convex, and Gateaux differentiable.

For $(\phi, y, \nu) \in \mathcal{A}d$, set

$$\bar{\phi} = \phi \text{ on } I_0^{t_1}, \quad (11.3.4)$$
$$\bar{\phi} = y \text{ on } I_{-r}^0. \quad (11.3.5)$$

Now, define a functional on $\mathcal{A}d$ by

$$\begin{aligned} F_K(\phi, y, \nu) = {} & J(\bar{\phi}, \nu) + \|\phi' - \phi_0'\|^2 + |\phi(0) - \phi_0(0)|^2 \\ & + \epsilon\|\nu - \nu_0\|_L + K\|d/dt(\bar{\phi}(t)) - f(\bar{\phi}(\cdot), \nu, t)\|^2 \\ & + \|y' - y_0'\|_*^2 + P_K(y) + KT^2(\phi(0), \phi(t_1)) \end{aligned} \quad (11.3.6)$$

Remark 11.3.2. In $B(\epsilon)$, we choose $0 < \epsilon < \epsilon_1$ to ensure that all admissible trajectories lie in a fixed compact set $\mathcal{X}' \subset \mathcal{X}$.

Lemma 11.3.3. *For any* $0 < \epsilon \le \epsilon_1$, $\exists\ K(\epsilon) > 0$ *such that* $F_{K(\epsilon)}(\phi, y, \nu) > 0$*,* $(\phi, y, \nu) \in B(\epsilon)$*, if any of the following inequalities is an equality:*

$$|\phi(0) - \phi_0(0)| \le \epsilon, \quad \|\phi' - \phi_0'\| \le \epsilon, \quad \|y' - y_0'\|_* \le \epsilon, \quad \|\nu - \nu_0\|_L \le \epsilon.$$

Proof. Suppose the lemma were false. Then, for any $0 < \epsilon \le \epsilon_1$ $\exists$ $\{\langle\phi_j, y_j, \nu_j\rangle\}$ $\subset B(\epsilon)$, K_j, $K_j \to \infty$, where ϕ_j, y_j, ν_j are such that at least one of the inequalities is an equality and $F_{K_j}(\phi_j, y_j, \nu_j) \le 0$. Then, it is easy to conclude that there exists $j_1 < j_2 < \cdots$ such that $\phi_{j_k}' \to \phi^{*'}$, $y_{j_k}' \to y^{*'}$ weakly in L_2, and $\phi_{j_k} \to \phi^*$, $y_{j_k} \to y^*$ uniformly, and $\nu_{j_k} \to \nu^*$ weak-*. Further, $(\bar{\phi}^*, \nu^*)$ is admissible where $\bar{\phi}^* = \phi^*$ on $I_0^{t_1}$ and $\bar{\phi}^* = y^*$ on I_{-r}^0. Note that $J(\bar{\phi}^*, \nu^*) + \epsilon^2 \le 0$. Since (ϕ_0, ν_0) is optimal, we have a contradiction and the lemma is proved. □

Lemma 11.3.4. *Let* $K(\epsilon)$ *be as in Lemma 11.3.3,* $0 < \epsilon < \epsilon_1$*. Then, the functional* $(\phi, y, \nu) \mapsto F_{K(\epsilon)}(\phi, y, \nu)$ *attains its minimum in* $B(\epsilon)$*. For* $0 < \epsilon < \epsilon_1$*, let* $(\phi_\epsilon, y_\epsilon, \nu^\epsilon) \in B(\epsilon)$ *be such that*

$$F_{K(\epsilon)}(\phi_\epsilon, y_\epsilon, \nu^\epsilon) = \inf\{F_{K(\epsilon)}(\phi, y, \nu) \mid (\phi, y, \nu) \in B(\epsilon)\}.$$

Then

$$\|\phi'_\epsilon - \phi'_0\| < \epsilon, \qquad |\phi_\epsilon(0) - \phi_0(0)| < \epsilon,$$
$$\|y' - y'_0\|_* < \epsilon, \qquad \|\nu^\epsilon - \nu_0\|_L < \epsilon.$$

Proof. The proof follows immediately from Lemma 11.3.3. □

For $0 < \epsilon \leq \epsilon_1$, let

$$\begin{aligned} V(\epsilon) = \{(\phi, y) \mid \phi \in \mathcal{AC}(I_0^{t_1}, \mathcal{X}), y \in H^1(-r, 0), \phi' \in L_2(I_0^{t_1}), \\ \phi(0) = y(0), \|\phi' - \phi'_0\| \leq \epsilon, \|y' - y'_0\|_* \leq \epsilon, |\phi(0) - \phi_0(0)| \leq \epsilon\} \end{aligned} \tag{11.3.7}$$

Consider the functional $H^j_{K(\epsilon)}(\phi, y)$ on $V(\epsilon)$ defined by

$$H^j_{K(\epsilon)}(\phi, y) = j \int_0^{t_1} \omega(G(t, \phi(t)))dt + F_{K(\epsilon)}(\phi, y, \nu^\epsilon) \tag{11.3.8}$$

where ν^ϵ is as in Lemma 11.3.4 and ω is a smooth convex function such that

$$\omega(t) = 0, \qquad t \leq 0,$$

$$\omega(t) > 0, \qquad t > 0.$$

One can verify that, for each $j = 1, 2, 3, \dots$ there exist $(\phi_j, y_j) \in V(\epsilon)$ such that

$$H^j_{K(\epsilon)}(\phi_j, y_j) = \inf\{H^j_{K(\epsilon)}(\phi, y) \mid (\phi, y) \in V(\epsilon)\} \tag{11.3.9}$$

Lemma 11.3.5. *Let (ϕ_j, y_j) be as in (11.3.9). Then, there exists a subsequence $\{j_{k_\ell}\}$ such that*

$$\|\phi'_{j_{k_\ell}} - \phi'_0\| < \epsilon, \qquad \|y'_{j_{k_\ell}} - y'_0\|_* < \epsilon, \qquad |\phi_{j_{k_\ell}}(0) - \phi_0(0)| < \epsilon.$$

Proof. For each $j = 1, 2, 3 \dots$, we have

$$H^j_{K(\epsilon)}(\phi_j, y_j) \leq H^j_{K(\epsilon)}(\phi_\epsilon, y_\epsilon) = F_{K(\epsilon)}(\phi_\epsilon, y_\epsilon, \nu^\epsilon).$$

Thus, we can infer that

$$\begin{aligned} &\overline{\lim_{j\to\infty}}\, j \int_0^{t_1} \omega(G(t, \phi_j(t)))dt + \underline{\lim}_{j\to\infty} \int_0^{t_1} f^0(\bar{\phi}_j(\cdot), \nu^\epsilon, t)dt + \underline{\lim}_{j\to\infty} \|\phi'_j - \phi'_0\|^2 \\ &\quad + \underline{\lim}_{j\to\infty} \|y'_j - y'_0\|_*^2 + \underline{\lim}_{j\to\infty} K(\epsilon)\|d/dt\,(\bar{\phi}_j) - f(\bar{\phi}_j(\cdot), \nu^\epsilon, t)\|^2 \\ &\quad + \underline{\lim}_{j\to\infty} |\phi_j(0) - \phi_0(0)|^2 + \epsilon\|\nu^\epsilon - \nu_0\|_L \leq F_{K(\epsilon)}(\phi_\epsilon, y_\epsilon, \nu^\epsilon), \end{aligned} \tag{11.3.10}$$

where

$$\bar{\phi}_j = \phi_j, \qquad \text{on } I_0^{t_1},$$

$$\bar{\phi}_j = y_j, \qquad \text{on } I^0_{-r}.$$

There exists a subsequence of $j = 1, 2, 3, \ldots,$ which we denote by $\{j_k\}$, $\widetilde{\phi}_\epsilon, \widetilde{y}_\epsilon$ such that $\phi'_{j_k} \to \widetilde{\phi}'_\epsilon$, $y'_{j_k} \to \widetilde{y}'_\epsilon$ weakly in L_2 in $I_0^{t_1}(I^0_{-r})$, respectively, and $\phi_{j_k} \to \widetilde{\phi}_\epsilon$, $y_{j_k} \to \widetilde{y}_\epsilon$ uniformly on $I_0^{t_1}(I^0_{-r})$, respectively. From (11.3.8) we see that

$$G(t, \widetilde{\phi}_\epsilon(t)) \leq 0, \qquad 0 \leq t \leq t_1.$$

Also, $(\widetilde{\phi}_\epsilon, \widetilde{y}_\epsilon) \in V(\epsilon)$, and we conclude that

$$F_{K(\epsilon)}(\widetilde{\phi}_\epsilon, \widetilde{y}_\epsilon, \nu^\epsilon) = F_{K(\epsilon)}(\phi_\epsilon, y_\epsilon, \nu^\epsilon).$$

Now, using Lemma 11.3.4, we have

$$\|\widetilde{\phi}'_\epsilon - \phi'_0\| < \epsilon, \qquad \|\widetilde{y}'_\epsilon - y'_0\|_* < \epsilon, \qquad |\widetilde{\phi}_\epsilon(0) - \phi_0(0)| < \epsilon.$$

Now, from (11.3.8) and the fact

$$F_{K(\epsilon)}(\widetilde{\phi}_\epsilon, \widetilde{y}_\epsilon, \nu^\epsilon) = F_{K(\epsilon)}(\phi_\epsilon, y_\epsilon, \nu^\epsilon)$$

the lemma follows. □

Remark 11.3.6. In the course of proving Lemma 11.3.5 we have come across $(\widetilde{\phi}_\epsilon, \widetilde{y}_\epsilon)$ satisfying

$$\|\widetilde{\phi}'_\epsilon - \phi'_0\| < \epsilon, \qquad \|\widetilde{y}'_\epsilon - y'_0\|_* < \epsilon, \qquad |\widetilde{\phi}_\epsilon(0) - \phi_0(0)\| < \epsilon,$$

$$\widetilde{\phi}_\epsilon(0) = \widetilde{y}_\epsilon(0), \qquad G(t, \widetilde{\phi}_\epsilon(t)) \leq 0, \qquad 0 \leq t \leq t_1,$$

$$F_{K(\epsilon)}(\widetilde{\phi}_\epsilon, \widetilde{y}_\epsilon, \nu^\epsilon) = F_{K(\epsilon)}(\phi_\epsilon, y_\epsilon, \nu^\epsilon).$$

In subsequent discussions, we keep this in mind. Again, for convenience we set

$$\bar{\phi}_\epsilon(t) = \begin{cases} \widetilde{\phi}_\epsilon(t), & 0 \leq t \leq t_1 \\ \widetilde{y}_\epsilon(t), & -r \leq t \leq 0. \end{cases}$$

Lemma 11.3.7. *Let $p, q \in L_1(0,1)$. Let the inequality*

$$\int_0^{t_1} \{pw + qw'\} dt \geq 0$$

hold for all w satisfying $w(0) = w(1) = 0$, $w \geq 0$, with w piecewise continuously differentiable. Then, the function

$$T(t) = q(t) - \int_0^t p(s) ds$$

is nonincreasing outside of a set of measure zero.

Proof. Let $t'' > t' \geq 0$ be Lebesgue points of q. Taking

$$w(t) = \begin{cases} (t - t')/\epsilon, & t' \leq t \leq t' + \epsilon \\ 1, & t' + \epsilon \leq t \leq t'' - \epsilon \\ (t'' - t)/\epsilon, & t'' - \epsilon \leq t \leq t'' \\ 0, & 0 \leq t \leq t', t'' \leq t \leq t_1 \end{cases}$$

gives the result. □

By the Riesz representation theorem, there exists a function $s \mapsto \widetilde{\Gamma}^i(t, \bar{\phi}_\epsilon(\cdot),\ \nu_t^\epsilon, s)$ defined for $s \in I_{-r}^{t_1}$, of bounded variation and continuous from the right such that, for each $\zeta \in C(I_{-r}^{t_1})$, we have

$$dh^i(t, \bar{\phi}_\epsilon(\cdot), \nu_t^\epsilon)(\zeta) = \int_{-r}^{t_1} \zeta(s) \cdot d_s\widetilde{\Gamma}^i(t, \bar{\phi}_\epsilon(\cdot), \nu_t^\epsilon, s) \tag{11.3.11}$$

Here, $dh^i(t, \bar{\phi}_\epsilon(\cdot), \nu_t^\epsilon)$ denotes the Frechet derivative of $h^i(t, \cdot, \nu_t^\epsilon)$ at $\bar{\phi}_\epsilon$. Since h^i does not depend on $\bar{\phi}_\epsilon(s)$ for $s > t$, $s \mapsto \widetilde{\Gamma}^i(t, \bar{\phi}_\epsilon(\cdot), \nu_t^\epsilon, s)$ is constant for $s > t$. We take this constant to be zero and uniquely determine $\widetilde{\Gamma}^i(t, \bar{\phi}_\epsilon(\cdot), \nu_t^\epsilon, s)$. Thus, we may write (11.3.11) as

$$dh^i(t, \bar{\phi}_\epsilon(\cdot), \nu_t^\epsilon)(\zeta) = \int_{-r}^{t} \zeta(s) \cdot d_s\widetilde{\Gamma}^i(t, \bar{\phi}_\epsilon(\cdot), \nu_t^\epsilon, s) = \int_{-r}^{t} \zeta(s) \cdot d_s\widetilde{\Gamma}^i(t, \epsilon; s) \tag{11.3.12}$$

Let

$$\bar{\Gamma}^i(t, \bar{\phi}_\epsilon(\cdot), \nu_t^\epsilon, s) = -\int_s^t g_x^i(t, \tau, \bar{\phi}_\epsilon(\cdot), \nu_t^\epsilon) d_\tau \omega^i(t, \tau), \tag{11.3.13}$$

for $-r \leq s \leq t$, and equal to zero for $s \geq t$. Let

$$\Gamma^i = \widetilde{\Gamma}^i + \bar{\Gamma}^i, \qquad i = 0, 1, 2, \ldots, n. \tag{11.3.14}$$

Denote the Frechet derivative of $f^i(\cdot, \nu_t^\epsilon, t)$ at $\bar{\phi}_\epsilon$ by $df^i(\bar{\phi}_\epsilon(\cdot), \nu_t^\epsilon, t)$. Then, for $\zeta \in C(I_{-r}^{t_1})$, we have

$$df^i(\bar{\phi}_\epsilon(\cdot), \nu_t^\epsilon, t)(\zeta) = \int_{-r}^{t} \zeta(s) \cdot d_s\Gamma^i(t, \bar{\phi}_\epsilon(\cdot), \nu_t^\epsilon, s) = \int_{-r}^{t} \zeta(s) \cdot d_s\Gamma^i(t, \epsilon; s) \tag{11.3.15}$$

At this point, let us note that for a.e. $t \in I_0^{t_1}$, $i = 0, \ldots, n$,

$$|\Gamma^i(t, \epsilon; s)| = |\Gamma^i(t, \epsilon; s) - \Gamma^i(t, \epsilon; t)| \leq V(\Gamma^i(t, \epsilon; \cdot)) \leq \Lambda(t),$$

where V stands for total variation.

Let

$$\Gamma(t, \epsilon; s) = (\Gamma^1(t, \epsilon; s), \ldots, \Gamma^n(t, \epsilon; s)) \tag{11.3.16}$$

where $\Gamma^1, \ldots, \Gamma^n$ are row vectors. If $v = (v_1, \ldots, v_n)$, then by $v \cdot \Gamma$ we mean

$v_1\Gamma^1 + \cdots + v_n\Gamma^n$. Thus, $v_1\Gamma_1 + \cdots + v_n\Gamma_n$ is an n-dimensional row vector. Let

$$\psi(\epsilon;t) = 2(\widetilde{\phi}'_\epsilon(t) - \phi'_0(t)) + 2K(\epsilon)\left[\frac{d}{dt}\bar{\phi}_\epsilon(t) - f(\epsilon;t)\right], \tag{11.3.17}$$

where we have written $f(\epsilon;t)$ for $f(\bar{\phi}_\epsilon(\cdot), \nu^\epsilon_t, t)$. We will use this sort of notation in the sequel.

We next let

$$\Psi(\epsilon;t) = \psi(\epsilon;t) - \int_t^{t_1} \Gamma^0(s,\epsilon;t)ds + \int_t^{t_1} [\psi(\epsilon;s) - 2(\widetilde{\phi}'_\epsilon(s) - \phi'_0(s))]\cdot\Gamma(s,\epsilon;t)ds. \tag{11.3.18}$$

Assumption 11.3.8. We assume that for $0 \leq t \leq t_1$ we have $\nabla_x G(t,x) \neq 0$ $\forall$ x such that $|\phi_0(t) - x| < \epsilon_2$, $\epsilon_2 > \epsilon_1$. $\nabla_x G(t,x) = (\nabla_{x_1} G(t,x), \ldots, \nabla_{x_n} G(t,x))$.

Let ζ be an absolutely continuous scalar function with square integrable derivative on $I_0^{t_1}$ such that

$$\zeta(t) \geq 0, \qquad 0 \leq t \leq t_1$$

$$\zeta(0) = \zeta(t_1) = 0.$$

Then, there exists $\theta_0 > 0$ such that for $0 \leq \theta \leq \theta_0$,

$$G(t, \widetilde{\phi}_\epsilon(t) + \theta\zeta(t)\xi_\epsilon(t)) \leq 0,$$

where

$$\xi_\epsilon(t) = -\nabla_x G(t, \widetilde{\phi}_\epsilon(t))/|\nabla_x G(t, \widetilde{\phi}_\epsilon(t))|^2.$$

From the fact that

$$dF_{K(\epsilon)}(\widetilde{\phi}_\epsilon + \theta\zeta\xi_\epsilon, \widetilde{y}_\epsilon, \nu^\epsilon)/d\theta|_{\theta=0^+} \geq 0 \tag{11.3.19}$$

we obtain

$$\int_0^{t_1} ((\Psi(\epsilon;t)\cdot\xi_\epsilon)\zeta' + (\Psi(\epsilon;t)\cdot\xi'_\epsilon)\zeta)dt \geq 0. \tag{11.3.20}$$

Thus, by Lemma 11.3.7, we obtain a nonincreasing function,

$$\lambda(\epsilon;t) = \Psi(\epsilon;t)\cdot\xi_\epsilon(t) - \int_0^t \Psi(\epsilon;s)\cdot\xi'_\epsilon(s)ds. \tag{11.3.21}$$

Let

$$h(t) = \nabla_x G(t, \widetilde{\phi}_\epsilon(t))/|\nabla_x G(t, \phi_\epsilon(t))|^2.$$

In Lemma 11.3.5, for simplicity of notation, we assume that the conclusion of the lemma is true for $\{j_k\}$ instead of $\{j_{k_\ell}\}$. Let $\widehat{h}(t)$ be defined by

$$\widehat{h}(t) = h(t), \qquad 0 \leq t \leq 1,$$

$$\widehat{h}(t) = \nabla_x G(0, \widetilde{\phi}_\epsilon(0))/|\nabla_x G(0, \widetilde{\phi}_\epsilon(0))|^2, \qquad -r \leq t \leq 0$$

For $\theta > 0$ small,

$$(\phi_{j_k} + \theta h, y_{j_k} + \theta \nabla_x G(0, \widetilde{\phi}_\epsilon(0))/|\nabla_x G(0, \widetilde{\phi}_\epsilon(0))|^2) \in V(\epsilon).$$

Then, using (11.3.8),

$$dH^{j_k}_{K(\epsilon)}(\phi_{j_k} + \theta h, y_{j_k} + \theta \nabla_x G(0, \widetilde{\phi}_\epsilon(0))/|\nabla G(0, \widetilde{\phi}_\epsilon(0))|^2)|_{\theta=0} = 0. \quad (11.3.22)$$

From (11.3.22), we can conclude that $j_k \int_0^{t_1} \omega'(G(t, \phi_{j_k}(t)))dt$ is bounded as $k \to \infty$.

Let ζ be a smooth vector function defined on $I_0^{t_1}$ and vanishing for $t \leq 0$ and $t \geq t_1$. Then let

$$w(t) = \zeta(t) + (\nabla_x G(t, \widetilde{\phi}_\epsilon(t)) \cdot \zeta)\xi_\epsilon, \quad (11.3.23)$$

where ξ_ϵ is defined as before. For $\theta > 0$ small enough, $(\phi_{j_k} + \theta w, y_{j_k}) \in V(\epsilon)$. Thus, again from (11.3.8) we have

$$dH^{j_k}_{K(\epsilon)}(\phi_{j_k} + \theta w, y_{j_k})/d\theta|_{\theta=0} = 0. \quad (11.3.24)$$

In (11.3.24), we let $k \to \infty$ after differentiating with respect to θ at $\theta = 0$ and obtain

$$\begin{aligned}
&\Phi(\epsilon; t) - \int_t^{t_1} \Gamma^0(s, \epsilon; t)ds \qquad (11.3.25)\\
&+ \int_t^{t_1} [\Phi(\epsilon; s) - 2(\widetilde{\phi}'_\epsilon(s) - \phi'_0(s))] \cdot \Gamma(s, \epsilon; t)ds\\
&- \int_t^{t_1} \lambda(\epsilon; s)\nabla_x G(s, \widetilde{\phi}_\epsilon(s)) \cdot \Gamma(s, \epsilon; t)ds\\
&+ \int_t^{t_1} \lambda(\epsilon; s)[dG_x(s, \widetilde{\phi}_\epsilon(s))/ds]ds = C_\epsilon,
\end{aligned}$$

where

$$\Phi(\epsilon; s) = \psi(\epsilon; s) + \lambda(\epsilon; s)\nabla_x G(s, \widetilde{\phi}_\epsilon(s)). \quad (11.3.26)$$

Note that from (11.3.25) we have

$$C_\epsilon = \Phi(\epsilon, t_1^-). \quad (11.3.27)$$

Lemma 11.3.9. *The nonincreasing function $\lambda(\epsilon, \cdot)$ defined in (11.3.21) is constant at points of $I_0^{t_1}$ where $G(t, \widetilde{\phi}_\epsilon(t)) < 0$.*

Proof. Suppose that $t_0 \in (0, t_1)$ is such that $G(t_0, \widetilde{\phi}_\epsilon(t_0)) < 0$. Then, there exist $0 < \alpha < \beta < t_1$ such that

$$G(t, \widetilde{\phi}_\epsilon(t)) < 0, \qquad \alpha \leq t \leq \beta.$$

Let $\eta \in C_0^\infty(\alpha, \beta)$. Using Remark 11.3.6, we have

$$dF_{K(\epsilon)}(\widetilde{\phi}_\epsilon + \theta\eta, \widetilde{y}_\epsilon, \nu^\epsilon)/d\theta|_{\theta=0} = 0,$$

from which we infer that $\Psi(\epsilon; \cdot)$ defined by (11.3.18) is constant over the interval (α, β). Then, from (11.3.21), we immediately see that

$$\lambda(\epsilon; t) = \lambda(\epsilon; \alpha), \qquad t \in (\alpha, \beta).$$

Let ζ be an absolutely continuous n-dimensional vector function on $I_{-r}^{t_1}$ such that $\zeta(-r) = 0$ and $\zeta = 0$ on $[0, t_1]$. From Remark 11.3.6 we have

$$dF_{K(\epsilon)}(\widetilde{\phi}_\epsilon, \widetilde{y}_\epsilon + \theta\zeta, \nu^\epsilon)/d\theta|_{\theta=0} = 0. \tag{11.3.28}$$

Then, from (11.3.28) we obtain

$$\int_{-r}^{0} \zeta'(t) \cdot \left\{ \int_0^{t_1} [-\Gamma^0(s, \epsilon; t) + (\psi(\epsilon; s) - 2(\widetilde{\phi}'_\epsilon(s) - \phi'_0(s))) \cdot \Gamma(s, \epsilon; t)] ds \right.$$
$$\left. + 2(\widetilde{y}'_\epsilon(t) - y'_0(t)) \right\} dt + \partial P_{K(\epsilon)}(\widetilde{y}_\epsilon)(\zeta) = 0 \tag{11.3.29}$$

Writing

$$\partial P_{K(\epsilon)}(\widetilde{y}_\epsilon)(\zeta) = \int_{-r}^{0} (b'_\epsilon \cdot \zeta' - B_\epsilon \cdot \zeta') ds,$$

where $B'_\epsilon = b_\epsilon$, $B_\epsilon(-r) = 0$, we conclude from (11.3.28) that, for a.e. t,

$$\int_0^{t_1} [\cdots] ds + 2(\widetilde{y}'_\epsilon(t) - y'_0(t)) + b'_\epsilon(t) - B_\epsilon(t) = D_\epsilon, \tag{11.3.30}$$

where D_ϵ is a constant.

Let ζ be a smooth scalar function such that $\zeta \geq 0$, $\zeta(t) = 1$, $0 \leq t \leq \delta/2$ and $\zeta(t) = 0$, $t \geq 3\delta/4$.

Let

$$\eta_N(t) = (\eta_1(t), \ldots, \eta_n(t))$$

such that

$$\eta_i(t) = e^{-Nt}\zeta(t), \quad \eta_j(t) = 0 \quad \text{if } j \neq i.$$

Let

$$\begin{aligned} \eta_{N+}(t) &= \eta_N(t), \quad 0 \leq t \leq t_1 \\ \eta_{N-}(t) &= \eta_N(0), \quad -r \leq t \leq 0 \end{aligned}$$

For the next results dealing with boundary conditions we assume that Assumption 11.4.1 is in force. Then, in

$$\left.\frac{dH^{j_k}_{K(\epsilon)}(\phi_{j_k} + \theta\eta_{N+}, y_{j_k} + \theta\eta_{N-})}{d\theta}\right|_{\theta=0} = 0,$$

we let $N \to \infty$. We carry out this operation for $i = 1, 2, \ldots, n$ and obtain

$$\begin{aligned}0 = &- \psi(\epsilon; 0) + \int_0^{t_1} \left\{\Gamma^0(t, \epsilon; 0) - \left[\psi(\epsilon; t) - 2(\widetilde{\phi}'_\epsilon(t) - \phi'_0(t))\right] \cdot \Gamma(t, \epsilon; 0)\right\} dt \\ &+ \int_0^{t_1} \left\{-\Gamma^0(t, \epsilon; -r) + \left[\psi(\epsilon; t) - 2(\widetilde{\phi}'_\epsilon(t) - \phi'_0(t))\right] \cdot \Gamma(t, \epsilon; -r)\right\} dt \\ &+ 2K(\epsilon)T(\widetilde{\phi}_\epsilon(0), \widetilde{\phi}_\epsilon(t_1))\partial_1 T(\widetilde{\phi}_\epsilon(0), \widetilde{\phi}_\epsilon(t_1)) + 2(\widetilde{\phi}_\epsilon(0) - \phi_0(0)) + B_\epsilon(0)\end{aligned} \tag{11.3.31}$$

Next, let ζ be a smooth scalar function such that $\zeta \geq 0$

$$\begin{aligned}\zeta(t) &= 1, \quad t_1 - \delta/2 \leq t \leq t_1 \\ \zeta(t) &= 0, \quad t \leq t_1 - 3\delta/4\end{aligned}$$

Now let

$$\eta_N(t) = (\eta_1(t), \ldots, \eta_n(t))$$

such that

$$\eta_i(t) = e^{-N(t_1 - t)}\zeta(t), \quad \eta_j(t) = 0 \quad \text{if } j \neq i.$$

$$\begin{aligned}\eta_{N+}(t) &= \eta_N(t), \quad 0 \leq t \leq t_1 \\ \eta_{N-}(t) &= 0, \quad -r \leq t \leq 0\end{aligned}$$

Then, in

$$\left.\frac{dH^{j_k}_{K(\epsilon)}(\phi_{j_k} + \theta\eta_{N+}, y_{j_k} + \theta\eta_{N-})}{d\theta}\right|_{\theta=0} = 0,$$

we let $N \to \infty$. We carry out this operation for $i = 1, 2, \ldots, n$ and obtain

$$\widetilde{\psi}(t_1; \epsilon) + 2K(\epsilon)T(\widetilde{\phi}_\epsilon(0)\widetilde{\phi}_\epsilon(t_1))\partial_2 T(\widetilde{\phi}_\epsilon(0), \widetilde{\phi}_\epsilon(t_1)) = 0 \tag{11.3.32}$$

According to Lemma 11.3.9,

$$\begin{aligned}\lambda(\epsilon; \cdot) &= \lambda(\epsilon; 0^+) \quad \text{ in } (0, \delta) \\ \lambda(\epsilon; \cdot) &= \lambda(\epsilon; t_1^-) \quad \text{ in } (t_1 - \delta, t_1)\end{aligned}$$

Now, we can rewrite (11.3.31) and (11.3.32)

$$\begin{aligned}\Phi(\epsilon; 0) = &\lambda(\epsilon; 0^+)\nabla G(0, \widetilde{\phi}_\epsilon(0)) + 2K(\epsilon)T(\widetilde{\phi}_\epsilon(0), \widetilde{\phi}_\epsilon(t_1))\partial_1 T(\widetilde{\phi}_\epsilon(0), \widetilde{\phi}_\epsilon(t_1)) \\ &+ \int_0^{t_1} \Big\{\Gamma^0(t, \epsilon; 0) - \left[\Phi(\epsilon; t) - 2(\widetilde{\phi}'_\epsilon(t) - \widetilde{\phi}'_0(t))\right] \cdot \Gamma(t, \epsilon; 0) \\ &+ \lambda(\epsilon; t)\nabla G(t, \widetilde{\phi}_\epsilon(t)) \cdot \Gamma(t, \epsilon; 0)\Big]\Big\} dt \\ &+ \int_0^{t_1} \Big\{-\Gamma^0(t, \epsilon; -r) + \left[\Phi(\epsilon; t) - 2(\widetilde{\phi}'_\epsilon(t) - \widetilde{\phi}'_0(t))\right] \cdot \Gamma(t, \epsilon; -r)\end{aligned}$$

$$- \lambda(\epsilon; t)\nabla G(t, \widetilde{\phi}_\epsilon(0)) \cdot \Gamma(t, \epsilon; -r)\Big\}dt$$
$$+ 2(\widetilde{\phi}_\epsilon(0) - \phi_0(0)) + B_\epsilon(0) \tag{11.3.33}$$

$$\Phi(\epsilon; t_1) = \lambda(\epsilon; t_1^-)\nabla G(t, \widetilde{\phi}_\epsilon(t_1))$$
$$- 2K(\epsilon)T(\widetilde{\phi}_\epsilon(0), \widetilde{\phi}_\epsilon(t_1))\partial_2 T(\widetilde{\phi}_\epsilon(0), \widetilde{\phi}_\epsilon(t_1)) \tag{11.3.34}$$

Next let ζ be an absolutely continuous vector function such that $\zeta \in L_2(I_{-r}^0)$ and $\zeta(t) = 0$, $0 \leq t \leq t_1$. Using (11.3.28) we obtain

$$\zeta(-r) \cdot \int_0^{t_1} [-\Gamma^0(t, \epsilon; -r) + \widetilde{\psi}(\epsilon; t) \cdot \Gamma(t, \epsilon; -r)]dt$$
$$+ \int_{-r}^0 \zeta'(t) \cdot \left\{ \int_0^{t_1} [-\Gamma^0(s, \epsilon; t) + \widetilde{\psi}(\epsilon; s) \cdot \Gamma(s, \epsilon; t)]ds \right.$$
$$\left. + 2(\widetilde{y}'_\epsilon - y'_0) + b'_\epsilon - B_\epsilon(t) \right\} dt = 0,$$

where
$$\widetilde{\psi}(\epsilon; t) = \psi(\epsilon; t) - 2(\widetilde{\phi}'_\epsilon(t) - \phi'_0(t)).$$

We have seen in (11.3.30) that the vector in braces is constant, and thus can be pulled out of the integral. Thus, we have

$$\zeta(-r) \cdot \left\{ \int_0^{t_1} [-\Gamma^0(t, \epsilon; -r) + \widetilde{\psi}(\epsilon; t) \cdot \Gamma(t, \epsilon; -r)]dt \right.$$
$$- \int_0^{t_1} [-\Gamma^0(s, \epsilon; t) + \widetilde{\psi}(\epsilon; s)\Gamma(s, \epsilon; t)]ds$$
$$\left. - 2(\widetilde{y}'_\epsilon(t) - y'_0(t)) - (b'_\epsilon(t) - B_\epsilon(t)) \right\} = 0$$

Thus,

$$b'_\epsilon(t) - B_\epsilon(t) - \int_0^{t_1} [-\Gamma^0(t, \epsilon; -r) + \widetilde{\psi}(\epsilon; t) \cdot \Gamma(t, \epsilon; -r)]dt$$
$$+ \int_0^{t_1} [-\Gamma^0(s, \epsilon; t) + \widetilde{\psi}(\epsilon; s)\Gamma(s, \epsilon; t)]ds + 2(\widetilde{y}'_\epsilon(t) - y'_0(t)) = 0$$

Since $B'_\epsilon(t) = b_\epsilon$ and $B_\epsilon(-r) = 0$ we have

$$B''_\epsilon(t) - B_\epsilon(t) + \int_0^{t_1} [-\Gamma^0(s, \epsilon; t) + \widetilde{\psi}(\epsilon; s) \cdot \Gamma(s, \epsilon; t)]ds$$
$$- \int_0^{t_1} [-\Gamma^0(t, \epsilon; -r) + \widetilde{\psi}(\epsilon; t)\Gamma(t, \epsilon; -r)]dt + 2(\widetilde{y}'_\epsilon(t) - y'_0(t)) = 0$$

$$B_\epsilon(-r) = 0, \tag{11.3.35}$$

$(B'_\epsilon, B_\epsilon) \in \partial P_{K(\epsilon)}(\widetilde{y}_\epsilon)$.

Next, for $0 < \epsilon < \epsilon_1$ and $0 \le \theta \le 1$, let

$$\nu(\theta) = \nu^\epsilon + \theta(\nu - \nu^\epsilon).$$

Since

$$\|\nu^\epsilon - \nu_0\|_L < \epsilon,$$

there exists $\theta_0 > 0$ such that $0 \le \theta \le \theta_0$ implies

$$\|\nu(\theta) - \nu_0\|_L < \epsilon.$$

Now, from Remark 11.3.6 and Lemma 11.3.4, we have

$$dF_{K(\epsilon)}(\widetilde{\phi}_\epsilon, \widetilde{y}_\epsilon, \nu(\theta))/d\theta|_{\theta=0} \ge 0. \tag{11.3.36}$$

Let

$$\rho_\epsilon(\theta) = \|\nu(\theta) - \nu_0\|_L.$$

The function $\theta \mapsto \rho_\epsilon(\theta)$ has a right derivative at $\theta = 0$. Thus, from (11.3.36) we obtain

$$\widetilde{H}(t, \bar{\phi}_\epsilon(\cdot), \nu^\epsilon_t) \ge \widetilde{H}(t, \bar{\phi}_\epsilon(\cdot), \nu_t) - \epsilon\rho'_\epsilon(0^+), \quad \text{a.e. } t, \tag{11.3.37}$$

where

$$\begin{aligned}\widetilde{H}(t, \bar{\phi}_\epsilon(\cdot), \sigma_t) &= -f^0(t, \bar{\phi}_\epsilon(\cdot), \sigma_t) \\ &\quad + [\Phi(\epsilon; t) - \lambda(\epsilon; t)\nabla G(t, \widetilde{\phi}_\epsilon(t)) - 2(\widetilde{\phi}'_\epsilon(t) - \phi'_0(t))] \cdot f(t, \bar{\phi}_\epsilon(\cdot), \sigma_t),\end{aligned} \tag{11.3.38}$$

and

$$\bar{\phi}_\epsilon(t) = \begin{cases} \widetilde{\phi}_\epsilon(t), & 0 \le t \le 1 \\ \widetilde{y}_\epsilon(t), & -r \le t \le 0 \end{cases} \tag{11.3.39}$$

$\square$

11.4 Limiting Operations

Before we proceed in this section we make Assumption 11.4.1.

Assumption 11.4.1. We make the assumption that there exists $\delta > 0$ such that for $t \in (0, \delta) \cup (t_1 - \delta, t_1)$, $0 < \delta < t_1$, we have $G(t, \phi_0(t)) < 0$.

Note that if ϵ, $0 < \epsilon \le \epsilon_1$, is sufficiently small,

$$G(t, \widetilde{\phi}_\epsilon(t)) < 0, \qquad t \in (0, \delta) \cup (t_1 - \delta, t_1).$$

According to Lemma 11.3.9,

$$\lambda(\epsilon; \cdot) = \lambda(\epsilon; 0^+), \qquad \text{in } (0, \delta),$$

$$\lambda(\epsilon; \cdot) = \lambda(\epsilon; t_1^-), \qquad \text{in } (t_1 - \delta, t_1).$$

Remark 11.4.2. We modify λ by subtracting $d \equiv \lambda(\epsilon; t_1^-)$ from it. Henceforth we assume that this has been done. Thus, $\lambda \ge 0$.

Remark 11.4.3. The effect of Remark 11.4.2 is as follows. Modifying λ as indicated will force us to modify $\Phi(\epsilon; t)$ by taking off $d\nabla_x G(t, \widetilde{\phi}_\epsilon(t))$ from it, and $d\nabla_x G(t_1, \widetilde{\phi}_\epsilon(t_1))$ from C_ϵ to maintain (11.3.25). We assume that these modifications have already been done in what follows. The modifications in (11.3.33) and (11.3.34) are similarly done. For example, (11.3.34) is modified by removing $d\nabla_x G(t_1, \widetilde{\phi}_\epsilon(t_1))$ from the right-hand side.

From (11.3.25) we can verify that

$$|\Phi(\epsilon; t)| \le C(d_1 + \lambda(\epsilon, 0^+)d_2 + |\Phi(\epsilon; 1^-)|), \tag{11.4.1}$$

where $C > 0$, $d_1 > 0$, $d_2 > 0$ are constants independent of ϵ, $0 < \epsilon \le \epsilon_1$. We may also verify that

$$V\Phi(\epsilon; \cdot) \le C_1(\widetilde{d}_1 + \lambda(\epsilon, 0^+)\widetilde{d}_2), \tag{11.4.2}$$

where $C_1, \widetilde{d}_1, \widetilde{d}_2$ are positive constants independent of ϵ, $0 < \epsilon \le \epsilon_1$, and $V\Phi(\epsilon; \cdot)$ is the total variation of $\Phi(\epsilon; \cdot)$. Motivated by (11.4.1) and (11.4.2) we define a constant

$$M(\epsilon) = 1 + |\Phi(\epsilon; 1^-)| + \lambda(\epsilon, 0^+) + |\beta_\epsilon|. \tag{11.4.3}$$

Using the Helley compactness theorem, there exists a subsequence $0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_n \to 0$ such that

$$\lambda(\epsilon_i; \cdot)/M(\epsilon_i) \to \lambda(\cdot), \qquad \text{a.e. } t \in [0, t_1] \tag{11.4.4}$$

$$\Phi(\epsilon_i; \cdot)/M(\epsilon_i) \to \Phi(\cdot), \qquad \text{a.e. } t \in [0, t_1]. \tag{11.4.5}$$

Clearly, we have

$$\int_0^{t_1} \int_0^{t_1} |\Gamma^i(s, \epsilon; t)|^2 ds\, dt \le \|\Lambda\|^2, \tag{11.4.6}$$

and thus we can extract a subsequence $\epsilon_1', \epsilon_2', \ldots, \epsilon_n', \ldots$ such that

$$\Gamma^i(\cdot, \epsilon_j', \cdot) \to \Gamma^i(\cdot, \cdot) \tag{11.4.7}$$

weakly in $L_2((0,t_1)\times(0,t_1))$. By Mazur's theorem, there exist $\alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nk_n}$, $n = 1,2,3,\ldots$, $\alpha_{nm} \geq 0$, $\sum_{m=1}^{k_n} \alpha_{nm} = 1$, such that

$$v_n^i = \sum_{m=1}^{k_n} \alpha_{nm}\Gamma^i(\cdot,\epsilon_m,\cdot) \to \Gamma^i(\cdot,\cdot), \qquad \text{in } L_2$$

We can extract a subsequence $n_1 < n_2 < \cdots$ such that

$$v_{n_j}^i \to \Gamma^i(\cdot,\cdot), \qquad \text{a.e.} \tag{11.4.8}$$

Suppose that $(s^*,t^*) \in [0,t_1] \times [0,t_1]$ such that

$$\lim_{j\to\infty} v_{n_j}^i(s^*,t^*) = \Gamma^i(s^*,t^*) \tag{11.4.9}$$

Note that we have

$$Vv_{n_j}^i(s^*,\cdot) \leq \Lambda(s^*), \tag{11.4.10}$$

where V denotes the total variation. By the Helley compactness theorem, there exists $j_1 < j_2 < \cdots$ such that, pointwise in the t-variable,

$$\lim_{k\to\infty} v_{n_{j_k}}^i(s^*,t) = \gamma_{s^*}(t), \tag{11.4.11}$$

where $\gamma_{s^*}(\cdot)$ is of bounded variation and

$$V\gamma_{s^*}(\cdot) \leq \Lambda(s^*) \tag{11.4.12}$$

Using (11.4.11), we see that we can modify $t \mapsto \Gamma^i(s^*,t)$ to agree with $\gamma_{s^*}(\cdot)$. Having done the modification, we can say $\Gamma^i(s^*,\cdot)$ is of bounded variation and

$$V\Gamma^i(s^*,\cdot) \leq \Lambda(s^*) \tag{11.4.13}$$

Let

$$\lambda^0 = 1/\varliminf_{\epsilon\to 0^+} M(\epsilon) \tag{11.4.14}$$

We can extract a subsequence $\epsilon_1'', \epsilon_2'', \ldots$ tending to zero such that

$$1/M(\epsilon_i'') \to \lambda^0, \tag{11.4.15}$$

$$\beta_{\epsilon_i}''/M(\epsilon_i'') \to \beta. \tag{11.4.16}$$

Clearly, we can take an appropriate subsequence of $\{\epsilon\}_{\epsilon>0}$ tending to zero so that all limiting operations involving ϵ tending to zero are simultaneously valid.

In (11.3.25), (11.3.33), (11.3.34), and (11.3.37), keeping in mind Remarks 11.4.2 and 11.4.3, we divide both sides of the equation/inequality by $M(\epsilon)$ and let $\epsilon \to 0^+$ through an appropriate subsequence to obtain the following theorem.

Theorem 11.4.4. *Suppose Assumption 11.4.1 is in force. Then, at any relaxed optimal pair* (ϕ_0, ν_0)*, the following conditions are met: There exist a function of bounded variation* Φ *on* $I_0^{t_1}$*, a bounded nonincreasing function* λ*,* $\lambda(\cdot) \geq 0$*,* $\lambda(t_1^-) = 0$*,* $\Gamma(t,s) = (\Gamma^1(t,s), \ldots, \Gamma^n(t,s))$*, and a function* Γ^0 *such that* $s \mapsto \Gamma^i(t,s)$*,* $s \in I_{-r}^{t_1}$*,* $i = 0, 1, \ldots, n$*, is of bounded variation satisfying (11.4.13), continuous from the right vanishing for* $s > t$*, and scalars* β*,* λ^0*,* $\lambda^0 \geq 0$*,* γ_0 *such that*

(i) $|\Phi(t_1^-)| + \lambda(0^+) + |\beta| + \lambda^0 = 1,$

(ii) $\Phi(t) - \lambda^0 \int_t^{t_1} \Gamma^0(s,t)ds + \int_t^{t_1} \Phi(s)\Gamma(s,t)ds - \int_t^{t_1} \lambda(s)\nabla G(s,\phi_0(s))\Gamma(s,t)ds$

$+ \int_t^{t_1} \lambda(s)[d\nabla G(s,\phi_0(s))/ds]ds = \Phi(t_1^-)$

(iii) $\Phi(0) = \lambda(0^+)\nabla G(0,\phi_0(0)) + \beta\partial_1 T(\phi_0(0),\phi_0(t_1)) + \int_0^{t_1}\{\lambda^0\Gamma^0(t,0) - \Phi(t)\cdot \Gamma(t,0) + \lambda(t)\nabla G(t,\phi_0(t))\cdot\Gamma(t,0)\}dt + \int_0^{t_1}\{-\lambda^0\Gamma^0(t,-r) + \Phi(t)\cdot\Gamma(t,-r) - \lambda(t)\nabla G(t,\phi_0(t))\cdot\Gamma(t,-r)\}dt + B(0)$

$B''(t) - B(t) + \int_0^{t_1}[-\lambda^0\Gamma^0(s,t) + \Phi(s)\cdot\Gamma(s,t) - \lambda(s)\nabla G(s,\phi_0(s))]ds$

$- \int_0^{t_1}[-\lambda^0\Gamma^0(t,-r) + \Phi(t)\cdot\Gamma(t,-r) - \lambda(t)\nabla G(t,\phi_0(t))\cdot\Gamma(t,-r)]dt = 0$

$B(-r) = 0$

$(B', B) \in \partial I_{\mathcal{M}}(y_0)$

(iv) $\Phi(t_1^-) = -\beta\partial_2 T(\phi(0),\phi_0(t_1)).$

(v) $H(t,\bar\phi(\cdot),\nu_{0t}) \leq H(t,\bar\phi(\cdot),\sigma_t)$*, a.e.* t*, where*

$H(t,\bar\phi(\cdot),\sigma_t) = -[\Phi(t) - \lambda(t)\nabla G(t,\phi_0(t))]\cdot f(t,\bar\phi(\cdot),\sigma_t) + \lambda^0 f^0(t,\bar\phi(\cdot),\sigma_t),$

$$\bar\phi(t) = \begin{cases} \phi_0(t), & 0 \leq t \leq t_1 \\ y_0(t), & -r \leq t \leq 0 \end{cases}$$

Remark 11.4.5. We have the condition

$$\|\Phi(\cdot) - \lambda(\cdot)\nabla G(\cdot,\phi_0(\cdot))\|_\infty + |\beta| + \lambda^0 \neq 0.$$

Otherwise,

$$\Phi(\cdot) = \lambda(\cdot)\nabla G(\cdot,\phi_0(\cdot))$$

Thus, $\Phi(t_1^-) = 0$, and (ii) of Theorem 11.4.4 says

$$\Phi(t) + \int_t^{t_1} \lambda(s)[d\nabla G(s,\phi_0(s))/ds] = 0,$$

and Φ is absolutely continuous. Thus,

$$\Phi'(t) = \lambda(t)[d\nabla G(t,\phi_0(t))/dt].$$

Thus,

$$\lambda' = 0.$$

Since $\lambda(t_1^-) = 0$, it follows that $\lambda \equiv 0$. In particular $\lambda(0^+) = 0$ and (i) of Theorem 11.4.4 is contradicted.

Exercise 11.4.6. Verify that $\lambda(0^+) + |\beta| + \lambda^0 \neq 0$.

11.5 The Bounded State Problem for Integrodifferential Systems

In this section we specialize the problem in Section 11.2. Consider the problem

$$\min \int_0^{t_1} f^0(\phi(t), u(t), t)dt \tag{11.5.1}$$

subject to

$$\frac{d}{dt}\phi^i(t) = f^i(t, \phi(t)) + \int_0^t g^i(t, s, \phi(s), u(s))ds, \qquad 1 \le i \le n \tag{11.5.2}$$

$$u(t) \in \Omega, \qquad 0 \le t \le t_1 \tag{11.5.3}$$

$$T(\phi(0), \phi(t_1)) = 0 \tag{11.5.4}$$

$$G(t, \phi(t)) \le 0, \qquad 0 \le t \le t_1 \tag{11.5.5}$$

The assumptions on T and G in (11.5.4) and (11.5.5) remain the same as in Section 11.2 and Ω is a fixed compact set in $\mathbb{R}^m$.

Assumption 11.5.1. For each compact set $\widetilde{\mathcal{X}} \subset \mathcal{X} \times \mathcal{U}$, we have

$$|f^0(x, u, t)| + |f^0_x(x, u, t)| \le \Lambda(t), \qquad (x, u, t) \in \widetilde{\mathcal{X}} \times [0, t_1].$$

$$|g^i(t, s, x, u)| + |g^i_x(t, s, x, u)| \le \Lambda(t),$$
$$(t, s, x, u) \in [0, t_1] \times [0, t_1] \times \widetilde{\mathcal{X}}, \quad 1 \le i \le n.$$

We also assume that f^0 and g^i, $1 \le i \le n$ are continuous in the u variable.

Assumption 11.5.2. In (11.5.2), $f^i : \mathcal{X} \times [0, t_1] \to \mathbb{R}$ is such that $f^i(\cdot, x)$ is measurable, $f^i(t, \cdot)$ is continuously differentiable satisfying

$$|f^i(t, x)| + |f^i_x(t, x)| \le \Lambda(t), \qquad \Lambda \in L^2(0, t_1).$$

Let $f_x = (f_x^1, \dots, f_x^n)$ where $f_x^i = (f_1^i, \dots, f_n^i)$. Similarly, let $g_x = (g_x^1, \dots, g_x^n)$. If $v = (v_1, \dots, v_n)$ by $v \cdot g_x$ we mean $v_1 g_x^1 + \cdots + v_n g_x^n$ where $v \cdot g_x^i$ is inner-product. Thus, $v_1 g_x^1 + \cdots + v_n g_x^n$ is an n-dimensional row vector.

The following theorem is a special case of Theorem 11.4.4.

Theorem 11.5.3. *Suppose Assumptions 11.4.1, 11.5.1, and 11.5.2 hold. At any relaxed pair (ϕ_0, ν_0) optimal for (11.5.1) to (11.5.5) the following conditions are met: There exist a function of bounded variation Φ on $I_0^{t_1}$, a bounded nonincreasing function λ, $\lambda \geq 0$, $\lambda(t_1^-) = 0$, scalars β, γ_0, λ^0 ($\lambda^0 \geq 0$) such that*

(i) $|\Phi(t_1^-)| + \lambda(0^+) + |\beta| + \lambda^0 = 1,$

(ii) $\Phi(t) + \int_t^{t_1} \lambda^0 f_x^0(\phi_0(s), \nu_{0s}, s)ds - \int_t^{t_1} \Phi(s) f_x(s, \phi_0(s))ds$

$+ \int_t^{t_1} \lambda(s) \nabla G(s, \phi_0(s)) f_x(s, \phi_0(s))ds$

$- \int_t^{t_1} \int_s^{t_1} \Phi(\tau) g_x(\tau, s, \phi_0(s), \nu_{0s}) d\tau\, ds$

$+ \int_t^{t_1} \int_s^{t_1} \lambda(\tau) \nabla G(\tau, \phi_0(\tau)) g_x(\tau, s, \phi_0(s), \nu_{0s}) d\tau\, ds$

$+ \int_t^{t_1} \lambda(s)[dG_x(s, \phi_0(s))/ds]ds = \Phi(t_1^-).$

(iii) $\Phi(0) = \lambda(0^+) \nabla G(0, \phi_0(0)) + \beta \partial_1 T(\phi_0(0), \phi_0(1))$

(iv) $\Phi(t_1^-) = -\beta \partial_2 T(\phi_0(0), \phi_0(1))$

(v) $H(t, \phi_0(t), \nu_{0t}) \geq H(t, \phi_0(t), \nu_t)$ *a.e., where*

$$\begin{aligned} H(t, \phi_0(t), \sigma_t) = {} & [\Phi(t) - \lambda(t) \nabla_x G(t, \phi_0(t))] \cdot (f(t, \phi_0(t)) \\ & + \int_0^t g(t, s, \phi_0(s), \sigma_s)ds - \lambda^0 f^0(\phi_0(t), \sigma_t, t) \end{aligned}$$

Remark 11.5.4. An equivalent condition to *(v)* of Theorem 11.5.3 is

(v′) $\widetilde{H}(t, \phi_0(t), \nu_{0t}) \geq \widetilde{H}(t, \phi_0(t), \nu_t)$ a.e., where

$$\begin{aligned} \widetilde{H}(t, \phi_0(t), \sigma_t) = {} & (\Phi(t) - \lambda(t) \nabla_x G(t, \phi_0(t))) \cdot f(t, \phi_0(t)) \\ & + \int_t^{t_1} [\Phi(s) - \lambda(s) \nabla_x G(s, \phi_0(s))] \cdot g(s, t, \phi_0(t), \sigma_t)ds \\ & - \lambda^0 f^0(\phi_0(t), \sigma_t, t) \end{aligned}$$

Remark 11.5.5. Let us remind the reader that Remark 11.4.5 is still valid.

Remark 11.5.6. In (11.5.2) we can replace $f^i(t, \phi(t))$ by $f^i(\phi(t), u(t), t)$. We only need to make appropriate assumptions on the functions $f^i(\cdot, \cdot, \cdot)$ (see Remark 11.2.1).

11.6 The Bounded State Problem for Ordinary Differential Systems

Again we specialize the problem in Section 11.2. In this case we consider the problem

$$\min \int_0^{t_1} f^0(\phi(t), u(t), t)dt \tag{11.6.1}$$

subject to

$$\phi'(t) = f(\phi(t), u(t), t), \tag{11.6.2}$$

$$u(t) \in \Omega, \qquad 0 \le t \le t_1, \tag{11.6.3}$$

$$T(\phi(0), \phi(1)) = 0, \tag{11.6.4}$$

$$G(t, \phi(t)) \le 0, \qquad 0 \le t \le t_1 \tag{11.6.5}$$

Assumption 11.6.1. Again, Ω is a fixed compact subset of $\mathbb{R}^m$, and the assumptions on T and G remain the same as in Section 11.2, and f in (11.6.2) has the same properties as f^0 (see Assumption 11.5.1).

Before we state the maximum principle in this situation, we make the following remark:

Remark 11.6.2. Repeating the procedure that was used to obtain (11.3.31) we obtain

$$\Phi(0) = \lambda(0^+)\nabla G(0, \phi_0(0)) + \beta\partial_1 T(\phi_0(0), \phi(t_1)) \tag{11.6.6}$$

Theorem 11.6.3. *Suppose Assumptions 11.4.1 and 11.6.1 hold. At any relaxed pair* (ϕ_0, ν_0) *optimal for* (*11.6.1*) *to* (*11.6.5*) *the following conditions are met: There exists an absolutely continuous function* Φ, *a bounded nonincreasing function* λ, $\lambda \ge 0$, $\lambda(t_1^-) = 0$, *scalars* β, γ_0, λ^0 ($\lambda^0 \ge 0$), *such that*

(i) $|\Phi(0)| + |\lambda(0^+)| + |\beta| + \lambda^0 = 1,$

(ii) $\Phi'(t) = \lambda^0 f^0_x(\phi_0(t), \nu_{0t}, t) - \Phi(t) \cdot f_x(\phi_0(t), \nu_{0t}, t)$

$\quad +\lambda(t)\nabla_x G(t, \phi_0(t)) \cdot f_x(\phi_0(t), \nu_{0t}, t) + \lambda(t)(d\nabla_x G(t, \phi_0(t))/dt)$

(iii) $\Phi(0) = \lambda(0^+)\nabla G(0, \phi_0(0)) + \beta\partial_1 T(\phi_0(0), \phi_0(t_1))$

(iv) $\Phi(t_1) = -\beta\partial_2 T(\phi_0(0), \phi_0(t_1))$

(v) $[\Phi(t) - \lambda(t)\nabla G(t, \phi_0(t))] \cdot f(\phi_0(t), \nu_{0t}, t) - \lambda^0 f^0(\phi_0(t), \nu_{0t}, t)$

$\quad \ge [\Phi(t) - \lambda(t)\nabla G(t, \phi_0(t))] \cdot f(\phi_0(t), \nu_t, t) - \lambda^0 f^0(\phi_0(t), \nu_t, t)$ *a.e. t.*

Remark 11.6.4. Remark 11.4.5 remains valid. Also, we can replace (i) by (i′) $|\Phi(t_1)| + \lambda(0^+) + |\beta| + \lambda^0 \ne 0$.

Exercise 11.6.5. Verify that $\lambda(0^+) + \lambda^0 + |\beta| \neq 0$.

The proof of Theorem 11.6.3 can be obtained from Theorem 11.4.4. The bounded state problem (11.6.1) to (11.6.5) is frequently encountered in applications. Thus, we will briefly present direct proof of Theorem 11.6.3 below. We follow the approach in [66].

Corresponding to the functional (11.3.6) we have

$$F_k(\phi, \nu) = \int_0^{t_1} f^0(\phi(t), \nu_t, t)dt + \|\phi' - \phi_0'\|^2 + |\phi(0) - \phi_0(0)|^2$$
$$+ \epsilon\|\nu - \nu_0\|_L + KT^2(\phi(0), \phi(t_1)) + K\|\phi'(t) - f(\phi(t), \nu_t, t\|^2$$

where (ϕ_0, ν_0) is an optimal pair, and we assume $F_K(\phi_0, \nu_0) = 0$. Lemma 11.3.3 remains valid. That is, for any $0 < \epsilon \leq \epsilon_1 \; \exists \; K(\epsilon)$ such that $F_{K(\epsilon)}(\phi, \nu) > 0$, $(\phi, \nu) \in B(\epsilon)$ if any of the inequalities is an equality:

$$|\phi(0) - \phi_0(0)| \leq \epsilon, \quad \|\phi' - \phi_0'\| \leq \epsilon, \quad \|\nu - \nu_0\|_L \leq \epsilon$$

where

$$B(\epsilon) = \{(\phi, \nu) \in \mathcal{B} \mid \|\phi' - \phi_0'\| \leq \epsilon, \quad |\phi(0) - \phi_0(0)| \leq \epsilon, \quad \|\nu - \nu_0\|_L \leq \epsilon\}$$

and

$$\mathcal{B} = \{(\phi, \nu) \mid \phi \in L_2(0, t_1)^n, \phi \text{ absolutely continuous,}$$
$$G(t, \phi(t)) \leq 0, 0 \leq t \leq t_1, \nu \text{ a relaxed control.}\}$$

Lemma 11.3.4 also remains valid in this case.

For $0 < \epsilon \leq \epsilon_1$, let

$$V(\epsilon) = \{\phi \mid \phi \text{ absolutely continuous, } \|\phi' - \phi_0'\| \leq \epsilon, \; |\phi(0) - \phi_0(0)| \leq \epsilon\}.$$

As in (11.3.8) we consider the functional $\phi \mapsto H^j_{K(\epsilon)}(\phi)$ on $V(\epsilon)$, where

$$H^j_{K(\epsilon)}(\phi) = j\int_0^{t_1} \omega(G(t, \phi(t))dt + F_{K(\epsilon)}(\phi, \nu^\epsilon)$$

Lemma 11.3.5 is also valid in this case. That is, there exists a subsequence $\{j_k\}$ such that

$$\|\phi'_{j_k} - \phi_0'\| < \epsilon, \quad |\phi_{j_k}(0) - \phi_0(0)| < \epsilon.$$

Remark 11.6.6. The proof of Lemma 11.3.5 also gives an absolutely continuous function $\widetilde{\phi}_\epsilon$ such that

$$\|\widetilde{\phi}'_\epsilon - \phi_0'\| < \epsilon, \quad |\widetilde{\phi}_\epsilon(0) - \phi_0(0)| < \epsilon,$$

$$G(t, \widetilde{\phi}_\epsilon(t)) \leq 0, \quad 0 \leq t \leq t_1,$$

$$F_{K(\epsilon)}(\widetilde{\phi}_\epsilon, \nu^\epsilon) = F_{K(\epsilon)}(\phi_\epsilon, \nu^\epsilon).$$

That is, $(\widetilde{\phi}_\epsilon, \nu^\epsilon)$ minimizes $F_{K(\epsilon)}(\phi, \nu)$ over $B(\epsilon)$.

Let ζ be an absolutely continuous scalar function on $[0, t_1]$ such that $\zeta \geq 0$ on $[0, t_1]$, $\zeta(0) = \zeta(t_1) = 0$. Let us set

$$\xi_\epsilon(t) = \frac{-\nabla G(t, \widetilde{\phi}_\epsilon(t))}{|\nabla G(t, \widetilde{\phi}_\epsilon(t))|^2}.$$

We can easily verify that $\exists\, \theta_0 > 0$ such that $0 < \theta \leq \theta_0$ implies

$$G(t, \widetilde{\phi}_\epsilon(t) + \theta\zeta(t)\xi_\epsilon(t)) \leq 0, \quad 0 \leq t \leq t_1$$

$$\|[\widetilde{\phi}_\epsilon(t) + \theta\zeta(t)\xi_\epsilon(t)]' - \phi_0'(t)\| < \epsilon$$

Then, using Remark 11.6.6 we have

$$\frac{dF_{K(\epsilon)}(\widetilde{\phi}_\epsilon + \theta\zeta\xi_\epsilon, \nu^\epsilon)}{d\theta}\Big|_{\theta=\theta^+} \geq 0$$

Now using Lemma 11.3.7 we obtain a nonincreasing function $\lambda(\epsilon; t)$ defined by

$$\begin{aligned} \lambda(\epsilon; t) = \psi(\epsilon; t) \cdot \xi_\epsilon(t) - \int_0^t \big\{ & f_1^0(\widetilde{\phi}_\epsilon(s), \nu_s^\epsilon, s) \cdot \xi_\epsilon(s) + \psi(\epsilon; s) \cdot \xi_\epsilon'(s) \\ & - [\psi(\epsilon; s) - 2(\widetilde{\phi}_\epsilon(s) - \phi_0(s))] \cdot f_1(\widetilde{\phi}_\epsilon(s), \nu_s^\epsilon, s) \cdot \xi_\epsilon(s) \big\} ds \end{aligned} \tag{11.6.7}$$

where

$$\psi_\epsilon(\epsilon; t) = 2(\widetilde{\phi}_\epsilon'(t) - \phi_0'(t)) + 2K(\epsilon)(\widetilde{\phi}_\epsilon'(t) - f(\widetilde{\phi}_\epsilon(t), \nu_t^\epsilon, t)) \tag{11.6.8}$$

In what follows we sometimes write $h(\epsilon; t)$ for $h(\widetilde{\phi}_\epsilon(t), \nu_t^\epsilon, t)$.

When we write out $dF_{K(\epsilon)}(\widetilde{\phi}_\epsilon + \theta\eta, \nu^\epsilon)/d\theta$ we get

$$\begin{aligned} J(\widetilde{\phi}_\epsilon, \eta) \equiv & \int_0^{t_1} f_1^0(\epsilon; t) \cdot \eta(t) dt + \int_0^{t_1} \psi(\epsilon; t) \cdot \eta'(t) dt \\ & - \int_0^{t_1} (\psi(\epsilon; t) - 2(\widetilde{\phi}_\epsilon' - \phi_0')) \cdot f_1(\epsilon; t) \cdot \eta(t) dt \end{aligned} \tag{11.6.9}$$

Integration by parts shows that

$$J(\widetilde{\phi}_\epsilon, \zeta\xi_\epsilon) = \int_0^{t_1} \lambda(\epsilon; t)\zeta'(t) dt \tag{11.6.10}$$

Let $h(t)$ be an absolutely continuous vector function such that $h \in L_2([0, t_1])^n$. Then, we have

$$\frac{dH^{j_k}_{K(\epsilon)}(\phi_{j_k} + \theta h)}{d\theta}\Big|_{\theta=0} = 0$$

If we take $h(t) = \nabla G(t, \widetilde{\phi}_\epsilon(t))/|\nabla G(t, \widetilde{\phi}_\epsilon(t))|^2$ we see that $j_k \int_0^{t_1} \omega'(G(t,$

$\phi_{j_k}(t))dt$ remain bounded as $k \to \infty$. If we next replace h by $\zeta(t) + [\nabla G(t, \widetilde{\phi}_\epsilon(t)) \cdot \zeta]\xi_\epsilon$, $\zeta(0) = \zeta(t_1) = 0$ we obtain

$$J(\widetilde{\phi}_\epsilon, \zeta) + J(\widetilde{\phi}_\epsilon, [\nabla G(\cdot, \widetilde{\phi}_\epsilon(\cdot)) \cdot \zeta]\zeta_\epsilon) = 0$$

Now, using (11.6.9) and (11.6.10) we obtain

$$\begin{aligned}
&\psi(\epsilon; t) + \lambda(\epsilon; t)\nabla G(t, \widetilde{\phi}_\epsilon(t)) \qquad (11.6.11)\\
&\quad = \int_0^t \Big\{ -[\psi(\epsilon; s) + \lambda(\epsilon; s)\nabla G(s, \widetilde{\phi}_\epsilon(s))] \cdot f_1(\epsilon; s) + f_1^0(\epsilon; s)\\
&\qquad + \lambda(\epsilon; s)\nabla G(s, \widetilde{\phi}_\epsilon(s)) \cdot f_1(\epsilon; s)\\
&\qquad + \lambda(\epsilon; s)\left[\frac{d\nabla G(s, \widetilde{\phi}_\epsilon(s))}{ds}\right] + 2(\widetilde{\phi}_\epsilon'(s) - \phi_0'(s)) \cdot f_1(\epsilon; s)\Big\} ds + c_\epsilon
\end{aligned}$$

We next obtain an ϵ-minimum principle. From Remark 11.6.6 we know that $F_{K(\epsilon)}(\widetilde{\phi}_\epsilon, \nu^\epsilon)$ minimizes $F_{K(\epsilon)}(\widetilde{\phi}, \nu)$ over $B(\epsilon)$. For $0 < \epsilon < \epsilon' < \epsilon$ and $0 \le \theta \le 1$, let

$$\nu(\theta) = \nu^\epsilon + \theta(\nu - \nu^\epsilon)$$

Since $\|\nu^\epsilon - \nu_0\| < \epsilon \; \exists \, \theta_0 > 0$ such that $0 \le \theta \le \theta_0$ implies that $\|\nu(\theta) - \nu_0\| < \epsilon$. Thus,

$$\left.\frac{dF_{K(\epsilon)}(\widetilde{\phi}_\epsilon, \nu(\theta))}{d\theta}\right|_{\theta = 0^+} \ge 0 \qquad (11.6.12)$$

Let

$$\rho_\epsilon(\theta) = \|\nu(\theta) - \nu_0\|_L$$

Thus, from (11.6.12) it follows that

$$\begin{aligned}
&-\int_0^{t_1} \psi(\epsilon; t) \cdot f(\widetilde{\phi}_\epsilon(t), \nu_t, t)dt + \int_0^{t_1} f^0(\widetilde{\phi}_\epsilon(t), \nu_t, t)dt\\
&+ \int_0^{t_1} 2(\widetilde{\phi}_\epsilon' - \phi_0') \cdot f(\widetilde{\phi}_\epsilon(t), \nu_t, t)dt\\
&\quad \ge -\int_0^{t_1} \psi(\epsilon; t) \cdot f(\widetilde{\phi}_\epsilon(t), \nu_t^\epsilon, t)dt + \int_0^{t_1} f^0(\widetilde{\phi}_\epsilon(t), \nu_t^\epsilon, t)dt\\
&+ \int_0^{t_1} 2(\widetilde{\phi}_\epsilon' - \phi_0') \cdot f(\widetilde{\phi}_\epsilon(t), \nu_t^\epsilon, t)dt - \epsilon\rho_\epsilon'(0^+) \qquad (11.6.13)
\end{aligned}$$

where $\psi(\epsilon; \cdot)$ is as in (11.6.8).

Finally we deal with end conditions. We enforce Assumption 11.4.1 here, too. We remark that Lemma 11.3.9 continues to hold here, too. According to Assumption 11.4.1 $\exists \, \delta > 0$ such that for $t \in (0, \delta) \cup (t_1 - \delta, t_1)$ we have $G(t, \phi_0(t)) < 0$. Note that if ϵ, $0 < \epsilon \le \epsilon_1$ is sufficiently small $G(t, \widetilde{\phi}_\epsilon(t)) < 0$, $t \in (0, \delta) \cup (t_1 - \delta, t_1)$. According to Lemma 11.3.9,

$$\lambda(\epsilon; \cdot) = \lambda(\epsilon; 0^+) \quad \text{in } (0, \delta)$$

$$\lambda(\epsilon;\cdot) = \lambda(\epsilon;t_1^-) \quad \text{in } (t_1-\delta, t_1)$$

Let ζ be a smooth scalar function such that $\zeta \geq 0$,

$$\begin{aligned} \zeta(t) &= 1 \quad 0 \leq t \leq \delta/2 \\ \zeta(t) &= 0 \quad t \geq 3\delta/4 \end{aligned}$$

For absolutely continuous vector function η such that $\eta' \in L_2([0,t_1])^n$, we have

$$\left.\frac{dF_{K(\epsilon)}(\widetilde{\phi}_\epsilon + \theta\zeta\eta, \nu^\epsilon)}{d\theta}\right|_{\theta=0} = 0 \tag{11.6.14}$$

In (11.6.14), let $\eta(t) = (\eta_1(t), \ldots, \eta_n(t))$ such that $\eta_i(t) = e^{-Nt}$, $\eta_j(t) = 0$ if $j \neq i$ and then let $N \to \infty$. Repeat this for $i = 1, \ldots, n$. We obtain

$$-\psi(\epsilon;0) + \widetilde{\phi}_\epsilon(0) - \phi_0(0) + 2K(\epsilon)\partial_1 T(\widetilde{\phi}_\epsilon(0), \widetilde{\phi}_\epsilon(t_1)) = 0 \tag{11.6.15}$$

Similarly we obtain

$$\psi(\epsilon;t_1) + 2K(\epsilon)\partial_2 T(\widetilde{\phi}_\epsilon(0), \widetilde{\phi}_\epsilon(t_1)) = 0 \tag{11.6.16}$$

Writing $\Phi(\epsilon;t)$ for $\psi(\epsilon;t)+\lambda(\epsilon;t)\nabla G(t,\widetilde{\phi}_\epsilon(t))$ where we have replaced $\lambda(\epsilon;\cdot)$ by $\lambda(\epsilon;\cdot)-\lambda(\epsilon;t_1^-)$, we can rewrite (11.6.11), (11.6.13), (11.6.15), and (11.6.16). Then, we can proceed with limiting operations.

Let

$$M(\epsilon) = 1 + |\Phi(\epsilon;0)| + \lambda(\epsilon;0^+) + 2K(\epsilon)|T\widetilde{\phi}_\epsilon(0), \widetilde{\phi}_\epsilon(t_1))|$$

$$\widetilde{\Phi}(\epsilon;\cdot) = \frac{\Phi(\epsilon;\cdot)}{M(\epsilon)}$$

$$\widetilde{\lambda}(\epsilon;\cdot) = \frac{\lambda(\epsilon;\cdot)}{M(\epsilon)}$$

$$\widetilde{\beta}_\epsilon = 2K(\epsilon)|T(\widetilde{\phi}_\epsilon(0), \widetilde{\phi}_\epsilon(t_1))|/M(\epsilon)$$

We take appropriate subsequences in $\{\widetilde{\Phi}_\epsilon\}_{0<\epsilon\leq\epsilon'}$, $\{\widetilde{\lambda}(\epsilon;0)\}_{0<\epsilon\leq\epsilon'}$, $\{\widetilde{\beta}_\epsilon\}_{0<\epsilon\leq\epsilon'}$ and verify that each of the assertions in Theorem 11.6.3 is true.

11.7 Further Discussion of the Bounded State Problem

This section should be considered an extension of Section 11.6. The purpose is to attempt to do away with Assumption 11.4.1.

Consider the problem

$$\min \int_0^{t_1} f^0(\phi(t), u(t), t)dt \tag{11.7.1}$$

subject to

$$\phi'(t) = f(\phi(t), u(t), t), \tag{11.7.2}$$

$$T(\phi(0)) = 0, \qquad \phi(t_1) = A \tag{11.7.3}$$

$$G(t, \phi(t)) = 0, \tag{11.7.4}$$

$$u(t) \in \Omega, \tag{11.7.5}$$

which is the same problem as (11.6.1) to (11.6.5) except for two changes. First, the state here is completely on the boundary, and the end conditions here are separated, and the state at t_1 is fixed. We could have fixed the state at $t = 0$ and require that $T(\phi(t_1)) = 0$.

The following theorem is useful here in obtaining necessary conditions.

Theorem 11.7.1 (Helley's Convergence Theorem). *Let $\{g_n\}$ be a sequence of functions of bounded variations on $[a, b]$ converging to g everywhere on $[a, b]$. Suppose $V(g_n) \leq C\ \forall\ n$. Then, g is also of bounded variation and*

$$\int_\Lambda f(x) dg_n \to \int_\Lambda f(x) dg$$

for any continuous function f.

Let (ϕ_0, ν_0) be optimal for (11.7.1) to (11.7.5). Define a functional

$$\begin{aligned} F_K(\phi, \nu) = \int_0^{t_1} & f^0(\phi(t), \nu_t, t) dt + \|\phi' - \phi_0'\|^2 + |\phi(0) - \phi_0(0)|^2 \\ & + \epsilon\|\nu - \nu_0\|_L + K(\epsilon) \left[T^2(\phi(0)) + \int_0^{t_1} G^2(t, \phi(t)) dt \right] \\ & + K\|\phi'(t) - f(\phi(t), \nu_t, t)\|^2. \end{aligned}$$

We proceed as in Section 11.3 or Section 11.6 and immediately obtain the following:

Theorem 11.7.2. *Let (ϕ_0, ν_0) be a relaxed optimal pair for (11.7.1) to (11.7.5). Then, there exist scalars $\lambda^0 \geq 0$, β, an absolutely continuous function ψ, a function of bounded variation σ such that*

(i) $\lambda^0 + |\psi(0)| + |\beta| + |d\sigma| = 1$, *$|d\sigma|$ = total variation of the measure $d\sigma$.*

(ii) $\psi(t) = \lambda^0 \int_0^t f_1^0(\phi_0(s), \nu_{0s}, s) ds - \int_0^t \psi(s) \cdot f_1(\phi_0(s), \nu_{0s}, s) ds$

$+ \int_0^t \nabla G(s, \phi_0(s)) d\sigma(s) + \beta \nabla T(\phi_0(0))$

(iii) $-\lambda^0 f^0(\phi_0(t), \nu_{0t}, t) + \psi(t) \cdot f(\phi_0(t), \nu_{0t}, t) \geq -\lambda^0 f^0(\phi_0(t), \nu_t, t) + \psi(t) \cdot f(\phi_0(t), \nu_t, t)$ *a.e. t.*

Now let us consider the problem

$$\min \int_0^{t_1} f^0(\phi(t), u(t), t)dt \tag{11.7.6}$$

subject to

$$\phi'(t) = f(\phi(t), u(t), t), \tag{11.7.7}$$

$$T(\phi(0)) = 0, \tag{11.7.8}$$

$$T(\phi(t_1)) = 0, \tag{11.7.9}$$

$$G(t, \phi(t)) \leq 0, \tag{11.7.10}$$

$$u(t) \in \Omega. \tag{11.7.11}$$

Now, suppose (ϕ_0, ν_0) is a relaxed optimal pair for (11.7.6) to (11.7.11). Suppose $G(t, \phi_0(t)) = 0$ for $t \in [0, \tau_1] \cup [\tau_2, t_1]$, $0 < \tau_1 < \tau_2 < t_1$. Suppose also $\exists\ \delta > 0$ such $\tau_1 + \delta < \tau_2 - \delta$ and $G(t, \phi_0(t)) < 0$ on $(\tau_1, \tau_1 + \delta) \cup (\tau_2 - \delta, \tau_2)$. On $[0, \tau_1]$ we consider the problem

$$\min \int_0^{\tau} f^0(\phi(t), u(t), t)dt \tag{11.7.12}$$

subject to

$$\phi'(t) = f(\phi(t), u(t), t), \tag{11.7.13}$$

$$T(\phi(0)) = 0, \tag{11.7.14}$$

$$\phi(\tau_1) = \phi_0(\tau_1), \tag{11.7.15}$$

$$G(t, \phi(t)) = 0, \tag{11.7.16}$$

$$u(t) \in \Omega,$$

and apply Theorem 11.7.1. Similarly on $[\tau_2, t_1]$. On $[\tau_1, \tau_2]$ we consider the problem

$$\min \int_{\tau_1}^{\tau_2} f^0(\phi(t), u(t), t)dt \tag{11.7.17}$$

subject to

$$\phi'(t) = f(\phi(t), u(t), t), \tag{11.7.18}$$

$$\phi(\tau_1) = \phi_0(\tau_1), \tag{11.7.19}$$

$$\phi(\tau_2) = \phi_0(\tau_2), \tag{11.7.20}$$

$$G(t, \phi(t)) \leq 0, \tag{11.7.21}$$

$$u(t) \in \Omega, \tag{11.7.22}$$

and apply Theorem 11.6.3.

11.8 Sufficiency Conditions

Consider the following problem

$$\min \int_0^{t_1} f^0(\phi(t), u(t), t)dt \tag{11.8.1}$$

subject to

$$\phi'(t) = f(\phi(t), u(t), t) \tag{11.8.2}$$

$$G(\phi(t)) \leq 0 \tag{11.8.3}$$

$$T_1(\phi(0)) = T_2(\phi(t_1)) = 0 \tag{11.8.4}$$

$$u(t) \in \Omega \tag{11.8.5}$$

This is the same problem as (11.6.1) to (11.6.5) except that (11.6.4) is replaced by (11.8.4), and (11.6.5) by (11.8.3). We also make assumptions additional to those stated in Theorem 11.6.3.

Assumption 11.8.1. Let the assumptions stated in Theorem 11.6.3 remain in force. In addition assume that f^0 and f are continuously differentiable in u. We also assume that G, T_1, and T_2 in (11.8.3) and (11.8.4) are twice continuously differentiable.

Using Theorem 11.6.3 and Assumption 11.8.1 we state the following:

Theorem 11.8.2. *Let Assumption 11.8.1 be in force. Suppose* (ϕ_0, ν_0) *is a relaxed optimal pair for* (*11.8.1*) *to* (*11.8.5*). *Then, there exist an adjoint variable* Φ*, a nonincreasing nonnegative function* λ*, multipliers* $\lambda^0 \geq 0$*,* β*,* $\lambda(t_1) = 0$ *such that*

(i) $\Phi'(t) = \lambda^0 f_x^0(\phi_0(t), \nu_{0t}, t) - \Phi(t) \cdot f_x(\phi_0(t), \nu_{0t}, t)$

$+\lambda(t)\nabla_x G(\phi_0(t)) \cdot f_x(\phi_0(t), \nu_{0t}, t) + \lambda(t) d\nabla_x G(\phi_0(t))/dt,$

λ *is constant where* $G(\phi_0(t)) < 0$.

(ii) $\Phi(0) = \lambda(0^+)\nabla G(\phi_0(0)) + \beta_1 \nabla T_1(\phi_0(0)),$

$\Phi(t_1) = -\beta_1 \nabla T_2(\phi_0(t_1)).$

(iii) $[\Phi(t) - \lambda(t)\nabla G(\phi_0(t))] \cdot f(\phi_0(t), \nu_{0t}, t) - \lambda^0 f^0(\phi_0(t), \nu_{0t}, t)$

$\geq [\Phi(t) - \lambda(t)\nabla G(\phi_0(t))] \cdot f(\phi_0(t), \nu_t, t) - \lambda^0 f^0(\phi_0(t), \nu_t, t)$ *a.e.*

We may write

$$\nu_{0t} = \sum_{i=1}^{n+1} \Pi_i^0(t)\delta_{u_i^0(t)}, \qquad \Pi_i^0 \geq 0, \qquad \sum \Pi_i^0 = 1,$$

and thus,

$$F^0(\phi_0(t), \nu_{0t}, t) = \sum \Pi_i^0(t) f^0(\phi_0(t), u_i^0(t), t).$$

Similarly,

$$F(\phi_0(t), \nu_{0t}, t) = \sum \Pi_i^0(t) f(\phi_0(t), u_i^0(t), t).$$

Then, we write

$$u^0(t) \equiv (\Pi_1^0(t), \ldots, \Pi_{n+1}^0(t), u_1^0(t), \ldots, u_{n+1}^0(t)),$$

$$\Pi_i^0 \geq 0, \qquad \sum \Pi_i^0 = 1, \qquad u_i^0 \in \Omega,$$

Remark 11.8.3. In general, in this section, when we write $u(t)$, we think of it in the form

$$u(t) \equiv (\Pi_1(t), \ldots, \Pi_{n+1}(t), u_1(t), \ldots, u_{n+1}(t)),$$

$$\Pi_i \geq 0, \qquad \sum \Pi_i = 1, \qquad u_i \in \Omega.$$

Then,

$$F^0(\phi(t), u(t), t) \equiv \sum \Pi_i(t) f^0(\phi(t), u_i(t), t),$$

$$F(\phi(t), u(t), t) \equiv \sum \Pi_i(t) f(\phi(t), u_i(t), t).$$

Now, in (i) above we assume $\lambda^0 = 1$, and let

$$\begin{aligned}\widetilde{\mathcal{H}}(\phi, u, t) = F^0(\phi(t), u(t), t) &- (\Phi(t) - \lambda(t)\nabla G(\phi(t))) \cdot F(\phi(t), u(t), t) \\ &- \dot{\Phi} \cdot \phi + \lambda(0^+)G(\phi(0))/t_1 \\ &- (\Phi(0) \cdot \phi(0) - \Phi(t_1) \cdot \phi(t_1))/t_1 \\ &+ [\beta_1 T(\phi(0)) + \beta_1 T(\phi(t_1))]/t_1\end{aligned}$$

Set

$$J(\phi, u) = \int_0^{t_1} \widetilde{\mathcal{H}}(\phi, u, t)dt.$$

Then, using (i) and (ii) of Theorem 11.8.2,

$$J(\phi_0, u^0) = \int_0^{t_1} F^0(\phi_0(t), u^0(t), t)dt,$$

and for (ϕ, u) admissible,

$$J(\phi, u) = \int_0^{t_1} F^0(\phi(t), u(t), t)dt - \int_0^{t_1} G(\phi(t))d\lambda(t)$$

$$\leq \int_0^{t_1} F^0(\phi(t), u(t), t)dt$$

Using (i) and (ii) of Theorem 11.8.2 we can verify that $\widetilde{\mathcal{H}}_\phi(\phi_0, u^0, t) = 0$. Thus,

$$J(\phi_0 + \delta\phi, u^0) - J(\phi_0, u^0) = O(\delta\phi^2).$$

Further,

$$\widetilde{\mathcal{H}}_\Phi(\phi_0, u^0, t) = \dot{\phi}_0(t) - F(\phi_0(t), u^0(t), t) = 0$$

and

$$\widetilde{\mathcal{H}}_u(\phi_0, u^0, t) = F_u^0(\phi_0(t), u^0(t), t) - (\Phi - \lambda\nabla G(t, \phi_0(t))) \cdot F_u(\phi_0(t), u^0(t), t).$$

Note that $\widetilde{\mathcal{H}}_u(\phi_0, u^0, t)$ is the same as the derivative of the Hamiltonian with respect to u. Using (iii) of Theorem 11.8.2, for u admissible, we should insist that

$$\widetilde{\mathcal{H}}_u(\phi_0, u_0, t)(u - u_0) \geq 0 \qquad \text{a.e. } t.$$

Thus, we may state the following:

Theorem 11.8.4. *Let Assumption 11.8.1 be in force. For problem (11.8.1)–(11.8.5) a sufficient condition for (ϕ_0, u_0) to be minimum is that*

$$\widetilde{\mathcal{H}}_u(\phi_0, u_0, t)(u - u_0) \geq 0 \qquad a.e.\ t.$$

and the matrix

$$\begin{pmatrix} \widetilde{\mathcal{H}}_{\phi\phi}(\phi_0, u^0, t) & \widetilde{\mathcal{H}}_{\phi u}(\phi_0, u^0, t) \\ \widetilde{\mathcal{H}}_{\phi u}(\phi_0, u^0, t) & \widetilde{\mathcal{H}}_{uu}(\phi_0, u^0, t) \end{pmatrix}$$

is positive definite a.e. t.

Next, we modify (11.8.1) to (11.8.5) as follows:

$$\min \int_0^{t_1} F^0(\phi(t), u(t), t)dt \tag{11.8.6}$$

subject to

$$\phi'(t) = F(\phi(t), u(t), t), \tag{11.8.7}$$

$$G(\phi(t)) \leq 0, \tag{11.8.8}$$

$$T_1(\phi(0)) = T_2(\phi(t_1)) = 0, \tag{11.8.9}$$

$$u(t) \in \Omega. \tag{11.8.10}$$

Let (ϕ_0, ν_0) be a relaxed optimal pair for (11.8.6) to (11.8.10). Again we apply Theorem 11.8.2. We write

$$u^0(t) = (\Pi_1^0(t), \ldots, \Pi_{n+1}^0(t), u_1^0(t), \ldots, u_{n+1}^0(t)),$$

$$\sum \Pi_i^0 = 1, \qquad \Pi_i \geq 0, \qquad u_i^0(t) \in \Omega$$

if

$$\nu_{0t} = \sum_{i=1}^{n+1} \Pi_i^0(t)\delta_{u_i^0(t)}$$

For (ϕ, u) admissible we have

$$\int_0^{t_1} F^0(\phi(t), u(t), t)dt \geq J(\phi, u).$$

Here, u is, in general, of the form

$$u(t) = \sum_{i=1}^{n+1} \Pi_i(t)\delta_{u_i(t)}, \qquad \Pi_i \geq 0, \quad \sum \Pi_i = 1, \quad u_i(t) \in \Omega.$$

Then, we have

$$\begin{aligned}
&\int_0^{t_1} F^0(\phi(t), u(t), t)dt - \int_0^{t_1} F^0(\phi_0(t), u_0(t), t)dt \geq J(\phi, u) - J(\phi_0, u_0) \\
&\quad = \int_0^{t_1} \left\{ \widetilde{\mathcal{H}}(\phi, u, t) - \widetilde{\mathcal{H}}(\phi_0, u_0, t) - \widetilde{\mathcal{H}}_\phi(\phi - \phi_0) \right\} dt \\
&\quad \geq \int_0^{t_1} \left\{ \widetilde{\mathcal{H}}(\phi, u, t) - \widetilde{\mathcal{H}}(\phi_0, u_0, t) - \widetilde{\mathcal{H}}_\phi(\phi - \phi_0) - \mathcal{H}_u(u - u^0) \right\} dt,
\end{aligned}$$

where we have used (i) and (ii) of Theorem 11.8.2 and the fact

$$\widetilde{\mathcal{H}}_u(\phi_0, u_0, t)(u(t) - u_0(t)) \geq 0 \qquad \text{a.e.,}$$

for u admissible.

11.9 Nonlinear Beam Problem

We now discuss the example presented in Section 1.8. We have to minimize the cost

$$I_\alpha(u) = \frac{1}{2}\int_0^1 u^2 dt + \alpha \int_0^1 \cos\theta \, dt$$

subject to

$$\frac{d}{dt}\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} \sin\theta \\ u \end{pmatrix},$$

$$x(0) = x(1) = 0$$

$$x^2(t) \leq (0.05)^2.$$

We use Theorem 11.6.3 to analyze this problem. Corollary 4.5.1 can also be used to simply look for ordinary controls. Let (ψ_1, ψ_2) be the adjoint variable. Then,

$$\begin{gathered}\dot{\psi}_1 = 2\lambda \sin\theta \\ \dot{\psi}_2 = -\alpha\lambda^0 \sin\theta - \psi_1 \cos\theta + 2\lambda x \cos\theta \\ \psi_2(0) = \psi_2(1) = 0.\end{gathered}$$

The Hamiltonian H is given by

$$H = -\frac{1}{2}\,\lambda^0 u^2 + \psi_2 u + (\psi_1 - 2\lambda x)\sin\theta - \alpha\lambda^0 \cos\theta.$$

On physical ground we eliminate $\lambda^0 = 0$ and thus we may take $\lambda^0 = 1$. Thus, we get

$$\begin{gathered}\dot{\psi}_1 = 2\lambda \sin\theta \\ \dot{\psi}_2 = -\alpha \sin\theta + (2\lambda x - \psi_1)\cos\theta \\ \psi_2(0) = \psi_2(1) = 0\end{gathered}$$

The Hamiltonian H is given by

$$H = -\frac{1}{2}\,u^2 + \psi_2 u + (\psi_1 - 2\lambda x)\sin\theta - \alpha \cos\theta$$

and

$$H(\psi, \lambda, x, \theta, u^0) \geq H(\psi, \lambda, x, \theta, u).$$

Thus,

$$\begin{gathered}-u + \psi_2 = 0 \\ \therefore\ u = \psi_2.\end{gathered}$$

Thus, we must solve the system of equations

$$\begin{aligned}\dot{x} &= \sin\theta \\ \dot{\theta} &= \psi_2 \\ \dot{\psi}_1 &= 2\lambda \sin\theta \\ \dot{\psi}_2 &= -\alpha \sin\theta + (2\lambda x - \psi_1)\cos\theta \\ x(0) = x(1) &= \psi_2(0) = \psi_2(1) = 0.\end{aligned} \tag{11.9.1}$$

Off the state constraint λ is constant. On the state constraint itself we can easily verify, using (11.9.1), that λ is constant. Thus,

$$\dot{\psi}_1 - 2\lambda \sin\theta = \dot{\psi}_1 - 2\lambda\dot{x} = (\psi_1 - 2\lambda x)^{\cdot} = 0$$

$$\therefore\ \psi_1 - 2\lambda x = \psi_1(0) \tag{11.9.2}$$

Since $x(0) = x(1) = 0$ it follows from (11.9.2) that

$$\psi_1(0) = \psi_1(1).$$

Thus, using (11.9.1) and (11.9.2) we have

$$\begin{aligned} \dot{\psi}_2 &= -\alpha \sin\theta - \psi_1(0)\cos\theta, \\ \dot{\theta} &= \psi_2. \end{aligned} \tag{11.9.3}$$

Also, using (11.9.1) and (11.9.3)

$$\begin{aligned} 0 = \int_0^1 \dot{\psi}_2 ds &= -\alpha \int_0^1 \sin\theta\, ds - \psi_1(0) \int_0^1 \cos\theta\, ds \\ &= -\alpha \int_0^1 \dot{x} ds - \psi_1(0) \int_0^1 \cos\theta\, ds \\ &= -\psi_1(0) \int_0^1 \cos\theta\, ds \\ &\therefore\ \psi_1(0) = 0. \end{aligned}$$

Thus, we rewrite (11.9.3) as

$$\begin{aligned} \dot{\theta} &= \psi_2 \\ \dot{\psi}_2 &= -\alpha \sin\theta \\ \psi_2(0) &= \psi_2(1) = 0 \end{aligned} \tag{11.9.4}$$

We have

$$x(t) = \int_0^t \sin\theta\, ds = -\frac{1}{\alpha} \int_0^t \ddot{\theta}\, ds = -\frac{1}{\alpha}\, [\dot{\theta}(t) - \dot{\theta}(0)],$$

and from the fact $\dot{\theta}(0) = \psi_2(0) = 0$, we have

$$x(t) = -\frac{1}{\alpha}\, \dot{\theta}(t). \tag{11.9.5}$$

Since we require $x^2(t) \leq 0.0025$ it follows that

$$|\dot{\theta}(t)| \leq \frac{\alpha}{20}.$$

Now, we go back to the cost

$$J = \frac{1}{2} \int_0^1 u^2 dt + \alpha \int_0^1 \cos\theta\, dt.$$

Since $\dot{\theta} = u$ and $u = \psi_2$, it follows that

$$J = \frac{1}{2} \int_0^1 \psi_2^2 dt + \alpha \int_0^1 \cos\theta\, dt.$$

Note that

$$\begin{aligned}
\dot{\psi}_2 &= -\alpha \sin\theta \\
\dot{\psi}_2\dot{\theta} &= -\alpha(\sin\theta)\dot{\theta} \\
\therefore\ \dot{\psi}_2\psi_2 &= -\alpha(\sin\theta)\dot{\theta}
\end{aligned}$$

$$\therefore\ \frac{d}{dt}\left(\frac{1}{2}\psi_2^2 - \alpha\cos\theta\right) = 0$$

$$\therefore\ \frac{1}{2}\psi_2^2 = \alpha\cos\theta - \alpha\cos\theta(0).$$

Thus, the cost becomes

$$J = 2\alpha\int_0^1 \cos\theta\,dt - \alpha\cos\theta(0)$$

where

$$\begin{aligned}
&\ddot{\theta} + \alpha\sin\theta = 0 \\
&\dot{\theta}(0) = \dot{\theta}(1) = 0, \qquad |\dot{\theta}| \le \frac{\alpha}{20}
\end{aligned} \tag{11.9.6}$$

Corresponding to the equation,

$$\ddot{\theta} + \alpha\sin\theta = 0.$$

We have the phase portrait shown in Fig. 11.1.

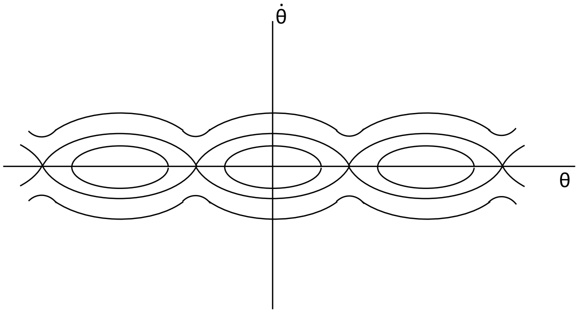

FIGURE 11.1

In (11.9.6) what we have is the familiar pendulum equation. For the undamped pendulum any periodic orbit must intersect the θ-axis of the phase portrait at two points, say $(-b, 0)$ and $(b, 0)$ and the period T is given by

$$T = \frac{4}{\sqrt{\alpha}}\int_0^{\pi/2} \frac{d\xi}{\sqrt{1 - \sin^2\left(\frac{b}{2}\right)\sin^2\xi}} \tag{11.9.7}$$

In the problem at hand, $\dot{\theta}(0) = \dot{\theta}(1) = 0$. Thus,

$$T = \frac{2}{k}, \qquad k = 1, 2, 3, \ldots \tag{11.9.8}$$

We verified previously that $|\dot{\theta}(t)| \leq \alpha/20$. Thus, it takes at least $20/\alpha$ units of time to traverse a closed orbit. Thus,

$$T > 20/\alpha \tag{11.9.9}$$

From (11.9.8) we see that (11.9.9) is impossible if $0 < \alpha \leq \pi^2$. From (11.9.6), if $0 < \alpha \leq \pi^2$, then $\theta \equiv 0$. Thus, in this case the control policy $u \equiv 0$ and the corresponding trajectory $x \equiv 0$. From (11.9.8) and (11.9.9) the same conclusion is reached if $\alpha \leq 10$. Thus, for a nontrivial control policy we should insist that $\alpha > 10$.

From (11.9.7) and (11.9.8) we have

$$\frac{2}{k} = \frac{4}{\sqrt{\alpha}} \int_0^{\pi/2} \frac{d\xi}{\sqrt{1 - \sin^2 \frac{b}{2} \, \sin^2 \xi}} \tag{11.9.10}$$

Since

$$\frac{4}{\sqrt{\alpha}} \int_0^{\pi/2} \frac{d\xi}{\sqrt{1 - \sin^2 \frac{b}{2} \, \sin^2 \xi}} \geq \frac{4}{\sqrt{\alpha}} \frac{\pi}{2} = \frac{2\pi}{\sqrt{\alpha}},$$

from (11.9.10) we have

$$\frac{2}{k} \geq \frac{2\pi}{\sqrt{\alpha}}$$

$$\alpha \geq k^2 \pi^2 \tag{11.9.11}$$

Thus, given α and $k = 0, 1, 2, \ldots$ such that $k^2\pi^2 \leq \alpha$ we use (11.9.10) to obtain a corresponding b. This procedure picks θ and hence a control policy $u = \dot{\theta}$.

Chapter 12

Hamilton-Jacobi Theory

12.1 Introduction

In Section 6.2 we used dynamic programming to derive the nonlinear partial differential equation (12.1.1) for the value function associated with an optimal control problem. This partial differential equation is called a Hamilton-Jacobi-Bellman (HJB) equation, also Bellman's equation. Typically, the value function W is not smooth, and (12.1.1) must be understood to hold in some weaker sense. In particular, under suitable assumptions W satisfies (12.1.1) in the Crandall-Lions viscosity solution sense (Section 12.5). Section 12.6 gives an alternate characterization (12.6.2) of the value function using lower Dini derivatives. This provides a control theoretic proof of uniqueness of viscosity solutions to the HJB equation with given boundary conditions.

From Section 6.2, we recall the optimal control problem formulation.

Problem 12.1.1. Minimize the functional

$$J(\phi, u) = g(t_1, \phi(t_1)) + \int_{\tau}^{t_1} f^0(t, \phi(t), u(t))dt$$

subject to the state equations

$$\frac{dx}{dt} = f(t, x, u(t)),$$

control constraints $u(t) \in \Omega(t)$, and end conditions

$$(t_0, \phi(t_0)) = (\tau, \xi) \qquad (t_1, \phi(t_1)) \in \mathcal{J}.$$

In Section 6.2 we assumed that at each point (τ, ξ) in a region $\mathcal{R}$ of (t, x)-space the above control problem has a unique solution. Under certain assumptions on the data of the problem and under the assumption that the value function W is $C^{(1)}$ we showed that W satisfies

$$W_\tau(\tau, \xi) = \max_{z \in \Omega(\tau)} [-f^0(\tau, \xi, z) - \langle W_\xi(\tau, \xi), f(\tau, \xi, z)\rangle]. \tag{12.1.1}$$

In this chapter we assume that $(\tau, \xi) \in \mathcal{R}_0$, where $\mathcal{R}_0$ is a region with properties described in part (vii) of Assumption 12.2.1. If we now denote a generic

point in $\mathcal{R}$ by (t,x) rather than by (τ,ξ), set

$$H(t,x,z,q^0,q) = q^0 f^0(t,x,z) + \langle q, f(t,x,z)\rangle$$

and set

$$\overline{H}(t,x,q) = \sup_{z\in\Omega(t)} H(t,x,z,-1,q),$$

then we may write (12.1.1) as

$$-W_t(t,x) + \overline{H}(t,x,-W_x(t,x)) = 0. \tag{12.1.2}$$

In this chapter we shall give conditions under which W is continuous and Lipschitz continuous. We shall then show that if W is Lipschitz continuous, then W satisfies (12.1.2) for a generalized notion of solution. We shall also consider the problem of determining an optimal synthesis and shall obtain the maximum principle in some cases.

12.2 Problem Formulation and Assumptions

We consider Problem 12.1.1 and assume the following.

Assumption 12.2.1. (i) The function $\widehat{f} = (f^0, f) = (f^0, f^1, \ldots, f^n)$ is continuous on $\mathcal{I}\times\mathbb{R}^n\times\mathcal{U}$, where $\mathcal{I}$ is a real compact interval $[0,T]$ and $\mathcal{U}$ is an interval in $\mathbb{R}^m$.

(ii) There exists a function β in $L_2[\mathcal{I}]$ such that $f^0(t,x,z) \geq \beta(t)$ for a.e. $t\in\mathcal{I}$, all $x\in\mathbb{R}^n$, and all $z\in\mathcal{U}$.

(iii) For each compact set $\mathcal{X}_0 \subset \mathbb{R}^n$ there exists a function K in $L_2[\mathcal{I}]$ such that for all x and x' in $\mathcal{X}_0$, all t in $\mathcal{I}$, and all z in $\mathcal{U}$

$$|\widehat{f}(t,x,z) - \widehat{f}(t,x',z)| \leq K(t)|x-x'|. \tag{12.2.1}$$

(iv) There exists a function M in $L_2[\mathcal{I}]$ such that for all x in $\mathbb{R}^n$, all t in $\mathcal{I}$, and all z in $\cup\{\Omega(t)\colon t\in\mathcal{I}\}$

$$|\widehat{f}(t,x,z)| \leq M(t)(|x|+1). \tag{12.2.2}$$

(v) The terminal set $\mathcal{J}$ is an r-dimensional manifold, $0\leq r\leq n$ of class $C^{(1)}$ in $\mathbb{R}\times\mathbb{R}^n$ with $\sup\{t_1\colon (t_1,x_1)\in\mathcal{J}\}\leq T$.

(vi) The function g is real valued and defined on an open set in $\mathbb{R}\times\mathbb{R}^n$ that contains $\mathcal{J}$.

(vii) There exists an open connected set $\mathcal{R}_0 \subset \mathcal{I} \times \mathbb{R}^n$ with the following properties.

(a) The terminal set $\mathcal{J}$ is a subset of the boundary of $\mathcal{R}_0$.

(b) At each point (t, x) in $\mathcal{R}_0$, let $\mathcal{A}(t, x)$ denote the set of admissible controls u for Problem 12.1.1 with initial point (t, x) such that *all* trajectories corresponding to u are admissible. Then $\mathcal{A}(t, x)$ is not empty.

(c) Problem 12.1.1 with initial point (t, x) in $\mathcal{R}_0$ has a solution (ϕ, u), not necessarily unique, such that $(r, \phi(r)) \in \mathcal{R}_0$ for $t \leq r < t_1$ and $(t_1, \phi(t_1)) \in \mathcal{J}$. In the rest of this chapter, when we say solution to Problem 12.1.1 we mean a solution with property (c).

Later we will specialize to consider terminal sets $\mathcal{J}$, which are relatively open subsets of a hyperplane $t_1 = T$ (Section 12.3), including the case when $\mathcal{J}$ is the entire hyperplane (Section 12.6).

Remark 12.2.2. In Chapter 4 and 5 existence theorems were proved that ensured the existence of a solution (ϕ, u). These theorems required the sets $Q^+(t, x)$ defined in Section 5.4 to be convex. To overcome this limitation, the notion of relaxed controls and the relaxed problem were introduced. As in earlier chapters, we denote a relaxed control by μ and a relaxed trajectory by ψ.

The relaxed problem corresponding to Problem 12.1.1 is the following:

Problem 12.2.1. Minimize the functional

$$J(\psi, \mu) = g(t_1, \psi(t_1)) + \int_\tau^{t_1} f^0(t, \psi(t), \mu_t) dt$$

subject to the state equation

$$\frac{d\psi}{dt} = f(t, \psi(t), \mu_t),$$

control constraint $\mu_t \in \Omega(t)$ and end conditions

$$(t_0, \psi(t_0)) = (\tau, \xi) \qquad (t_1, \psi(t_1)) \in \mathcal{J}.$$

Remark 12.2.3. Statements (i) to (vii-a) in Assumption 12.2.1 hold for the relaxed problem, and therefore can be taken as assumptions for the relaxed problem. Assumptions for the relaxed problem corresponding to (vii-b) and (vii-c) are obtained by replacing $\mathcal{A}(t, x)$ by $\mathcal{A}_r(t, x)$, by replacing u by μ, and replacing ϕ by ψ. *Assumptions (i) to (vii-c) for the relaxed problem will be called* **Assumption 12.2.1-r**.

Remark 12.2.4. In some of the existence theorems in Chapter 5 for problems with non-compact constraints, the mapping Q_r^+ is required to have the weak Cesari property. We showed that if f is of slower growth than f^0 and if the function identically one is of slower growth than f^0, then the weak Cesari property holds. If we assume that these growth conditions hold, then there exists a positive constant A such that $|f(t,x,z)| \leq A + |f^0(t,x,z)|$. Hence we need only require (12.2.2) to hold for f^0 for it to hold for $\widehat{f}$.

Remark 12.2.5. If all the constraint sets $\Omega(t)$ are compact and the mapping Ω is u.s.c.i, then by Lemma 3.3.11 for any compact set $\mathcal{X}_0 \subseteq \mathcal{X}$, the set

$$\Delta = \{(t,x,z)\colon t \in [0,T],\ x \in \mathcal{X}_0,\ z \in \Omega(t)\} \tag{12.2.3}$$

is compact. It then follows from the continuity of $\widehat{f}$ that (12.2.2) holds with M a constant, for all (t,x,z) in Δ.

Remark 12.2.6. In the HJB equation for the relaxed problem, $\overline{H}$ in (12.1.2) is replaced by $\overline{H}_r$ where

$$\overline{H}_r(t,x,q) = \sup_{\mu}[\langle -f^0(t,x,\cdot),\mu\rangle + \langle q\cdot f(t,x,\cdot),\mu\rangle] \tag{12.2.4}$$

with μ any probability measure on $\Omega(t)$. It is easy to show that $\overline{H} = \overline{H}_r$. Thus, the ordinary and relaxed control problems have the same HJB equation.

12.3 Continuity of the Value Function

We define the *value function* W on $\mathcal{R}_0$ by

$$W(\tau,\xi) = \min\{J(\psi,\mu)\colon (\psi,\mu) \in \mathcal{A}_r(\tau,\xi)\}.$$

We are justified in writing "min" by virtue of (vii-c) of Assumption 12.2.1-r.

We first establish Bellman's "Principle of Optimality."

Theorem 12.3.1 (Principle of Optimality). *Let the relaxed problem with initial point (τ,ξ) have a solution $(\overline{\psi},\overline{\mu})$ with terminal time $\overline{t}_1$. If (t,x) is a point on the trajectory $\overline{\psi}$, then $(\overline{\psi},\overline{\mu})$ restricted to $[t,\overline{t}_1]$ is optimal for the relaxed problem with initial point (t,x). If (ψ,μ) is a control trajectory pair with μ in $\mathcal{A}_r(\tau,\xi)$ and with terminal time t_1, then for any $\tau \leq t \leq t_1$*

$$W(\tau,\xi) \leq \int_\tau^t f^0(s,\psi(s),\mu_s)ds + W(t,\psi(t)). \tag{12.3.1}$$

Equality holds if and only if (ψ,μ) is optimal.

Proof. We have that

$$W(\tau, \xi) = \int_{\tau}^{t} f^0(s, \overline{\psi}(s), \overline{\mu}_s)ds + \int_{t}^{\overline{t}_1} f^0(s, \overline{\psi}(s), \overline{\mu}_s)ds + g(\overline{t}_1, \overline{\psi}(\overline{t}_1)). \quad (12.3.2)$$

If $(\overline{\psi}, \overline{\mu})$ were not optimal for the problem with initial point $(t, \overline{\psi}(t))$, then

$$W(t, \overline{\psi}(t) < \int_{t}^{\overline{t}_1} f^0(s, \overline{\psi}(s), \overline{\mu}_s)ds + g(\overline{t}_1, \overline{\psi}(\overline{t}_1)). \quad (12.3.3)$$

Let (ψ_1, μ_1) be optimal for the problem with initial point $(t, \overline{\psi}(t))$ and let $(t_1, \psi_1(t_1))$ be the terminal point of (ψ_1, μ_1). Define a control μ on $[\tau, t_1]$ by

$$\mu_s = \overline{\mu}_s \text{ on } [\tau, t] \qquad \mu_s = \mu_{1s} \text{ on } [t, t_1].$$

Let ψ be the trajectory corresponding to μ. Then by virtue of (12.2.1) and the uniqueness theorem for solutions of ordinary differential equations, $\psi(s) = \overline{\psi}(s)$ on $[\tau, t]$. Then $\mu \in \mathcal{A}_r(\tau, \xi)$ and

$$J(\psi, \mu) = \int_{\tau}^{t} f^0(s, \overline{\psi}(s), \overline{\mu}_s)ds + W(t, \overline{\psi}(t)).$$

Combining this with (12.3.3) and (12.3.2) gives $J(\psi, \mu) < W(\tau, \xi)$, which contradicts the definition of W and establishes the first conclusion. Inequality (12.3.1) and the statement about equality in (12.3.1) are obvious. □

Theorem 12.3.2. *Let Assumption* 12.2.1-r *hold. Let g be continuous. Let $\mathcal{J}$ be a relatively open set in the hyperplane $t_1 = T$. Let the sets $\mathcal{A}_r(t, x)$ depend only on t; that is, $\mathcal{A}_r(t, x) = \mathcal{A}_r(t)$. Then the value function W is continuous on $\mathcal{R}_0$. If g is locally Lipschitz continuous, then W is locally Lipschitz continuous.*

Remark 12.3.3. The value function is continuous or Lipschitz continuous when the terminal set has a different structure from the one assumed here. See [5], Section 2.10 of [35], [94] and the references therein.

Proof. To simplify the proof we transform the problem to the Mayer form, as in Section 2.4. Assumption 12.2.1-r is valid for the Mayer form. The value $W(t, \widehat{x})$ of the Mayer problem is related to the value $W(t, x)$ of the Bolza problem by the relation $W(t, \widehat{x}) = W(t, x) + x^0$, where $\widehat{x} = (x^0, x)$. Hence if a continuity property holds for the value in Mayer form, it holds for the value in Bolza form and vice versa. We henceforth assume the problem to be in Mayer form. □

We shall need the following result.

Lemma 12.3.4. *Let $\mu \in \mathcal{A}_r(\tau)$ be an admissible relaxed control and let $\psi(\cdot, \tau, x)$ and $\psi(\cdot, \tau, x')$ be corresponding trajectories with initial conditions (τ, x) and (τ, x'). Then there exists a positive constant A such that for all $\tau \leq t \leq T$*

$$|\psi(t, \tau, x) - \psi(t, \tau, x')| \leq A|x - x'|. \quad (12.3.4)$$

Proof. To simplify notation we suppress the dependence on τ and write $\psi(\cdot, x)$ for $\psi(\cdot, \tau, x)$ and $\psi(\cdot, x')$ for $\psi(\cdot, \tau, x')$. From (12.2.1) and Gronwalls' Lemma we get that for $\tau \leq t \leq T$

$$\begin{aligned} |\psi(t,x) - \psi(t,x')| &\leq |x - x'| + \int_\tau^t |(f(s,\psi(s,x),\mu_s) - f(s,\psi(s,x'),\mu_s)|ds \\ &\leq |x - x'| + \int_\tau^t K(s)|\psi(s,x) - \psi(s,x')|ds \end{aligned}$$

Thus,

$$|\psi(t,x) - \psi(t,x')| \leq |x - x'| \exp \int_\tau^t K(s)ds \leq |x - x'|A,$$

where $A = \exp \int_{\mathcal{I}} K(s)ds$. □

We return to the proof of the theorem. Let (τ, ξ) be a point in $\mathcal{R}_0$ and let $(\overline{\psi}, \overline{\mu}) = (\overline{\psi}(\cdot, \tau, \xi, \overline{\mu}), \overline{\mu}(\tau, \xi))$ be an optimal relaxed pair for Problem 12.2.1. Since the problem is in Mayer form

$$W(\tau, \xi) = g(T, \overline{\psi}(T, \tau, \xi)) = g(T, \overline{\psi}(T)). \tag{12.3.5}$$

Now let (τ', ξ') be another point in $\mathcal{R}_0$ with $\tau' > \tau$. If $\tau' < \tau$, interchange the roles of (τ, ξ) and (τ', ξ'). Then

$$\begin{aligned} |W(\tau,\xi) - W(\tau',\xi')| \leq\ & |W(\tau,\xi) - W(\tau', \overline{\psi}(\tau',\tau,\xi))| \\ &+ |W(\tau', \overline{\psi}(\tau',\tau,\xi)) - W(\tau',\xi')|. \end{aligned}$$

It follows from (12.3.5) and the Principle of Optimality (Theorem 12.3.1) that the first term on the right is zero. Hence

$$|W(\tau,\xi) - W(\tau',\xi')| \leq |W(\tau', \overline{\psi}(\tau',\tau,\xi)) - W(\tau',\xi')|. \tag{12.3.6}$$

We now estimate the term on the right in (12.3.6). To simplify notation, let $x' = \overline{\psi}(\tau', \tau, \xi)$. Let $\psi = \psi(\cdot, \tau', \xi', \overline{\mu})$ denote the trajectory on the interval $[\tau', T]$ with initial point (τ', ξ') corresponding to the optimal $\overline{\mu} = \overline{\mu}(\cdot, \tau', x')$ for the problem with initial point (τ', x'). Then since $\overline{\mu} \in \mathcal{A}_r(\tau')$, the pair $(\psi, \overline{\mu})$ is admissible for the problem with initial point (τ', ξ'). Then

$$\begin{aligned} W(\tau',\xi') - W(\tau',x') &\leq J(\psi,\overline{\mu}) - W(\tau',x') \\ &= g(T,\psi(T)) - g(T,\overline{\psi}(T)), \end{aligned}$$

the equality following from (12.3.5) and Theorem 12.3.1. Let $\varepsilon > 0$ be given. It then follows from (12.3.4) with $x = \xi'$ and the continuity of g that there exists a $\delta > 0$ such that if $|\xi' - x'| < \delta$, then

$$|g(T,\psi(T)) - g(T,\overline{\psi}(T)| < \varepsilon,$$

and so

$$W(\tau', \xi') - W(\tau', x') < \varepsilon. \tag{12.3.7}$$

If g is Lipschitz continuous with Lipschitz constant L, we get that

$$W(\tau', \xi') - W(\tau', x') \leq LA|\xi' - x'|. \tag{12.3.8}$$

A similar argument applied to $W(\tau', x') - W(\tau', \xi)$ gives (12.3.7) and (12.3.8) with the left side replaced by $W(\tau', x') - W(\tau', \xi')$. Hence we have shown that

$$\lim_{x' \to \xi'} W(\tau', x') = W(\tau', \xi'), \tag{12.3.9}$$

and that if g is Lipschitz continuous

$$|W(\tau', \xi') - W(\tau', x')| \leq LA|\xi' - x'|. \tag{12.3.10}$$

We conclude the proof by relating $x' - \xi'$ to $\xi - \xi'$. Recall that $x' = \overline{\psi}(\tau', \tau, \xi)$. Hence

$$|x' - \xi'| \leq |\xi - \xi'| + \int_{\tau}^{\tau'} |f(t, \overline{\psi}(t, \tau, \xi), \overline{\mu}_t)|dt.$$

It follows from (12.2.2) and Lemma 4.3.14 that for all ξ, ξ' in a compact set, $|\overline{\psi}(t)|$ is bounded by a positive constant B, which depends on the compact set. Since f is continuous there exists a positive constant C such that $|f(t, \overline{\psi}(t, \tau, \xi), \overline{\mu}_t)| \leq C$. Therefore,

$$|x' - \xi'| \leq |\xi - \xi'| + C|t - t'|. \tag{12.3.11}$$

From (12.3.11) we get that

$$x' \to \xi' \quad \text{as} \quad (\tau, \xi) \to (\tau', \xi'). \tag{12.3.12}$$

The continuity of W now follows from (12.3.9) and (12.3.12); the Lipschitz continuity follows from (12.3.10) and (12.3.11).

Corollary 12.3.5. *Let every point of $\mathcal{J}$ be the terminal point of an optimal trajectory with initial point in $\mathcal{R}_0$. For each such initial point, let the optimal trajectory be unique. For $(T, x_1) \in \mathcal{J}$ let*

$$W(T, x_1) = g(T, x_1) = g(T, \overline{\psi}(T, t', x')), \tag{12.3.13}$$

where $\overline{\psi}(\ , t', x')$ is the unique optimal trajectory with initial point (t', x') in $\mathcal{R}_0$ terminating at x_1. Then W is continuous on $\mathcal{R}_0 \cup \mathcal{J}$.

Proof. Let (t, x) in $\mathcal{R}_0$ tend to (T, x_1) in $\mathcal{J}$. Then to prove the corollary we must show that

$$W(t, x) \to W(T, x_1) = g(T, x_1) = g(T, \overline{\psi}(T)), \tag{12.3.14}$$

where $\overline{\psi}(\ ) = \overline{\psi}(\ , t', x')$ and $x_1 = \overline{\psi}(T)$. Let $\psi(\ , t, x)$ denote the optimal trajectory with initial point (t, x) that terminates at $\mathcal{J}$ and let μ be the associated optimal control. Then

$$W(t, x) = g(T, \psi(T, t, x)), \tag{12.3.15}$$

and

$$\psi(T, t, x) = x + \int_t^T f(s, \psi(s, t, x), \mu_s)ds.$$

From (12.2.2) we get that the absolute value of the integral does not exceed

$$\int_t^T M(s)(|\psi(s, t, x)| + 1)ds. \tag{12.3.16}$$

Since we are considering points (t, x) approaching a point (T, x_1) in $\mathcal{J}$, we may suppose that all points (t, x) are in a compact set $\mathcal{K}$ containing (T, x_1). It then follows from Lemma 4.3.14 that there exists a positive constant B such that $|\psi(s, t, x)| \leq B$ for all $(t, x) \in \mathcal{K}$ and $t \leq s \leq T$. Therefore, the integral in (12.3.16) tends to zero as $t \to T$. By hypothesis $x \to x_1$. Hence

$$\psi(T, t, x) \to x_1 \quad \text{as} \quad (t, x) \to (T, x_1).$$

Since g is continuous we have shown that $g(T, \psi(T, t, x)) \to g(T, x_1)$ as $(t, x) \to (T, x_1)$. Hence, by (12.3.13) and (12.3.15) we have that $W(t, x) \to W(T, x_1)$, and the corollary is established. $\square$

Remark 12.3.6. The argument also shows that if $\mathcal{K}_T$ is a compact set contained in $\mathcal{J}$, then the convergence $W(t, x) \to W(T, x_1)$ is uniform on $\mathcal{K}_T$.

12.4 The Lower Dini Derivate Necessary Condition

In this section we shall develop a necessary condition that a Lipschitz continuous value function satisfies. This necessary condition involves the lower Dini directional derivate, which we now define.

Definition 12.4.1. Let L be a real valued function defined on an open set in $\mathbb{R} \times \mathbb{R}^n$. The *lower Dini derivate of* L *at the point* (t, x) *in the direction* $(1, h)$, where $h \in \mathbb{R}^n$, is denoted by $D^- L(t, x; 1, h)$ and is defined by

$$D^- L(t, x; 1, h) = \liminf_{\delta \downarrow 0} [L(t + \delta, x + \delta h) - L(t, x)]\delta^{-1}. \tag{12.4.1}$$

The *upper Dini derivate of* L *at* (t, x) *in the direction* $(1, h)$ is denoted by $D^+ L(t, x; 1, h)$ and is defined as in (12.4.1) with lim inf replaced by lim sup. The function L is said to have a *directional derivative at* (t, x) *in the direction* $(1, h)$ if $D^+ L(t, x; 1, h) = D^- L(t, x; 1, h)$. We denote the directional derivative by $DL(t, x; 1, h)$.

Remark 12.4.2. If L is differentiable at (t,x), then $DL(t,x;1,h)$ exists for every $h \in \mathbb{R}^n$ and is given by

$$DL(t,x;1,h) = L_t(t,x) + \langle L_x(t,x), h\rangle.$$

We shall be concerned with the value function W for the relaxed problem, Problem 12.2.1. We shall take the formulation of the relaxed problem to be that given in Section 5.4. Recall that in this formulation the control variable is $\overline{z} = (\zeta, \pi)$, where $\zeta = (z_1, \ldots, z_{n+1})$, with $z_i \in \mathbb{R}^m$ and $\pi = (\pi^1, \ldots, \pi^{n+1})$, with $\pi^i \in \mathbb{R}$. The constraint set $\widetilde{\Omega}$ is given by $\widetilde{\Omega} = \Gamma_{n+1} \times (\prod_{n+1} \Omega)$, where Ω is a constraint set in $\mathbb{R}^m$, $\prod_{n+1}$ denotes the $(n+1)$-fold Cartesian product and

$$\Gamma_{n+1} = \left\{ \pi\colon \pi = (\pi^1, \ldots, \pi^{n+1}), \pi^i \geq 0, \sum_{i=1}^{n+1} \pi^i = 1 \right\}.$$

To simplify notation we shall omit the subscript r and write (f^0, f) instead of (f^0_r, f_r). We shall, as in (5.4.1), write the control function as v, and shall write Q and Q^+ for Q_r and Q^+_r, respectively. In this context,

$$Q(t,x) = \{\widehat{y} = (y^0, y)\colon \widehat{y} = \widehat{f}(t,x,\overline{z})\colon \overline{z} \in \widetilde{\Omega}(t,x)\}$$
$$Q^+(t,x) = \{\widehat{y} = (y^0, y)\colon y^0 \geq f^0(t,x,\overline{z}),\ y = f(t,x,\overline{z}),\ \overline{z} \in \widetilde{\Omega}(t,x)\}.$$

Since the problem is a relaxed problem, both $Q(t,x)$ and $Q^+(t,x)$ are convex.

For problems in Mayer form with initial point (τ, ξ) we shall consider

$$D^- W(\tau, \xi; 1, h) = \liminf_{\delta \downarrow 0} [W(\tau + \delta, \xi + \delta h) - W(\tau, \xi)]\delta^{-1} \tag{12.4.2}$$

for $h \in Q(\tau, \xi)$. If we transform a problem in Lagrange or Bolza form to Mayer form, then an initial point $(\tau, \widehat{\xi}) = (\tau, \xi^0, \xi)$ for the transformed problem always has $\xi^0 = 0$. Therefore, $W(\tau, \widehat{\xi}) = W(\tau, \xi)$. Thus, for the problem in Bolza or Lagrange form we shall consider

$$D^- W(\tau, \xi; 1, \widehat{h}) = \liminf_{\delta \downarrow 0} [\delta h^0 + W(\tau + \delta, \xi + \delta h) - W(\tau, \xi)]\delta^{-1},$$

for $\widehat{h} = (h^0, h) \in Q(\tau, \xi)$.

The principal result of this section is the following theorem:

Theorem 12.4.3. *Let Assumption 12.2.1-r hold, with the function M in (12.2.2) in $L_\infty[\mathcal{I}]$. Let the value function W, defined on $\mathcal{R}_0$, be Lipschitz continuous on compact subsets of $\mathcal{R}_0$. Let the mapping Q^+ possess the Cesari property at all points of $\mathcal{R}_0$. For each τ in $[0,T]$ and each $\overline{z}$ in $\widetilde{\Omega}(\tau)$ let there exist a δ_0 and a control v defined on $[\tau, \tau + \delta_0]$ such that $\lim_{t \to \tau + 0} v(t) = \overline{z}$. Let $\widehat{h} = (h^0, h)$, with $h^0 \in \mathbb{R}$ and $h \in \mathbb{R}^n$. Then for each (τ, ξ) in $\mathcal{R}_0$*

$$\min[D^- W(\tau, \xi; 1, \widehat{h})\colon \widehat{h} \in Q(\tau, \xi)] = 0. \tag{12.4.3}$$

Remark 12.4.4. If $\widetilde{\Omega}$ is a constant map, then for $\overline{z} \in \widetilde{\Omega}$, the control v can be $v(t) = \overline{z}$, for $\tau \leq t \leq \tau + \delta$.

Proof. As in the proof of Theorem 12.3.2 we shall assume that the problem is in Mayer form. Then (12.4.3) becomes

$$\min[D^{-}W(\tau, \xi; 1, h) \colon h \in Q(\tau, \xi)] = 0.$$

The assumption that Q^{+} has the Cesari property at each point of $\mathcal{R}_0$ becomes the assumption that Q does.

We first show that

$$\inf[D^{-}W(\tau, \xi; 1, h) \colon h \in Q(\tau, \xi)] \geq 0. \tag{12.4.4}$$

Let $h \in Q(\tau, \xi)$. Then there exists a $\overline{z}$ in $\widetilde{\Omega}(\tau)$ such that $h = f(\tau, \xi, \overline{z})$. Also, there exists a $\delta_0 > 0$ and a control v defined on $[\tau, \tau + \delta_0]$ with $v(\tau) = \overline{z}$ and that is continuous from the right at $t = \tau$. The control can be extended to $[\tau, T]$ and we denote the extended control also by v. Let ψ denote the trajectory corresponding to v and having initial point (τ, ξ). Then for $\delta > 0$

$$\begin{aligned} \psi(\tau + \delta) &= \xi + \int_{\tau}^{\tau+\delta} f(s, \psi(s), v(s)) ds \qquad (12.4.5) \\ &= \xi + \int_{\tau}^{\tau+\delta} [f(\tau, \xi, \overline{z}) + o(1)] ds \\ &= \xi + \delta f(\tau, \xi, \overline{z}) + o(\delta), \end{aligned}$$

where $o(\delta)$ is as $\delta \to 0$.

By the Principle of Optimality ((12.3.1) with $f^0 = 0$),

$$[W(\tau + \delta, \psi(\tau + \delta)) - W(\tau, \xi)]\delta^{-1} \geq 0.$$

If we substitute the rightmost member of (12.4.5) into this inequality and use the Lipschitz continuity of W, we get that

$$[W(\tau + \delta, \xi + \delta h) - W(\tau, \xi)]\delta^{-1} + o(1) \geq 0,$$

where $h = f(\tau, \xi, \overline{z})$. From this, (12.4.4) follows. $\square$

We next show that there exists an $h^{*} \in Q(\tau, \xi)$ such that $D^{-}W(\tau, \xi; 1, h^{*}) \leq 0$. This in conjunction with (12.4.4) will establish the theorem.

Let ψ now denote an *optimal* trajectory for Problem 12.2.1 with initial point (τ, ξ). Then

$$\psi(\tau + \delta) = \xi + \int_{\tau}^{\tau+\delta} \psi'(s) ds,$$

with $\psi'(s) \in Q(s, \psi(s))$ for almost all $t \in [\tau, T]$. Let $N_{\varepsilon}(\tau, \xi) = \{(t, x) \colon (t, x) \in$

$\mathcal{R}_0,\ |(t,x)-(\tau,\xi)|<\varepsilon\}$. Since ψ is continuous, given an $\varepsilon'>0$, there exists a $\delta(\varepsilon')$ with $0<\delta(\varepsilon')<\varepsilon'$ such that if $\tau\le s\le\tau+\delta$, then

$$\psi'(s)\in Q(N_\varepsilon(\tau,\xi)) \qquad \text{a.e.}, \tag{12.4.6}$$

where $\varepsilon=\delta(\varepsilon')$. Let K_ε denote the set of points in $[\tau,\tau+\delta]$ at which the inclusion (12.4.6) holds. Then the Lebesgue measure of K_ε equals δ. Thus,

$$\int_\tau^{\tau+\delta}\psi'(s)ds=\delta\int_\tau^{\tau+\delta}\psi'(s)\left(\frac{ds}{\delta}\right)=\delta\int_{K_\varepsilon}\psi'(s)\left(\frac{ds}{\delta}\right).$$

From (12.4.6) we get that

$$\text{cl co }\{\psi'(s)\colon s\in K_\varepsilon\}\subseteq \text{cl co }\{Q(N_\varepsilon(\tau,\xi))\}.$$

From Lemma 3.2.9 we get that

$$\text{cl co }\{\psi'(s)\colon s\in K_\varepsilon\}=\ \text{cl}\left\{\int_{K_\varepsilon}\psi'(s)d\mu\colon \mu\in\mathcal{P}(K_\delta)\right\},$$

where $\mathcal{P}(K_\delta)$ is the set of probability measures on K_δ. Hence

$$\int_{K_\varepsilon}\psi'(s)\left(\frac{ds}{\delta}\right)\in\ \text{cl co }\{Q(N_\varepsilon(\tau,\xi))\}.$$

Let

$$h_\delta\equiv\int_\tau^{\tau+\delta}\psi'(s)\left(\frac{ds}{\delta}\right).$$

We have shown that for each $\varepsilon'>0$ there exists a $\delta(\varepsilon')$, with $0<\delta(\varepsilon')<\varepsilon'$, and a point h_δ such that

$$\psi(\tau+\delta)=\xi+\delta h_\delta \quad h_\delta\in\ \text{cl co }(Q(N_\varepsilon(\tau,\xi))), \tag{12.4.7}$$

where $\varepsilon=\delta(\varepsilon')$. We also have that

$$|h_\delta|\le\delta^{-1}\int_\tau^{\tau+\delta}|\psi'(s)|ds\le\delta^{-1}\int_\tau^{\tau+\delta}|(1+\psi(s))|M(s)ds,$$

the last inequality following from (12.2.2). Since M is in $L_\infty[0,T]$ and $|\psi(s)|$ is bounded on $[\tau,T]$, we get that there exists a positive constant A such that $|h_\delta|\le A$ for all $\delta=\delta(\varepsilon)$.

Let $\{\varepsilon'_k\}$ be a decreasing sequence of positive terms with $\varepsilon'_k\to0$. We can take the corresponding sequence $\delta(\varepsilon'_k)$ also to be decreasing. Then the sequence $\{\varepsilon_k\}$ is also decreasing, and $\varepsilon_k\to0$. Let $h_k=h_{\delta(\varepsilon_k)}$ be the corresponding sequence given by (12.4.7). Thus, $h_k\in$ cl co $(Q(N_{\varepsilon_k}(\tau,\xi)))$. Since $|h_k|\le A$ for all k, there exists a subsequence, that we relabel as h_k, that converges to an element h^* in $\mathbb{R}^n$. Since for $0<\rho<\rho'$

$$Q(N_\rho(\tau,\xi))\subset Q(N_{\rho'}(\tau,\xi)), \tag{12.4.8}$$

it follows that for all k,

$$\text{cl co } (Q(N_{\varepsilon_{k+1}}(\tau,\xi))) \subseteq \text{ cl co } (Q(N_{\varepsilon_k}(\tau,\xi))).$$

Thus, for fixed k_0 we have that for all $k \geq k_0$

$$h_k \in \text{ cl co } (Q(N_{\varepsilon_{k_0}}(\tau,\xi))).$$

Hence $h^* \in$ cl co $(Q(N_{\varepsilon_{k_0}}(\tau,\xi)))$. But k_0 is an arbitrary positive integer, so for *all* k

$$h^* \in \text{ cl co } (Q(N_{\varepsilon_k}(\tau,\xi))). \tag{12.4.9}$$

It follows from (12.4.8) that

$$\bigcap_k \text{ cl co } (Q(N_{\varepsilon_k}(\tau,\xi))) = \bigcap_{\delta>0} \text{ cl co } (Q(N_{\delta}(\tau,\xi))).$$

From this, from (12.4.9), and from the fact that Q has the Cesari property we get that $h^* \in Q(\tau,\xi)$.

We return to our sequences $\{\delta_k\}$ and $\{h_k\}$. From the definition of h^* and from (12.4.7) we get that

$$\psi(\tau+\delta_k) = \xi + \delta_k h^* + o(\delta_k). \tag{12.4.10}$$

From the Principle of Optimality we have that

$$[W(\tau+\delta_k, \psi(\tau+\delta_k)) - W(\tau,\xi)]\delta_k^{-1} = 0.$$

Substituting (12.4.10) into this equation and using the Lipschitz continuity of W, we get that

$$\lim_{k\to\infty} [W(\tau+\delta_k, \xi+\delta_k h^*) - W(\tau,\xi)]\delta_k^{-1} = 0.$$

Hence

$$\liminf_{\delta\to 0+} [W(\tau+\delta, \xi+\delta h^*) - W(\tau,\xi)]\delta^{-1} \leq 0.$$

Thus, $D^-W(\tau,\xi;1,h^*) \leq 0$. This proves the theorem for the problem in Mayer form. Note that our argument shows that $D^-W(\tau,\xi;1,h^*) = 0$, so we are justified in writing min in (12.4.2).

Remark 12.4.5. In Theorem 12.4.3 we do not assume that the sets $\widetilde{\Omega}(t)$ are compact. Instead we assume that the sets $Q(t,x)$ possess the Cesari property. Growth conditions and a regularity condition on the constraint mapping $\widetilde{\Omega}$ guaranteeing that the Cesari property holds are given in Lemma 5.4.6. If the mapping $\widetilde{\Omega}$ is u.s.c.i. and the sets $\Omega(t)$ are compact, then the sets $Q(t,x)$ possess the Cesari property. (See the proof of Theorem 5.6.1.)

12.5 The Value as Viscosity Solution

The definition of viscosity solution of a nonlinear partial differential equation was first given by Crandall and Lions in [28]. For subsequent developments in the theory of viscosity solutions the reader is referred to Crandall, Ishii, and Lions [29], Bardi and Capuzzo-Dolcetta [5], and Fleming and Soner [35]. We shall confine our attention to Eq. (12.1.2), which for the reader's convenience we present again here, rather than consider the general nonlinear partial differential equation.

Let

$$H(t, x, \overline{z}, q^0, q) = q^0 f^0(t, x, \overline{z}) + \langle q, f(t, x, \overline{z}) \rangle$$

and let

$$\overline{H}(t, x, q) = \sup_{\overline{z} \in \widetilde{\Omega}(t)} H(t, x, \overline{z}, -1, q).$$

We consider the nonlinear partial differential equation

$$-V_t(t, x) + \overline{H}(t, x, -V_x(t, x)) = 0. \tag{12.5.1}$$

This equation is a Hamilton-Jacobi-Bellman equation. We consider this equation on $\mathcal{R}_0$.

Definition 12.5.1. A continuous function V on $\mathcal{R}_0$ is a *viscosity subsolution* of (12.5.1) on $\mathcal{R}_0$ if for each function ω in $C^{(1)}(\mathcal{R}_0)$

$$-\omega_t(\overline{t}, \overline{x}) + \overline{H}(\overline{t}, \overline{x}, -\omega_x(\overline{t}, \overline{x})) \leq 0$$

at each point $(\overline{t}, \overline{x})$ in $\mathcal{R}_0$ at which $V - \omega$ has a local maximum.

A continuous function V on $\mathcal{R}_0$ is a *viscosity supersolution* of (12.5.1) on $\mathcal{R}_0$ if for each function ω in $C^{(1)}(\mathcal{R}_0)$

$$-\omega_t(\overline{t}, \overline{x}) + \overline{H}(\overline{t}, \overline{x}, -\omega_x(\overline{t}, \overline{x})) \geq 0$$

at each point $(\overline{t}, \overline{x})$ in $\mathcal{R}_0$ at which $V - \omega$ has a local minimum.

A continuous function V on $\mathcal{R}_0$ is a *viscosity solution* on $\mathcal{R}_0$ if it is both a viscosity subsolution and a viscosity supersolution.

The functions ω are called *test functions.*

Remark 12.5.2. If V is a viscosity solution of (12.5.1) and the partial derivatives of V exist at a point $(\overline{t}, \overline{x})$ in $\mathcal{R}_0$, then V satisfies (12.5.1) in the usual sense at $(\overline{t}, \overline{x})$. To see this, let ω be a test function such that $V - \omega$ has a local maximum at $(\overline{t}, \overline{x})$. Then $V_t(\overline{t}, \overline{x}) = \omega_t(\overline{t}, \overline{x})$ and $V_x(\overline{t}, \overline{x}) = \omega_x(\overline{t}, \overline{x})$. Since W is a viscosity solution of (12.5.1), it is a subsolution, and so $-\omega_t(\overline{t}, \overline{x}) + \overline{H}(\overline{t}, \overline{x}, -\omega_x(\overline{t}, \overline{x})) \leq 0$. Hence $-V_t(\overline{t}, \overline{x}) + \overline{H}(\overline{t}, \overline{x}, -V_x(\overline{t}, \overline{x})) \leq 0$. By considering a test function ω such the $V - \omega$ has a local minimum at $(\overline{t}, \overline{x})$, we get the reverse inequality. Thus, V satisfies (12.5.1) at $(\overline{t}, \overline{x})$.

If V is a solution of (12.5.1) in the ordinary sense, then V is also a viscosity solution of (12.5.1) on $\mathcal{R}_0$. To see this, let $(\bar{t}, \bar{x})$ be a point in $\mathcal{R}_0$ and let ω be a test function such that $V - \omega$ has a relative maximum (minimum at $(\bar{t}, \bar{x})$). Then again $V_t = \omega_t$ and $V_x = \omega_x$. Since V is an ordinary $C^{(1)}$ solution of (12.5.1), then

$$0 = -V_t(\bar{t}, \bar{x}) + \overline{H}(\bar{t}, \bar{x}, -V_x(\bar{t}, \bar{x})) = -\omega_t(\bar{t}, \bar{x}) + \overline{H}(\bar{t}, \bar{x}, \omega_x(\bar{t}, \bar{x})).$$

Thus, V is a viscosity solution.

Theorem 12.5.3. *Let the hypotheses of Theorem 12.4.3 hold. Then the value function W is a viscosity solution on $\mathcal{R}_0$ of the Hamilton-Jacobi equation (12.5.1).*

Theorem 12.5.3 is an immediate consequence of Theorem 12.4.3 and the following lemma.

Lemma 12.5.4. *Let V be a continuous function such that at each point (τ, ξ) in $\mathcal{R}_0$*

$$\inf[D^-(V(\tau, \xi; 1, \widehat{h})\colon \widehat{h} \in Q(\tau, \xi)] = 0. \tag{12.5.2}$$

Then V is a viscosity solution of (12.5.1)*.*

Proof. Assume that the problem is in Mayer form.

We first show that (12.5.2) implies that V is a subsolution of (12.5.1). Let ω be a test function such that $V - \omega$ has a local maximum at (τ, ξ). Then for all (t, x) sufficiently close to (τ, ξ)

$$V(t, x) - \omega(t, x) \leq V(\tau, \xi) - \omega(\tau, \xi).$$

Hence for fixed h in $Q(\tau, \xi)$ and sufficiently small $\delta > 0$,

$$[V(\tau + \delta, \xi + \delta h) - V(\tau, \xi)]\delta^{-1} \leq [\omega(\tau + \delta, \xi + \delta h) - \omega(\tau, \xi)]\delta^{-1}.$$

Letting $\delta \to 0$ gives, for fixed $h \in Q(\tau, \xi)$,

$$\begin{aligned} D^-V(\tau, \xi; 1, h) &\leq \omega_t(\tau, \xi) + \langle \omega_x(\tau, \xi), h \rangle \\ &= -[-\omega_t(\tau, \xi) + \langle -\omega_x(\tau, \xi), h \rangle]. \end{aligned}$$

If we now take the infimum over $h \in Q(\tau, \xi)$ and use (12.5.2) we get that

$$\begin{aligned} 0 &\leq \inf\{-[-\omega_t(\tau, \xi) + \langle -\omega_x(\tau, \xi), h \rangle]\colon h \in Q(\tau, \xi)\} \\ &= -\sup\{[-\omega_t(\tau, \xi) + \langle -\omega_x(\tau, \xi), h \rangle]\colon h \in Q(\tau, \xi)\} \\ &= \omega_t(\tau, \xi) - \overline{H}(\tau, \xi, -\omega_x(\tau, \xi)), \end{aligned}$$

the last equality following from the definition of $\overline{H}$. Hence

$$-\omega_t(\tau, \xi) + \overline{H}(\tau, \xi, -\omega_x(\tau, \xi)) \leq 0,$$

and so V is a subsolution.

We now show that V is a supersolution. Let ω be a test function such that $V - \omega$ has a local minimum at (τ, ξ). Then for fixed $h \in Q(\tau, \xi)$ and δ sufficiently small,

$$V(\tau + \delta, \xi + \delta h) - \omega(\tau + \delta, \xi + \delta h) \geq V(\tau, \xi) - \omega(\tau, \xi).$$

Hence

$$D^- V(\tau, \xi; 1, h) \geq \omega_t(\tau, \xi) + \langle \omega_x(\tau, \xi), h \rangle,$$

and so

$$-\omega_t(\tau, \xi) + \langle -\omega_x(\tau, \xi), h \rangle \geq -D^- V(\tau, \xi; 1, h).$$

If we now take the supremum over $h \in Q(t, x)$, we get that

$$\begin{aligned} -\omega_t(\tau, \xi) + \overline{H}(\tau, \xi, -\omega_x(\tau, \xi)) &\geq \sup_h [-D^- V(\tau, \xi; 1, h)] \\ &= -\inf_h [D^- V(\tau, \xi; 1, h)] = 0. \end{aligned}$$

Hence V is a supersolution, and the lemma is proved. $\square$

Lemma 12.5.4 and Theorem 12.4.3 justify calling a continuous function V that satisfies (12.5.2) a generalized solution of the Hamilton-Jacobi equation.

Definition 12.5.5. A continuous function V defined on $\mathcal{R}_0$ that satisfies (12.5.2) will be called a *lower Dini solution* of the Hamilton-Jacobi equation.

Lemma 12.5.4 has a partial converse.

Lemma 12.5.6. *Let V be a Lipschitz continuous viscosity solution of* (12.5.1). *Let $\widehat{f}$ be continuous, and let the constraint mapping $\widetilde{\Omega}$ be u.s.c.i and such that each set $\widetilde{\Omega}(t)$ is compact. Then V is a lower Dini solution.*

Proof. We again take the problem to be in Mayer form. Let (t, x) be an arbitrary point in $\mathcal{R}_0$. We first show that for all $h \in Q(t, x)$

$$D^- V(t, x; 1, h) \geq 0. \tag{12.5.3}$$

Fix $h \in Q(t, x)$. There exists a decreasing sequence of positive numbers δ_k such that $\delta_k \to 0$ and

$$D^- V(t, x; 1, h) = \lim_{k \to \infty} [V(t + \delta_k, x + \delta_k h) - V(t, x)] \delta_k^{-1}. \tag{12.5.4}$$

Since V is Lipschitz continuous, by Rademacher's theorem, V is differentiable almost everywhere. Hence there exists a sequence $\{(t_k, x_k)\}$ such that V is differentiable at each point (t_k, x_k) and

$$|t_k - t| \leq \delta_k^2 \qquad |x - x_k| < \delta_k^2. \tag{12.5.5}$$

Then

$$\begin{aligned}V(t+\delta_k, x+\delta_k h) - V(t,x) &= \{V(t+\delta_k, x+\delta_k h) - V(t_k+\delta_k, x_k+\delta_k h)\} \\ &+ \{V(t_k+\delta_k, x_k+\delta_k h) - V(t_k,x_k)\} + \{V(t_k,x_k) - V(t,x)\} \\ &\equiv A_k + B_k + C_k\end{aligned}$$

From the Lipschitz continuity of V and (12.5.5) we have that

$$(A_k + C_k) = O(|t-t_k|) + O(|x-x_k|) = O(\delta_k^2)$$

Hence

$$\lim_{k\to\infty} (A_k + C_k)\delta_k^{-1} = 0. \tag{12.5.6}$$

Since V is differentiable at (t_k, x_k),

$$\begin{aligned}B_k &= [V_t(t_k,x_k) + \langle V_x(t_k,x_k), h\rangle]\delta_k + o(\delta_k(1+h)) \\ &\geq [V_t(t_k,x_k) + \inf_{h\in Q(t,x)} \langle V_x(t_k,x_k), h\rangle]\delta_k + o(\delta_k).\end{aligned}$$

Hence

$$B_k \geq -[-V_t(t_k,x_k) + \sup_{h\in Q(t,x)} \langle -V_x(t_k,x_k), h\rangle]\delta_k + o(\delta_k).$$

At points of differentiability a viscosity solution satisfies (12.5.1) in the ordinary sense. Hence

$$\liminf_{k\to\infty} B_k\delta_k^{-1} \geq 0. \tag{12.5.7}$$

From (12.5.4), (12.5.6), and (12.5.7) we get that

$$\begin{aligned}D^-V(t,x;1,h) &= \lim_{k\to\infty} (A_k + B_k + C_k)\delta_k^{-1} \\ &\geq \lim_{k\to\infty} (A_k + C_k)\delta_k^{-1} + \liminf_{k\to\infty} B_k\delta_k^{-1} \geq 0,\end{aligned}$$

which establishes (12.5.3).

To complete the proof it suffices to show that there exists an $h_0 \in Q(t,x)$ such that

$$D^-V(t,x;1,h_0) \leq 0, \tag{12.5.8}$$

for this in conjunction with (12.5.3) will establish (12.5.2).

As above, let $\{(t_k,x_k)\}$ be a sequence of points in $\mathcal{R}_0$ satisfying (12.5.5) and such that V is differentiable at each point (t_k,x_k). Since V is differentiable at (t_k,x_k), for each h in $Q(t,x)$,

$$D^-V(t_k,x_k,1,h) = V_t(t_k,x_k) + \langle V_x(t_k,x_k), h\rangle.$$

Hence

$$\inf_{h\in Q(t_k,x_k)} D^-V(t_k,x_k;1,h) = -[-V_t(t_k,x_k) \tag{12.5.9}$$

$$+ \sup_{h \in Q(t_k, x_k)} \langle -V_x(t_k, x_k), h \rangle] = 0,$$

the last equality being valid because V is a viscosity solution and is differentiable at (t_k, x_k). Since f is continuous, $\widetilde{\Omega}(t_k)$ is compact and $Q(t_k, x_k) = \{h \colon h = f(t_k, x_k, \overline{z}),\ \overline{z} \in \widetilde{\Omega}(t)\}$, it follows that there is a point $\overline{z}_k$ in $\widetilde{\Omega}(t_k)$ at which the supremum in (12.5.9) is attained. Let $h_k \in Q(t_k, x_k)$ be defined by $h_k = f(t_k, x_k, \overline{z}_k)$. Then

$$-V_t(t_k, x_k) + \langle -V_x(t_k, x_k), h_k \rangle = 0. \tag{12.5.10}$$

The sequence of points $\{(t_k, x_k, h_k)\}$ are in the set Δ defined in (12.2.3) which, as noted in Remark 12.2.5, is compact in the present case. It then follows from (12.5.5) that there is a subsequence, which we relabel as (t_k, x_k, h_k) that converges to the point (t, x, h_0) in Δ. Thus, $h_0 \in Q(t, x)$.

With the sequence $\{\delta_k\}$ related to (t_k, x_k) as in (12.5.5), we have that

$$\begin{aligned}
D^- V(t, x; 1, h_0) &\le \liminf_{k \to \infty} [V(t + \delta_k, x_k + \delta_k h_0) - V(t, x)]\delta_k^{-1} \\
&= \liminf_{k \to \infty} [\{V(t + \delta_k, x + \delta_k h_0) - V(t_k + \delta_k, x_k + \delta_k h_k)\} \\
&\quad + \{V(t_k + \delta_k, x_k + \delta_k h_k) - V(t_k, x_k)\} + \{V(t_k, x_k) - V(t, x)\}]\delta_k^{-1} \\
&\equiv \liminf_{k \to \infty} [A_k + B_k + C_k]\delta_k^{-1}.
\end{aligned}$$

From (12.5.5), the Lipschitz continuity of V and $h_k \to h_0$ we get that

$$(A_k + C_k)\delta_k^{-1} = [O(|t - t_k|) + O(|x - x_k| + \delta_k |h_0 - h_k|)]\delta_k^{-1} \tag{12.5.11}$$

as $k \to \infty$. From the differentiability of V at (t_k, x_k) we get that

$$B_k = [V_t(t_k, x_k) + \langle V_x(t_k, x_k), h_k \rangle]\delta_k + o(\delta_k |1 + h_k|).$$

By (12.5.10), the expression in square brackets is zero, and thus $B_k \delta_k^{-1} = o(1)$. Therefore, from (12.5.11), we get that

$$\begin{aligned}
\liminf_{k \to \infty} [(A_k + C_k) + B_k]\delta_k^{-1} &\le \limsup_{k \to \infty} (A_k + C_k)\delta_k^{-1} \\
&\quad + \liminf_{k \to \infty} B_k \delta_k^{-1} = 0.
\end{aligned}$$

Hence (12.5.8) holds and the lemma is proved. □

12.6 Uniqueness

After showing that under appropriate hypotheses the value function is a viscosity solution of the Hamilton-Jacobi equation, the question arises whether

this is the only solution. This question is answered in the affirmative in the viscosity solution literature. See Fleming and Soner [35] and Bardi and Capuzzo-Dolcetti [5] and the references given therein. In this section we shall also answer the uniqueness question affirmatively, but without using the analytical techniques or results of the viscosity solution literature. Rather, we shall use the techniques of control theory to give a self-contained presentation.

The following modification of Assumption 12.2.1-r will hold in this section.

Assumption 12.6.1. The function K in (12.2.1) is the constant function K. In conditions (v) to (vii) the set $\mathcal{R}_0 = [0, T) \times \mathbb{R}^n$, where we have taken $\mathcal{I} = [0, T]$. The terminal set $\mathcal{J} = \{T\} \times \mathbb{R}^n$.

Theorem 12.6.2. *Let Assumption 12.6.1 hold. Let the sets $\widetilde{\Omega}(t),\ 0 \le t \le T$ be a fixed compact set C. Let g be locally Lipschitz continuous. Then the value function W is the unique locally Lipschitz continuous viscosity solution of the boundary value problems*

$$\begin{aligned} -V_t(t,x) + \overline{H}(t,x,-V_x) &= 0 \qquad (t,x) \in \mathcal{R}_0 \\ V(T,x) &= g(x). \end{aligned} \tag{12.6.1}$$

The value function W is the unique locally Lipschitz continuous solution of

$$\begin{gathered} \min_{\overline{z} \in C} D^- V(t, x; 1, \widehat{f}(t, x, \overline{z})) = 0 \qquad (t,x) \in \mathcal{R}_0 \\ V(T,x) = g(x). \end{gathered} \tag{12.6.2}$$

Proof. In view of Lemma 12.5.6 to prove the theorem it suffices to show that W is the unique solution of (12.6.2). That is, we must show that if (t_0, x_0) is an arbitrary point of $\mathcal{R}_0$ and V is a locally Lipschitz solution of (12.6.2), then $V(t_0, x_0) = W(t_0, x_0)$. To simplify the argument, we shall assume that the problem is in Mayer form. We showed in Section 2.4 that there is no loss of generality in making this assumption. □

We first show that

$$V(t_0, x_0) \le W(t_0, x_0) \qquad (t_0, x_0) \in \mathcal{R}_0. \tag{12.6.3}$$

Let (ψ, v) be an arbitrary admissible pair for the control problem with initial point (t_0, x_0). Since V is locally Lipschitz continuous and ψ is absolutely continuous, the function $t \to V(t, \psi(t))$ is absolutely continuous. Hence

$$V(T, \psi(T)) - V(t_0, x_0) = \int_{t_0}^{T} \left(\frac{d}{dt}\, V(t, \psi(t) \right) dt. \tag{12.6.4}$$

Let s be a point in $[t_0, T]$ at which $t \to V(t, x(t))$ is differentiable, $v(s) \in C$, and s is a Lebesgue point for $t \to f(t, \psi(t), v(t))$. The set of such points has measure $T - t_0$, and

$$\frac{d}{dt}\, V(t, \psi(t))|_{t=s} = \lim_{\delta \to 0} \delta^{-1} [V(s + \delta, \psi(s + \delta)) - V(s, \psi(s))]$$

$$
\begin{aligned}
&= \lim_{\delta \to 0} \ \delta^{-1}[V(s+\delta, \psi(s) + \int_s^{s+\delta} f(\sigma, \psi(\sigma), v(\sigma))d\sigma) - V(s, \psi(s))] \\
&= \lim_{\delta \to 0} \delta^{-1}[V(s+\delta, \psi(s) + \delta f(s, \psi(s), v(s)) + o(\delta)) - V(s, \psi(s))] \\
&\geq D^- V(s, \psi(s); 1, f(s, \psi(s), v(s)) \geq 0,
\end{aligned}
$$

where the next to the last inequality follows from the Lipschitz continuity of V and the definition of lower Dini derivate. The last inequality follows from (12.6.2). Substituting this into (12.6.4) gives

$$V(t_0, x_0) \leq V(T, \psi(T)) = g(\psi(T)).$$

Since (ψ, v) is an arbitrary admissible pair and $J(\psi, v) = g(\psi(T))$, we get that (12.6.3) holds.

To prove the theorem we need to show that

$$V(t_0, x_0) \geq W(t_0, x_0). \tag{12.6.5}$$

The calculation in the preceding of $dV(t, \psi(t))/dt$ and (12.6.2) suggest that we attempt to find an admissible trajectory ψ such that at points $(t, \psi(t))$ of the trajectory

$$D^- V(t, \psi(t); 1, f(t, \psi(t), v(t)) = 0.$$

For then,

$$V(t_0, x_0) = V(T, \psi(T)) = g(\psi(T)) = J(\psi, v) \geq W(t_0, x_0).$$

This in turn suggests taking ψ to be a solution of the differential inclusion $\psi' = F(t, x)$, with initial condition $\psi(t_0) = x_0$, where

$$F(t, x) = \{w \colon w = \text{argmin } D^- V(t, x; 1, f(t, x, \overline{z})), \overline{z} \in C\}. \tag{12.6.6}$$

Unfortunately, we cannot guarantee the existence of a solution on $[t_0, T]$ of the differential inclusion $x' \in F(t, x)$ with $\psi(t_0) = x_0$ for F defined in (12.6.6). We do, however, have the following result.

Lemma 12.6.3. *Let V be a locally Lipschitz continuous function that satisfies* (12.6.2). *Then for each $\varepsilon > 0$ there exists an admissible pair (ψ, v) such that*

$$V(t_0, x_0) \geq g(\psi(T)) - \varepsilon. \tag{12.6.7}$$

Remark 12.6.4. Since $g(\psi(T)) = J(\psi) \geq W(t_0, x_0)$, and since $\varepsilon > 0$ is arbitrary, the inequality (12.6.7) implies (12.6.5).

The proof of Lemma 12.6.3 requires two preparatory lemmas, whose statements and proofs are facilitated by the introduction of additional notation. Let $\psi(\cdot; \tau, \xi, v)$ denote an admissible trajectory with initial point (τ, ξ) corresponding to the admissible control v. Let $\mathcal{R}(\tau, \xi)$ denote the set of points reachable by admissible trajectories with initial point (τ, ξ). Thus,

$$\mathcal{R}(\tau, \xi) = \{(t, x) \colon t > \tau \quad x = \psi(t, \tau, \xi, v), \quad v \text{ admissible}\}$$

Let $v_{\overline{z}}$ denote the control on an interval with $v_{\overline{z}}(t) = \overline{z} \in C$ for all t in the interval. Let $\overline{B}$ denote the closed unit ball in $\mathbb{R}^n$ and let L denote the Lipschitz constant for V.

Lemma 12.6.5. *There exists a positive number M and a positive integer i_0 with $T - t_0 > i_0^{-1}$ such that for all (τ, ξ) in $[t_0, T] \times M\overline{B}$, all $\overline{z}$ in C and all $[\tau, \sigma] \subset [t_0, T]$ with $|\sigma - \tau| < i^{-1}$ for all $i \geq i_0$:*

$$\text{(i)} \quad |\psi(\sigma; \tau, \xi, v_{\overline{z}})| \leq 2M \qquad \text{(ii)} \quad |\xi + (\sigma - \tau) f(\tau, \xi, \overline{z})| \leq 2M \tag{12.6.8}$$
$$\text{(iii)} \quad |\xi + (\sigma - \tau) f(\tau, \xi, \overline{z}) - \psi(\sigma; \tau, \xi, v_{\overline{z}})| \leq |\sigma - \tau|/2L.$$

Proof. It follows from (12.2.2) and Lemma 4.3.14 that all admissible trajectories with initial point (t_0, x_0) lie in a compact set $[t_0, T] \times M\overline{B}$ for some $M > 0$. Thus, $\mathcal{R}(t_0, x_0) \subseteq [t_0, T] \times M\overline{B}$.

The set $\Sigma \equiv [t_0, T] \times M\overline{B} \times C$ is compact, and since f is continuous it is bounded on Σ. Therefore, there exists a positive integer i_2 such that for $i > i_2$, $(\tau, \xi, \overline{z})$ in Σ and σ in $[t_0, T]$ with $|\sigma - \tau| < i^{-1}$, (ii) holds.

The set $[0, T] \times M\overline{B}$ is compact, so again by (12.2.2) and Lemma 4.3.14, there exists a constant $M_1 > 0$ such that for any point (τ, ξ) in $[t_0, T] \times M\overline{B}$, an admissible trajectory starting at (τ, ξ) lies in $[t_0, T] \times M_1\overline{B}$. Thus, for any $\overline{z}$ in C,

$$\begin{aligned} |\psi(\sigma, \tau, \xi, v_{\overline{z}})| &= |\xi + \int_\tau^\sigma f(s, \psi(s, \tau, \xi, v_{\overline{z}}), \overline{z}) ds| \\ &\leq M + \int_\tau^\sigma M_1 ds. \end{aligned}$$

Therefore, there exists an i_1 such that for $i > i_1$ and $|\sigma - \tau| < i^{-1}$, (i) holds for $i > i_1$. □

The functions f and ψ are uniformly continuous on compact sets. Hence for $\tau \leq s \leq \sigma$,

$$f(s, \psi(s; \tau, \xi, v_{\overline{z}}), \overline{z}) = f(\tau, \xi, \overline{z}) + \eta(s; \tau, \xi, \overline{z}),$$

where $\eta(s) \to 0$ as $\sigma \to \tau$, uniformly in $(\tau, \xi, \overline{z})$ on $[t_0, T] \times M\overline{B} \times C$. Consequently, there exists an i_3 such that $|\eta(s)| \leq 1/2L$ whenever $i > i_3$ and $|\sigma - \tau| < i^{-1}$. The left-hand side of (iii) is

$$|(\sigma - \tau) f(\tau, \xi, \overline{z}) - \int_\tau^\sigma f(s, \psi(s; \tau, \xi, v_{\overline{z}}), \overline{z}) ds|,$$

and so is less than or equal to

$$\int_\tau^\sigma |\eta(s)| ds \leq |\sigma - \tau|/2L,$$

which establishes (iii) for $i > i_3$. If we set $i_0 = \max(i_1, i_2, i_3)$, then (i), (ii), and (iii) all hold for $i > i_0$.

Lemma 12.6.6. *Let V be a locally Lipschitz continuous function that satisfies* (12.6.2). *Then there exists a positive number M, an integer $i_0 > 0$, and a sequence $\{\overline{\delta}_i\}_{i=1}^{\infty}$, of positive numbers such that for all $i \geq i_0$*

$$0 < \overline{\delta}_i < i^{-1}$$

the following holds. Corresponding to each integer $i \geq i_0$ and point $(\tau, \xi) \in [0, T - i^{-1}] \times M\overline{B}$ there exists a number $\delta \in (\overline{\delta}_i, i^{-1})$ and a point $\overline{z}$ in C such that

$$\delta^{-1}(V(\tau + \delta, \psi(\tau + \delta; \tau, \xi, v_{\overline{z}})) - V(\tau, \xi)) < i^{-1}. \tag{12.6.9}$$

Proof. Let i_0 be as in Lemma 12.6.5. Choose any $i \geq i_0$ and let $(\tau, \xi) \in [0, T] \times M\overline{B}$. The i chosen will be fixed for the remainder of this proof. Then since V satisfies (12.6.2), there exists a $\delta_{\tau,\xi} \in (0, i^{-1})$ and a $\overline{z}_{\tau,\xi} \in C$ such that

$$\delta_{\tau,\xi}^{-1}[V(\tau + \delta_{\tau,\xi}, \xi + \delta_{\tau,\xi} f(\tau, \xi, \overline{z}_{\tau,\xi})) - V(\tau, \xi)] < i^{-1}/2.$$

From this, from (12.6.8), and from the Lipschitz continuity of V we get that

$$\delta_{\tau,\xi}^{-1}[V(\tau + \delta_{\tau,\xi}, \psi(\tau + \delta_{\tau,\xi}; \tau, \xi, v_{\overline{z}_{\tau,\xi}})) - V(\tau, \xi)] < i^{-1}.$$

For each point (τ, ξ) in $[0, T - i^{-1}] \times M\overline{B}$ consider the set

$$\mathcal{O}_{\tau,\xi} = \{(\sigma, \eta) \colon \delta_{\tau,\xi}^{-1}[V(\sigma + \delta_{\tau,\xi}, \psi(\sigma + \delta_{\tau,\xi}; \sigma, \eta, v_{\overline{z}_{\tau,\xi}})) - V(\sigma, \eta)] < i^{-1}\}.$$

Each set $\mathcal{O}_{\tau,\xi}$ contains (τ, ξ) and is open by the continuity of the functions involved. Thus, the family of sets $\{\mathcal{O}_{\tau,\xi} \colon (\tau, \xi) \in [0, T - i^{-1}] \times M\overline{B}\}$ is an open cover of the compact set $[0, T - i^{-1}] \times M\overline{B}$. Hence there exists a finite subcover. That is, there exists a positive integer N, numbers $\delta_j \in (0, i^{-1})$, points $\overline{z}_j \in C$, and sets $\mathcal{O}_j$, $j = 1, \ldots, N$, where

$$\mathcal{O}_j = \{(\sigma, \eta) \colon \delta_j^{-1}[V(\sigma + \delta_j, \psi(\sigma + \delta_j; \sigma, \eta, v_{\overline{z}_j})) - V(\sigma, \eta)] < i^{-1}\},$$

such that

$$[0, T - i^{-1}] \times M\overline{B} \subseteq \bigcup_{j=1}^{N} \mathcal{O}_j. \tag{12.6.10}$$

Let $\overline{\delta}_i = \min\{\delta_1, \ldots, \delta_j\}$. Then $\overline{\delta}_i \in (0, i^{-1})$, since this is true for each δ_j. Now let (t, x) be an arbitrary point in $[0, T - i^{-1}] \times M\overline{B}$. Then by (12.6.10), $(t, x) \in \mathcal{O}_j$ for some $j = 1, \ldots, N$. But then by the definition of $\mathcal{O}_j$, (12.6.9) is true with $\delta = \delta_j$ and $\overline{z} = \overline{z}_j$. Since $\overline{\delta}_i \leq \delta_j \leq i^{-1}$, we have that $\delta \in [\overline{\delta}_i, i^{-1})$ as required.

We now take up the proof of Lemma 12.6.3. □

We noted in the proof of Lemma 12.6.5 that there exists an $M > 0$ such that for any admissible trajectory $\psi(\ ; t_0, x_0, v)$, the inequality $|\psi(s; t_0, x_0, v)| \leq M$ holds for all $t_0 \leq s \leq T$. It then follows from the continuity of f that all admissible trajectories $\psi(\cdot; t_0, x_0, v)$ satisfy a uniform Lipschitz condition on $[t_0, T]$.

We shall now obtain the admissible pair (ψ, v) whose existence is asserted in (12.6.7). For an arbitrary positive integer $i \geq i_0$, where i_0 is as in Lemma 12.6.6, a set of mesh points $\{t_0, t_1, \ldots, t_{N_{i+1}} = T\}$ will be chosen together with a control v with $v(t) = \overline{z}_j \in C$ for $t_{j-1} \leq t < t_j$, $j = 1, \ldots, N_{i+1}$ so that the corresponding trajectory $\psi(\ ; t_0, x_0, v)$ satisfies (12.6.7).

We define the sequences $\{t_j\}$ and $\{z_j\}$ recursively. Suppose that $\{t_0, t_1, \ldots, t_k\}$ and $\{\overline{z}_1, \ldots, \overline{z}_k\}$ have been defined, where $t_{j-1} < t_j$, where $t_k < T - i^{-1}$, and where $\overline{z}_j \in C$, $j = 1, \ldots, k$. Let $v(t) = \overline{z}_j$ for $t \in [t_{j-1}, t_j], j = 1, \ldots, k$ and let $\psi(\ ; t_0, x_0, v)$ be the corresponding admissible trajectory defined on $[0, t_k]$. Let $\xi_k = \psi(t_k; t_0, x_0, v)$.

By Lemma 12.6.6 we can choose a $\overline{z}_{k+1} \in C$ and a mesh point t_{k+1} so that

$$t_k + \overline{\delta}_k < t_{k+1} < t_k + i^{-1} \tag{12.6.11}$$

and

$$(t_{k+1} - t_k)^{-1}[V(t_{k+1}, \xi_{k+1}) - V(t_k, \xi_k)] < i^{-1}, \tag{12.6.12}$$

where

$$\xi_{k+1} = \psi(t_{k+1}; t_k, \xi_k, v_{\overline{z}_{k+1}}) = \psi(t_{k+1}; t_0, x_0, v).$$

Note that $|\xi_{k+1}| \leq M$. The recursion is terminated when we arrive at the first index value $k = N_i$ for which

$$T - i^{-1} \leq t_{N_i} \leq T. \tag{12.6.13}$$

This will occur by virtue of (12.6.11). On the interval $[T_{N_i}, T]$ take $v(t) = \overline{z}_T$, an arbitrary element of C.

From the Lipschitz continuity of V and from (12.6.12) we have that

$$\begin{aligned} V(T, \psi(T)) - V(t_0, x_0) &= V(T, \psi(T)) - V(t_{N_i}, \psi(t_{N_i})) \\ &\quad + \sum_{k=0}^{N_{i-1}} [V(t_{k+1}, \psi(t_{k+1})) - V(t_k, \psi_k(t))] \\ &\leq K|(T, \psi(T)) - (T_{N_i}, \psi(T_{N_i})| + \sum_{k=0}^{N_{i-1}} (t_{k+1} - t_k) i^{-1}. \end{aligned}$$

From (12.6.13) we have that $T - T_{N_i} < i^{-1}$. Let A denote the uniform Lipschitz constant for all admissible trajectories with initial point (t_0, x_0). Then

$$V(T, \psi(T)) - V(t_0, x_0) \leq i^{-1}(K(1 + A) + T).$$

Given an $\varepsilon > 0$, we may choose i so that $i^{-1}(K(1 + A) + T) < \varepsilon$. Since $V(T, \psi(T)) = g(\psi(T))$, we get that (12.6.7) holds.

12.7 The Value Function as Verification Function

In Problem 12.2.1 let $\overline{\psi}$ be a trajectory with control $\overline{v}$ that satisfies the Maximum Principle. In the absence of other information such as, for example, the existence of a solution and the uniqueness of a solution that satisfies the Maximum Principle, we cannot conclude that $\overline{\psi}$ solves Problem 12.2.1. Thus, it is desirable to have a condition that *verifies* the optimality of a suspect solution. Theorem 12.7.1 will furnish such a test, which involves the value function. Hence the value function can be considered to be a *verification function.* We take Problem 12.2.1 to be in Mayer form.

Theorem 12.7.1. *Let Assumption 12.2.1-r hold. Let the value function W be Lipschitz continuous on compact subsets of $\mathcal{R}_0$ and continuous on $\mathcal{R}_0 \cup \mathcal{J}$. For $(t_1, x_1) \in \mathcal{J}$ let*

$$W(t_1, x_1) = g(t_1, x_1). \tag{12.7.1}$$

Let $\psi(\ ;\tau,\xi)$ be an admissible trajectory for the problem with initial point (τ,ξ) in $\mathcal{R}_0$, and let t_1 be the terminal time of $\psi(\ ;\tau,\xi)$. Let $\omega(t;\psi) = W(t,\psi(t))$ on $[\tau, t_1)$. If

$$D^- W(t,\psi(t); 1, \psi'(t)) = 0 \quad \textit{a.e. on} \quad [\tau, t_1). \tag{12.7.2}$$

Then ψ is optimal for Problem 12.2.1.

Proof. Since W is Lipschitz continuous, the function $\omega(\ ;\psi)$ is absolutely continuous. Thus, almost all points of (τ, t_1) are simultaneously Lebesgue points of $\psi(\ ;\tau,\xi)$ and points of differentiability of $\omega(\ ;\psi)$.

At such points we have

$$\begin{aligned} \frac{d\omega}{dt} &= \lim_{\delta\to 0} [W(t+\delta, \psi(t+\delta)) - W(t,\psi(t))]\delta^{-1} \qquad (12.7.3) \\ &= \lim_{\delta\to 0}[W(t+\delta, \psi(t) + \int_t^{t+\delta} \psi'(s)ds) - W(t,\psi(t))]\delta^{-1} \\ &= \lim_{\delta\to 0}[W(t+\delta, \psi(t) + \delta\psi'(t) + o(\delta)) - W(t,\psi(t))]\delta^{-1} \\ &= DW(t,\psi(t); 1, \psi'(t)), \end{aligned}$$

where in passing to the last line we used the Lipschitz continuity of W and the fact that we know that the limit in the next to the last line exists. $\square$

From the definition of $\omega(\ ;\psi)$ we get that

$$W(t_1, \psi(t_1)) - W(\tau,\xi) = \omega(t_1;\psi) - \omega(\tau,\xi) = \int_\tau^{t_1} \frac{d\omega}{dt}\ (t;\psi)ds.$$

It then follows from (12.7.1) and (12.7.3) that if (12.7.2) holds, then

$$g(t_1, \psi(t_1)) = W(t_1, \psi(t_1)) = W(\tau, \xi). \tag{12.7.4}$$

Now let $\psi_0(\ ;\tau,\xi)$ be an arbitrary admissible trajectory for Problem 12.2.1 with initial point (τ,ξ). The chain of equalities in (12.7.3) is valid for the function $\omega(\ ;\psi_0)$ in place of $\omega(\ ;\psi)$. From Theorem 12.4.3 we get that for almost all t in $[\tau, t_{01})$, where t_{01} is the terminal time of ψ_0,

$$DW(t,\psi_0(t);1,\psi_0'(t)) \geq 0.$$

From this and from (12.7.4) we see that

$$g(t_{10},\psi_0(t_{01})) = W(t_{10},\psi_0(t_{01})) \geq W(\tau,\xi) = g(t_1,\psi(t_1)).$$

Since for any admissible pair (ψ_0, v_0), $J(\psi_0,v_0) = g(t_{01},\psi_0(t_{01}))$, the optimality of (ψ, v) follows.

Remark 12.7.2. Clearly, the assumption (12.7.2) can be replaced by $d\omega(t,\psi(t))/dt = 0$ a.e.

The next result states that if $\overline{\psi}$ is optimal, then (12.7.2) holds. Thus, (12.7.2) can be said to characterize an optimal control.

Corollary 12.7.3. *Let $\overline{\psi}(\ ) = \overline{\psi}(\ ;\tau,\xi)$ be an optimal trajectory with initial point (τ,ξ) and terminal time $\overline{t}_1$. Then for almost all t in $[\tau,\overline{t}_1]$ and all h in $Q(t,\overline{\psi}(t))$*

$$0 = \frac{d\omega}{dt} = D^-W(t,\overline{\psi}(t);\ 1,\overline{\psi}'(t)) \leq D^-W(t,\overline{\psi}(t);1,h).$$

If $\overline{\psi}(\ ) = \overline{\psi}(\cdot,\tau,\xi)$ is an optimal trajectory for the problem with initial point (τ,ξ), then by the Principle of Optimality,

$$W(\tau+\delta,\overline{\psi}(\tau+\delta)) - W(t,\psi(t)) = 0$$

for all $\tau \leq t \leq T$. Hence (12.7.3) with $\psi = \overline{\psi}$ gives

$$DW(t,\overline{\psi}(t);1,\overline{\psi}'(t)) = 0.$$

Combining this with (12.4.3) gives the result.

Remark 12.7.4. We emphasize that the conclusion (12.4.3) of Theorem 12.4.3 holds at all points of $\mathcal{R}_0$, while the conclusion of the corollary above holds at almost all t along an optimal trajectory.

12.8 Optimal Synthesis

We continue to consider the control problem in Mayer form. Taken together, Theorem 12.7.1 and Corollary 12.7.3 state that under appropriate

hypotheses, if $\psi(\ ;\tau,\xi)$ is an admissible trajectory then $\psi(\ ;\tau,\xi)$ is optimal for the problem with initial point (τ,ξ) if and only if

$$0 = D^{-}W(t,\psi(t);1,\psi'(t)) \leq D^{-}W(t,\psi(t);1,h)$$

for almost all t in $[\tau,T]$ and all h in $Q(t,\psi(t))$. Thus, if we set

$$U(t,x) = \text{ argmin } [D^{-}W(t,x;1,h)\colon h \in Q(t,x)],$$

then the vectors in $f(t,x,U(t,x))$ would contain the tangent vectors of all optimal trajectories passing through (t,x). The control vectors v in $U(t,x)$ would be the set of optimal controls at (t,x) for optimal trajectories starting at (t,x). We would expect to obtain an optimal trajectory for the problem with initial condition (τ,ξ) by solving the differential inclusion

$$x' \in f(t,x,U(t,x)) \qquad x(\tau) = \xi.$$

We noted earlier, in connection with a related question, that to carry out this program, regularity conditions must be imposed on the possibly set valued function $(t,x) \to U(t,x)$. The behavior of U, however, cannot be determined *a priori* from the data of the problem. Instead, to obtain an optimal trajectory, we shall construct a sequence of pairs $\{\psi_k, v_k\}$ such that ψ_k converges to an optimal trajectory.

Assumption 12.8.1. (i) Statements (i) to (iv) of Assumption 12.2.1-r hold with $K(t) = K$ and $M(t) = M$.

(ii) The set $\mathcal{R}_0 = [0,T] \times \mathbb{R}^n$.

(iii) The terminal set $\mathcal{J} = \{T\} \times \mathbb{R}^n$.

(iv) $\Omega(t) = C$, a fixed compact set.

(v) The sets $A_r(t)$ are non-empty.

(vi) The function g is locally Lipschitz continuous.

It follows from (iv) of Assumption 12.8.1 and from the definition of $\widetilde{\Omega}$, that the constraint set $\widetilde{\Omega}$ is a constant compact set. Let $\widetilde{C} = \widetilde{\Omega}$.

The next theorem will be used to show that our sequence of trajectories $\{\psi_k\}$ converges to an optimal trajectory ψ. The theorem states that for points (t,x) in a compact set, the minimum over h in $Q(t,x)$ of the difference quotient in (12.4.1) is uniformly small in absolute value for δ sufficiently small. This allows for the interchange of order $\min(\liminf) = \liminf(\min)$ in (12.4.3).

Theorem 12.8.2. *Let Assumption 12.8.1 hold. Let $\overline{B}_L$ denote the closed ball in $\mathbb{R}^n$ centered at the origin with radius L. Let $\mathcal{R}_{OL} = [0,T] \times \overline{B}_L$. Then for each $\varepsilon > 0$ there exists a $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, all (τ,ξ) in $\mathcal{R}_{OL}$ and all $\tau \leq t \leq \tau + \delta$*

$$\min_{\overline{z}\in\widetilde{\Omega}} |\delta^{-1}(W(\tau+\delta,\xi+\delta f(\tau,\xi,\overline{z})) - W(\tau,\xi))| < \varepsilon. \tag{12.8.1}$$

Proof. Under Assumption 12.8.1 the relaxed control problem has a solution for each initial point (t, x) in $\mathcal{R}_0$ and the value function is Lipschitz continuous on compact subsets of $\mathcal{R}_0$. (See Theorem 4.3.5 and Theorem 12.3.2.) $\square$

Let $\overline{z}$ be an arbitrary element of $\widetilde{C}$. Let $\psi(\ ;\tau,\xi,\overline{z})$ denote the trajectory with initial point (τ, ξ) and $v(t) = \overline{z}$. It follows from (12.2.2) and Corollary 4.3.15 that there exists a constant $A > 0$ such that for all (τ, ξ) in $\mathcal{R}_{OL}$ and all $\overline{z}$ in $\widetilde{C}$, $|\psi(t;\tau,\xi,\overline{z})| \leq A$ for all $\tau \leq t \leq T$. The function f is uniformly continuous on $[0, T] \times \overline{B}_A \times \widetilde{C}$ where $\overline{B}_A$ is the closed ball in $\mathbb{R}^n$ of radius A with center at the origin. From the solution, which holds for all admissible pairs (ψ, v),

$$\psi(t;\tau,\xi) = \xi + \int_{\tau}^{t} f(s, \psi(s;\tau,\xi), v(s))ds \tag{12.8.2}$$

it follows that $\psi(t,\tau,\xi) \to \xi$ as $t \to \tau$, uniformly for (τ, ξ) in $\mathcal{R}_{OL}$. It further follows that if we take $v(t) = \overline{z}$ for $\tau \leq t \leq T$ in (12.8.2), then given an $\varepsilon > 0$ there exists a $\delta_0(\varepsilon)$, independent of (τ, ξ) in $\mathcal{R}_{OL}$ and $\overline{z}$ in $\widetilde{C}$ such that whenever $0 < \delta < \delta_0(\varepsilon)$, we have

$$\psi(\tau+\delta;\tau,\xi,\overline{z}) = \xi + \int_{\tau}^{\tau+\delta} [f(\tau,\xi,\overline{z}) + \varepsilon(s)]ds,$$

with $|\varepsilon(s)| < \varepsilon$, for all (τ, ξ) in $\mathcal{R}_{OL}$ and $\overline{z}$ in $\widetilde{C}$. Hence

$$\psi(\tau+\delta;\tau,\xi,\overline{z}) = \xi + \delta f(\tau,\xi,\overline{z}) + o(\delta), \tag{12.8.3}$$

where $o(\delta)$ is uniform for (τ, ξ) in $\mathcal{R}_{OL}$ and $\overline{z}$ in $\widetilde{C}$.

By the Principle of Optimality we have

$$W(\tau+\delta, \psi(\tau+\delta;\tau,\xi,\overline{x})) - W(\tau,\xi) \geq 0.$$

From (12.8.3) and the Lipschitz continuity of W on compact sets we get that

$$W(\tau+\delta, \xi+\delta f(\tau,\xi,\overline{z})) - W(\tau,\xi) \geq o(\delta),$$

where $o(\delta)$ is uniform with respect to (τ, ξ) in $\mathcal{R}_{OL}$ and $\overline{z}$ in C. Since $\overline{z}$ in $\widetilde{C}$ is arbitrary we get that

$$\min_{\overline{z}\in\widetilde{C}} W(\tau+\delta, \xi+\delta f(\tau,\xi,\overline{z})) - W(\tau,\xi) \geq o(\delta), \tag{12.8.4}$$

where $o(\delta)$ is independent of (τ, ξ) in $\mathcal{R}_{OL}$.

Now let (ψ, v) be optimal for the problem with initial point (τ, ξ). Then $|\psi(t,\tau,\xi)| \leq A$ for all (τ, ξ) in $\mathcal{R}_{OL}$ and $\tau \leq t \leq T$. The function f is uniformly continuous on $[0, T] \times \overline{B}_A \times \widetilde{C}$. It therefore follows from (12.8.2) that for each $\varepsilon > 0$ there exists a $\delta_0 = \delta_0(\varepsilon)$ such that for $0 < \delta < \delta_0$ and all (τ, ξ) in $\mathcal{R}_{OL}$

$$\psi(\tau+\delta;\tau,\xi) = \xi + \int_{\tau}^{\tau+\delta} [f(\tau,\xi,v(s)) + \varepsilon(s;\tau,\xi)]ds, \tag{12.8.5}$$

where $|\varepsilon(s;\tau,\xi)| < \varepsilon$ for all $0 \le s < \delta$ and all (τ,ξ) in $\mathcal{R}_{OL}$.

We have

$$\int_{\tau}^{\tau+\delta} f(\tau,\xi,v(s))ds = \delta \int_{\tau}^{\tau+\delta} f(\tau,\xi,v(s))\ \frac{ds}{\delta}, \qquad (12.8.6)$$

where $v(s) \in \widetilde{C}$ for a.e. s in $[\tau,\tau+\delta]$. Since $\widetilde{C}$ is a fixed compact set,

$$f(\tau,\xi,v(s)) \in Q(\tau,\xi), \qquad \text{a.e.},$$

where $Q(\tau,\xi)$ is a closed convex set. Let K_δ denote the set of points in $[\tau,\tau+\delta]$ at which the inclusion holds. Then

$$\text{cl co}\{f(\tau,\xi,v(s))\colon s \in K_\delta\} \in \ \text{cl co}\ Q(\tau,\xi) = Q(\tau,\xi).$$

From Lemma 3.2.9 we get that

$$\text{cl co}\ \{f(\tau,\xi,v(s))\colon s \in K_\delta\} = \ \text{cl}\ \{\int_{K_\delta} f(\tau,\xi,v(s))d\mu_s \colon \mu \in P(K_\delta)\},$$

where $P(K_\delta)$ is the set of probability measures in K_δ. Thus,

$$\int_{\tau}^{\tau+\delta} f(\tau,\xi,v(s))\ \frac{ds}{\delta} \in Q(\tau,\xi).$$

Hence there exists a $\overline{z}_\delta$ in $\widetilde{C}$ such that

$$\int_{\tau}^{\tau+\delta} f(\tau,\xi,v(s))\ \frac{ds}{\delta} = f(\tau,\xi,\overline{z}_\delta).$$

From this and from (12.8.6) and (12.8.5) we get that

$$\psi(\tau+\delta,\tau,\xi) = \xi + \delta f(\tau,\xi,\overline{z}_\delta) + o(\delta), \qquad (12.8.7)$$

where $o(\delta)$ is uniform with respect to $\overline{z}$ in $\widetilde{C}$ and (τ,ξ) in $\mathcal{R}_{OL}$.

Since ψ is optimal, by the Principle of Optimality

$$W(\tau+\delta,\psi(\tau+\delta)) - W(\tau,\xi) = 0.$$

From (12.8.7) we get that

$$W(\tau+\delta,\xi+\delta f(\tau,\xi,\overline{z}_\delta) + o(\delta)) - W(\tau,\xi) = 0.$$

From this and from the Lipschitz continuity of W we get that

$$W(\tau+\delta,\xi+\delta f(\tau,\xi,\overline{z}_\delta)) - W(\tau,\xi) = o(\delta),$$

where $o(\delta)$ is uniform with respect to (τ,ξ) in $\mathcal{R}_{OL}$. Hence

$$\min_{\overline{z}\in C}[W(\tau+\delta,\xi+f(\tau,\xi,\overline{z})) - W(\tau,\xi)] \le o(\delta), \qquad (12.8.8)$$

where $o(\delta)$ is uniform with respect to (τ, ξ) in $\mathcal{R}_{OL}$.

The theorem now follows from (12.8.8) and (12.8.4).

We next present an algorithm for generating a sequence $\{(\psi_k, v_k)\}$ of admissible pairs that will furnish the desired approximation to an optimal pair (ψ, v).

Algorithm 12.8.3. Consider Problem 12.2.1 in Mayer form with initial point (τ, ξ). Let Assumption 12.8.1 hold. For each positive integer k let $\{t_{k,0} = \tau, t_{k,1}, \ldots, t_{k,k-1}, t_{k,k} = T\}$ be a uniform partition of $[\tau, T]$. Let $\delta_k = (T-\tau)/k$.

We define (ψ_k, v_k) on $[t_{k,0}, t_{k,1}]$. Let $x_{k,0} = \xi$ and let

$$v_{k,0} = \arg\min_{\overline{z}\in\widetilde{C}}[W(t_{k,0}, x_{k,0} + \delta f(t_{k,0}, x_{k,0}, \overline{z})) - W(t_{k,0}, x_{k,0})].$$

For $t \in [t_{k,0}, t_{k,1})$ let $v_{k,0}(t) = v_{k,0}$. Define $\psi_{k,0}$ on $[t_{k,0}, t_{k,1}]$ to be the solution of

$$\psi'_{k,0}(t) = f(t, \psi_{k,0}(t), v_{k,0}(t)) \qquad \psi_{k,0}(t_{k,0}) = x_{k,0}.$$

Now suppose that (ψ_k, v_k) has been defined on $[t_{k,0}, t_{k,i}]$, for i in the range $1 \le i \le k-1$. We shall define an admissible pair $(\psi_{k,i}, v_{k,i})$ on $[t_{k,i}, t_{k,i+1}]$ in such a way that if we extend (ψ_k, v_k) to $[t_{k,i}, t_{k,i+1}]$ by setting $\psi_k(t) = \psi_{k,i}(t)$ and $v_k(t) = v_{k,i+1}(t)$ for $t_{k,i} \le t \le t_{k,i+1}$, we shall have an admissible pair defined on $[t_{k,0}, t_{k,i+1})$. Let $x_{k,i} = \psi(t_{k,i})$. Let

$$v_{k,i} = \arg\min_{\overline{z}\in\widetilde{C}}[W(t_{k,i}, x_{k,i} + \delta f(t_{k,i}, x_{k,i}, \overline{z})) - W(t_{k,i}, x_{k,i})]. \tag{12.8.9}$$

For $t \in [t_{k,i}, t_{k,i+1})$ let $v_{k,i}(t) = v_{k,i}$. Define $\psi_{k,i}$ on $[t_{k,i}, t_{k,i+1})$ to be the solution of

$$\psi'_{k,i}(t) = f(t, \psi_{k,i}(t), v_{k,i}(t)) \quad \psi_{k,i}(t_{k,i}) = x_{k,i}.$$

Define $\psi_{k,i}(t_{k+1}) = \lim_{t\to(t_{k+1}-0)} \psi_k(t)$. That this limit exists follows from the uniform boundedness of all admissible trajectories, the continuity of f, and the Cauchy criterion.

Since f is continuous and since $v_{k,i}(t) = v_{k,i}$ on $[t_{k,i}, t_{k,i+1})$ it follows that for $t \in [t_{k,i}, t_{k,i+1})$

$$\psi_{k,i}(t) = \psi_{k,i}(t_{k,i}) + (t - t_{k,i}) f(t_{k,i}, \psi(t_{k,i}), v_{k,i}) + o(t - t_{k,i}).$$

Hence

$$\psi(t_{k,i+1}) = \psi(t_{k,i}) + \delta_k f(t_{k,i}, \psi(t_{k,i}), v_{k,i}) + o(\delta_k). \tag{12.8.10}$$

Theorem 12.8.4. *Let Assumption 12.8.1 hold. Then the sequence $\{\psi_k\}$ generated by Algorithm 12.8.3 has subsequences $\{\psi_{k_j}\}$ that converge uniformly to absolutely continuous functions ψ. Corresponding to each such ψ there exists an admissible control v such that (ψ, v) is admissible and is optimal for Problem 12.2.1.*

Proof. It again follows from (12.2.2) and Corollary 4.3.15 that the functions ψ_k are uniformly bounded. It then follows from the continuity of f that $\{|f(t, \psi_k(t), v_k(t)|\}$ is uniformly bounded for $\tau \leq t \leq T$. From this and from

$$\psi_k(t) = \xi + \int_\tau^t f(s, \psi_k(s), v_k(s))ds \tag{12.8.11}$$

it follows that the functions ψ_k are equi-absolutely continuous. Hence there exists a subsequence, again denoted by $\{\psi_k\}$, that converges uniformly to an absolutely continuous function ψ. □

We may write (12.8.11) as

$$\psi_k(t) = \xi + \int_\tau^t f(s, \psi_k(s), \mu_{ks})ds, \tag{12.8.12}$$

where μ_k is a relaxed control given by

$$\mu_{ks} = \sum_{i=1}^{n+1} p_i(s)\delta_{v_{k_i}(s)}$$

concentrated on the compact set C and defined on the compact interval $[\tau, T]$. By Theorem 3.3.6 the sequence $\{\mu_k\}$ is weakly compact. Hence there exists a subsequence $\{\mu_k\}$ that converges weakly to a relaxed control μ concentrated in C. Corresponding to μ_k is the subsequence $\{\psi_k\}$ that converges uniformly to ψ. Relation (12.8.12) holds for the sequence $\{(\psi_k, \mu_k)\}$. From Lemma 4.3.3 and the relation $\psi(t) = \lim_{k\to\infty} \psi_k(t)$ uniformly we conclude that for all t in $[T, \tau]$

$$\begin{aligned}\psi(t) &= \lim_{k\to\infty} \psi_k(t) = \lim_{k\to\infty} [\xi + \int_\tau^t f(s, \psi_k(s), \mu_{ks})ds] \\ &= \xi + \int_\tau^t f(s, \psi(s), \mu_s)ds.\end{aligned}$$

Thus, (ψ, μ) is an admissible pair.

It remains to show that ψ is optimal. Since $g(\psi_k(T)) = J(\psi_k, \mu_k)$ and $\lim\limits_{k\to\infty} g(\psi_k(T)) = g(\psi(T)) = J(\psi, \mu)$, to show that (ψ, μ) is optimal it suffices to show that

$$\lim_{k\to\infty} g(\psi_k(T)) = W(\tau, \xi). \tag{12.8.13}$$

We noted previously that there exists a compact set $\mathcal{R}_{OC} \subseteq \mathcal{R}_0$ such that for all admissible trajectories ψ with initial point (τ, ξ), the points $(t, \psi(t))$ lie in $\mathcal{R}_{OC}$. The function f is bounded by some constant A on $\mathcal{R}_{OC} \times \widetilde{C}$ and is uniformly continuous there. The value function W is Lipschitz continuous on $\mathcal{R}_{OC}$ with Lipschitz constant B.

We have

$$
\begin{aligned}
&\limsup_{k\to\infty} |g(\psi_k(T)) - W(\tau,\xi)| \qquad (12.8.14)\\
&= \limsup_{k\to\infty} |W(t_{k,k}, \psi(t_{k,k})) - W(\tau,\xi)|\\
&\le \limsup_{k\to\infty} \sum_{j=1}^{k} |W(t_{k,j}, \psi_k(t_{k,j})) - W(t_{k,j-1}, \psi_k(t_{k,j-1}))|\\
&= \limsup_{k\to\infty} \sum_{j=1}^{k} |W(t_{k,j-1} + \delta_k, \psi(t_{k,j-1})\\
&\quad + \delta_k f(t_{k,j-1}, \psi(t_{k,j-1}), v_{k,j-1}))\\
&\quad - W(t_{k,j-1}, \psi_k(t_{k,j-1}))| + o(\delta_k)
\end{aligned}
$$

where the last equality follows from (12.8.10) and $\delta_k = (t_j - t_{j-1})$. It then follows from the Lipschitz continuity of W, the definition of δ_k as $\delta_k = (T-\tau)/k$, the definition of $v_{k,j-1}$ in (12.8.9), and Theorem 12.8.2 that *each summand* in the rightmost side of (12.8.14) is $o(1/k)$ as $k \to \infty$. Hence the rightmost side of (12.8.14) is $o(1)$ as $k \to \infty$, and thus (12.8.13) and Theorem 12.8.4 are established.

12.9 The Maximum Principle

In Section 6.2 we derived the maximum principle for a class of problems under the assumption that the value function is of class $C^{(2)}$. In this section we shall derive the maximum principle for a certain class of problems under the assumption that the value function is a Lipschitz continuous viscosity solution of the Hamilton-Jacobi equation (12.5.1). We assume that the problem at hand is the relaxed problem, Problem 12.2.1. We consider this problem to be an ordinary problem as in Section 5.4.

Assumption 12.9.1. Assumption 12.2.1-r is in force with the following changes.

(i) For fixed (t, z) the function $\widehat{f}(t, \cdot, z)$ is of class $C^{(1)}$ on $\mathbb{R}^n$.

(ii) The set $\mathcal{R}_0 = [0, T] \times \mathbb{R}^n$.

(iii) The terminal set $\mathcal{T}$ is $\{T\} \times \mathbb{R}^n$. Thus, the function g is a function of x alone.

(iv) For all (t, x) in $\mathcal{R}_0$, the sets $\Omega(t, x)$ are a fixed compact set, C.

Remark 12.9.2. Assumption 12.9.1(i) implies Assumption 12.2.1-r (iii). Assumption 12.9.1 and Theorems 12.5.3 and 12.6.2 imply that the value function W is the unique viscosity solution of the Hamilton-Jacobi equation (12.5.1) with boundary condition $V(T,x) = g(x)$.

Remark 12.9.3. In the definition of viscosity solution the test functions were taken to be of class $C^{(1)}$, as in the definition given by Crandall and Lions. An examination of the proofs in Sections 12.5 and 12.6 shows that we did not use the continuity of the partial derivatives of the test functions. All we used was the existence of partial derivatives. Therefore, for our purposes we could have restricted ourselves to test functions that possess partial derivatives. The requirement of continuous differentiability is needed for the general nonlinear partial differential equation, but is not needed for the Hamilton-Jacobi equation (12.5.1).

Let (τ_0, ξ_0) be a point in $[0,T) \times \mathbb{R}^n$. Let $\psi^*(\ ) = \psi^*(\ ;\tau_0,\xi_0)$ be an optimal trajectory for the problem with initial point (τ_0,ξ_0) and let $v^*(\ ) = v^*(\ ;\tau_0,\xi_0)$ be the corresponding optimal control. Let $\widehat{f} = (f^0, f)$ and for $t \geq \tau$ and $x \in \mathbb{R}^n$ let

$$\widehat{F}(t,x) = \widehat{f}(t,x,v^*(t)). \tag{12.9.1}$$

Then for each $\tau_0 \leq t \leq T$ the function $\widehat{F}(t,\cdot)$ is of class $C^{(1)}$ on $\mathbb{R}^n$ and the function $\widehat{F}(\cdot,x)$ is measurable on $[\tau_s, T]$ for each fixed x in $\mathbb{R}^n$.

We consider the differential equation

$$x' = F(t,x) \quad x(\tau) = \xi \tag{12.9.2}$$

with $\tau \geq T_0$ and ξ in $\mathbb{R}^n$. It follows from Assumption 12.2.1-r that Eq. (12.9.2) has a unique solution $\psi(\ ) = \psi(\ ;\tau,\xi)$ defined on $[\tau, T]$. Note that $\psi^*(\ ) = \psi^*(\ ;\tau_0,\xi_0) = \psi(\ ;\tau_0,\xi_0)$.

For $\tau \geq \tau_0$ and $\xi \in \mathbb{R}^n$ we define a function ω as follows.

$$\omega(\tau,\xi) = g(\psi(T;\tau,\xi)) + \int_\tau^T F^0(t,\psi(t;\tau,\xi))dt, \tag{12.9.3}$$

where ψ is the unique solution of (12.9.2). It follows from the definitions of $W, \widehat{F}, \psi^*$, and v^* that

$$\omega(\tau_0,\xi_0) = W(\tau_0,\xi_0) = J(\psi^*(\ ;\tau_0,\xi_0), v^*(\ ;\tau_0,\xi_0)). \tag{12.9.4}$$

From the Principle of Optimality it further follows that

$$\omega(t,\psi^*(t;\tau_0,\xi_0)) = W(t,\psi^*(t;\tau_0,\xi_0)) \tag{12.9.5}$$

for all $\tau_0 \leq t \leq T$. For Problem 12.2.1 with initial point $(\tau,\xi), \tau_0 \leq \tau < T$, the control v^* need not be optimal. Therefore, since $\omega(\tau,\xi)$ is the payoff for this choice of control, we have that

$$\omega(\tau,\xi) \geq W(\tau,\xi), \quad \tau_0 \leq \tau < T, \quad \xi \in \mathbb{R}^n. \tag{12.9.6}$$

From (12.9.5) and (12.9.6) we get that at all points $(t, \psi^*(t; t_0, x_0))$, $\tau_0 \le t \le T$, on the optimal trajectory $W - \omega$ have a maximum. Lemma 12.9.4 states that at almost all points $(t, \psi^*(t))$ of the optimal trajectory the function ω is a test function as defined in Remark 12.9.3. Assume this to be true. Since W is a viscosity solution, it is also a subsolution. Hence for almost all t in $[\tau_0, T]$

$$-\omega_t(t, \psi^*(t)) + \overline{H}(t, \psi^*(t), -\omega_x(t, \psi^*(t))) \le 0. \tag{12.9.7}$$

Lemma 12.9.4. *At almost all points $(t, \psi^*(t))$, $\tau_0 \le t \le T$ the partial derivatives ω_τ and ω_ξ exist.*

Proof. From (12.9.3) we get that

$$\omega_\xi(\tau, \xi) = g_x(\psi(T; \tau, \xi))\psi_\xi(T; \tau, \xi) + \int_\tau^T F_x^0(t, \psi(t; \tau, \xi))\psi_\xi(t; \tau, \xi)dt. \tag{12.9.8}$$

The matrix $\psi_\xi(\ ; \tau, \xi)$ is a solution of the system

$$\gamma' = F_x(t, \psi(t; \tau, \xi))\gamma \qquad \gamma(\tau) = I. \tag{12.9.9}$$

Thus, $\psi_\xi(\ ; \tau, \xi)$ is an absolutely continuous matrix function defined on $[\tau, T]$. Hence standard theorems on differentiation under the integral justify the formula in Eq. (12.9.8). $\square$

We now consider the function ω evaluated along the optimal trajectory $\psi^*(\ ) = \psi^*(\ ; \tau_0, \xi_0)$. Designate the trajectory with initial point $(t, \psi^*(t))$ resulting from the control $v^*(\ )$ on $[t, T]$ by $\overline{\psi}(\ ) = \overline{\psi}(\ ; t, \psi^*(t))$. Then by the uniqueness theorem for solutions of ordinary differential equations and by the Principle of Optimality we get that for all $t \le s \le T$

$$\overline{\psi}(s; t, \psi^*(t)) = \psi^*(s; t, \psi^*(t)) = \psi^*(s; \tau_0, \xi_0). \tag{12.9.10}$$

Thus, from (12.9.8) we get that

$$\begin{aligned} \omega_\xi(t, \psi^*(t)) &= g_x(\psi^*(T))\psi_\xi^*(T; t, \psi^*(t)) \\ &\quad + \int_t^T F_x^0(s, \psi^*(s))\psi_\xi^*(s; t, \psi^*(t))ds. \end{aligned} \tag{12.9.11}$$

It follows from (12.9.11) that for fixed t in (τ, T), the function ω_ξ is continuous in a neighborhood of $(t, \psi^*(t))$. We shall not be able to conclude that ω_t, whose existence at almost all t we show next, is continuous.

From (12.9.3) and (12.9.10) we get that

$$\omega(t, \psi^*(t)) = g(\psi^*(T)) + \int_t^T F^0(s, \psi^*(s; \tau_0, \xi_0))ds.$$

Thus, $t \to \omega(t, \psi^*(t))$ is absolutely continuous and

$$d\omega(t, \psi^*(t))/dt = -F^0(t, \psi^*(t; \tau_0, \xi_0)) \tag{12.9.12}$$

for almost all t in $[\tau_0, T]$.

Let $t \in [\tau_0, T]$ be a Lebesgue point of ψ^* and a point of differentiability of $t \to \omega(t, \psi^*(t))$. The set of such points has full measure. We have for $\delta > 0$

$$[\omega(t+\delta, \psi^*(t)) - \omega(t, \psi^*(t))]\delta^{-1} = A\delta^{-1} + B\delta^{-1},$$

where

$$A = \omega(t+\delta, \psi^*(t)) - \omega(t+\delta, \psi^*(t+\delta)) \quad B = \omega(t+\delta, \psi^*(t+\delta)) - \omega(t, \psi^*(t)).$$

From (12.9.12) we have that $\lim_{\delta \to 0} B\delta^{-1} = -F^0(t, \psi^*(t))$.

Since t is a Lebesgue point of ψ^* we have that

$$A = \omega(t+\delta, \psi^*(t)) - \omega(t+\delta, \psi^*(t) + \delta\psi^{*'}(t) + o(\delta)).$$

Since $\omega_\xi(t+\delta, \cdot)$ is continuous in a neighborhood of $\psi^*(t)$,

$$A = -\langle \omega_\xi(t+\delta, \psi^*(t)), \psi^{*'}(t) + o(\delta)\rangle\delta + o(\delta).$$

Hence

$$\lim_{\delta \to 0} A\delta^{-1} = -\langle \omega_\xi(t, \psi^*(t)), \psi^{*'}(t)\rangle = -\langle \omega_\xi(t, \psi^*(t)), F(t, \psi^*(t))\rangle.$$

Thus, $\omega_\tau(t, \psi^*(t))$ exists and

$$\omega_\tau(t, \psi^*(t)) = -F^0(t, \psi^*(t)) - \langle \omega_\xi(\tau, \psi^*(t)), F(t, \psi^*(t))\rangle \qquad (12.9.13)$$

for almost all t in $[\tau_0, T]$.

Having established Lemma 12.9.4, we have established (12.9.7). We rewrite (12.9.7) using the definitions in Section 12.5 to get that for almost all t in $[\tau_0, T]$

$$-\omega_\tau(t, \psi^*(t)) + \max_{\overline{z} \in \widetilde{C}}[-f^0(t, \psi^*(t), \overline{z}) - \langle \omega_\xi(t, \psi^*(t)), f(t, \psi^*(t), \overline{z})\rangle \leq 0. \qquad (12.9.14)$$

We rewrite (12.9.13) using the definition of $\widehat{F}$ in (12.9.1) to get that for almost all t in $[\tau_0, T]$

$$-\omega_\tau(t, \psi^*(t)) - f^0(t, \psi^*(t), v^*(t)) - \langle \omega_\xi(t, \psi^*(t)), f(t, \psi^*(t), v^*(t))\rangle = 0. \qquad (12.9.15)$$

From (12.9.14) and (12.9.15) we get that for almost all t in $[\tau_0, T]$

$$\begin{aligned} &\max_{\overline{z} \in \widetilde{C}}[-f^0(t, \psi^*(t), \overline{z}) - \langle \omega_\xi(t, \psi^*(t)), f(t, \psi^*(t), \overline{z})\rangle \\ &\quad = -f^0(t, \psi^*(t), v^*(t)) - \langle \omega_\xi[t, \psi^*(t)), f(t, \psi^*(t), v^*(t))\rangle. \end{aligned} \qquad (12.9.16)$$

We next introduce the multipliers $\lambda(\ ) \equiv \lambda(\ ; \tau_0, \xi_0)$ via the adjoint equations and show that $\omega_\xi(t, \psi^*(t)) = -\lambda(t)$ a.e. on $[\tau_0, T]$. Let $\lambda(\ ; \tau_0, \xi_0)$ be the unique solution on $[\tau_0, T]$ of the linear system

$$\frac{dp}{dt} = f_x^0(t, \psi^*(t), v^*(t)) - pf_x(t, \psi^*(t), v^*(t)) \qquad (12.9.17)$$

$$p(T) = -g_x(\psi^*(T)).$$

Then for fixed t in $[\tau_0, T)$ and all almost all $t \le s \le T$

$$\begin{aligned}\lambda'(s;\tau_0,\xi_0)\psi^*_\xi(s;t,\psi^*(t)) &= f^0_x(s,x(s),v^*(s))\psi^*_\xi(s;t,\psi^*(t)) \\ &\quad - \lambda(s;\tau_0,\xi_0)f_x(s,\psi^*(s),v^*(s))\psi^*_\xi(s;t,\psi^*(t)).\end{aligned}$$

Also,

$$\lambda(T;\tau_0,\xi_0)\psi^*_\xi(T;t,\psi^*(t)) = -g_x(\psi^*(T))\psi^*_\xi(T;t,\psi^*(t)).$$

Recalling the definition of $\widehat{F}$ in (12.9.1) and substituting into (12.9.11) and using (12.9.9) gives

$$\begin{aligned}\omega_\xi(t,\psi^*(t)) &= -\lambda(T)\psi^*_\xi(T;t,\psi^*(t)) \\ &\quad + \int_t^T [\lambda'(s)\psi^*_\xi(s;t,\psi^*(t)) \\ &\quad + \lambda(s)f_x(s,\psi^*(s),v^*(s))\psi^*_\xi(s;t,\psi^*(t))]ds \\ &= -\lambda(T)\psi^*_\xi(T;t,\psi^*(t)) + \int_t^T d(\lambda(s)\psi^*_\xi(s;t,\psi^*(t))) \\ &= -\lambda(t)\psi^*_\xi(t;t,\psi^*(t)) = -\lambda(t)I.\end{aligned} \tag{12.9.18}$$

Combining (12.9.16), (12.9.17), and (12.9.18) gives the following theorem, which is the maximum principle in this case.

Theorem 12.9.5. *Let (ψ^*, v^*) be an optimal pair for Problem 12.2.1 with initial point (τ, ξ). Then there exists an absolutely continuous function $\lambda(\ ) = \lambda(\ ;\tau,\xi)$ such that for almost all t in $[\tau, T]$*

$$\begin{aligned}\lambda'(t) &= f^0_x(t,\psi^*(t),v^*(t)) - \lambda(t)f_x(t,\psi^*(t),v^*(t)) \\ \lambda(T) &= -g_x(\psi^*(T)).\end{aligned}$$

Moreover, for almost all t in $[\tau, T]$

$$\begin{aligned}&\max_{\overline{z}\in\widetilde{C}}[-f^0(t,\psi^*(t),\overline{z}) + \langle\lambda(t), f(t,\psi^*(t),\overline{z})\rangle] \\ &\quad = [-f^0(t,\psi^*(t),v^*(t)) + \langle\lambda(t), f(t,\psi^*(t),v^*(t))\rangle].\end{aligned}$$

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